ELECTROMAGNETIC ENGINEERING

I. Fundamentals

RADIO COMMUNICATION SERIES

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ELECTROMAGNETIC ENGINEERING

Vol. I. Fundamentals.

BY

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First Edition
Third Impression

PREFACE

For many years the Cruft Memorial Laboratory at Harvard University has offered courses in electronics, communication engineering, and applied physics. The aim of these courses has been and continues to be the analysis of practical problems from the point of view of fundamental physical theory. "Electromagnetic Engineering," a series of advanced textbooks, is designed to follow closely, although in a somewhat expanded form, the subject matter presented in courses offered at the Cruft Memorial Laboratory. In the normal sequence they are intended for graduate students and advanced seniors, but they have been taught with success to large groups of undergraduate concentrators in electronic physics working under the accelerated programs of wartime. Volume I, "Fundamentals," contains largely work of the first semester in the sequence. It is designed to provide the physical and mathematical background prerequisite to the systematic study of such fields as antennas and wave propagation, transmission circuits including wave guides, and transit-time electronics with application to microwave generators, which are the subjects of later volumes.

The approach to electromagnetic theory departs from the classical, more or less historical method in order to provide a logically coordinated, moderately comprehensive introduction to a difficult subject. Since it begins with first principles, it does not depend directly upon any previous work in electricity and magnetism, although an intermediate course including alternating currents is a desirable preparation. A working knowledge of calculus and of complex algebra, and some familiarity with elementary differential equations is presupposed.

Chapter I serves a double purpose. Although directed primarily toward defining density functions and formulating the principle of conservation of electricity, it also introduces vector operators in terms of fundamental *electromagnetic* concepts, rather than merely as a mathematical symbolism. Instead of summarizing vector analysis in an introductory chapter or in an

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appendix, it is made an integral part of the formulation of electromagnetic principles. In this way, it becomes associated in the student's mind with definite physical pictures that are basic in the very subject he is studying. Chapters II and III contain the outline of classical electromagnetism. Chapter IV deals with electromagnetic phenomena in unbounded regions and with asso·ciated general theorems. Chapters V and VI are concerned with the electromagnetic foundations of electric circuit theory. Rationalized m.k.s. units are used throughout. No claim is made for completeness. Many topics that might properly be called fundamentals are included in a logical way in Volume II, "Antennas," and in Volume III, "Transmission Circuits and Wave Guides."

In treating an old subject in a somewhat unconventional way the writer is happy to acknowledge with gratitude the rare opportunity he had at the University of Wisconsin where he learned electromagnetic theory rigorously and critically from Professor Warren Weaver of the Department of Mathematics, and practically and systematically from Professor Edward Bennett of the Department of Electrical Engineering. The first draft of the manuscript was prepared while the writer enjoyed unusual intellectual privileges as a John Simon Guggenheim Memorial Fellow in Berlin and Munich; its final preparation was aided by a grant from the Milton Fund of Harvard University. Many improvements are the result of constructive criticism by members of the faculty and students at Harvard University, and in particular, due to the gracious cooperation of Dr. P. LeCorbeiller who read the entire manuscript. The writer is happy to acknowledge the help and encouragement of Justine M. King who typed the first draft and facilitated and inspired its completion. The following instructors and graduate students devoted many hours to the careful reading of the proof; they made numerous valuable suggestions and corrections: H. F. Maling, the galley proof; W. M. Gottschalk, Chapter I; R. P. Lett, Chapter II; C. T. Tai, Chapters III and IV and the Appendices; B. C. Dunn, Chapter V; J. Porter and W. D. Woo, Chapter VI.

RONOLD W. P. KING.

NOTE CONCERNING THE NUMBERING
OF EQUATIONS AND FIGURES
AND THE NOTATION

Chapters are numbered in Roman; sections are numbered in Arabic beginning with 1 in each chapter; equations are numbered consecutively (1), (2), . . . , in each section with no reference to the section number. At the top of each left-hand page is the chapter number; at the top of each right-hand page is the section number. When reference is made to an equation in the same section, only the equation number is given, *e.g.* (5). When reference is made to an equation in another section in the same chapter, the section and equation numbers are given in the form (7.14). When reference is made to an equation in another chapter, the chapter number, section number, and equation number are given, *e.g.* (II.7.14). Figures are numbered giving both section and figure number; thus, Fig. 6.2 is the second figure in Sec. 6. Reference to figures in another chapter includes the chapter number, *e.g.*, Fig. II.6.6. By referring to chapter and section numbers at the tops of the pages, any equation is quickly found.

The symbolism includes italic, Greek, and Gothic letters in lightface and boldface. It is described in the Index of Symbols on page 543. The alignment of the Gothic type is slightly lower than the italic. This is not intentional but unavoidable. Care must be exercised not to read a slightly lower Gothic letter as a subscript. Gothic letters are never used as subscripts, so that no confusion should arise.

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INTRODUCTION

THE PHYSICAL WORLD AND ITS ANALOGUE

Physical science attempts to describe natural phenomena in terms of pictures and models which may be mathematical or physical but which are usually combinations of both. Engineering science makes use of such models in the solution of practical problems. A typical model consists of a logical sequence of mathematical symbols and operations together with an explanatory text and schematic diagrams describing a mechanism or structure which may be based in varying degrees upon observation and imagination. To be practically valuable such a model must be constructed so that it is possible to establish a correspondence or analogy between some of its parts or symbols and an array of tabulated pointer readings. These readings are obtained by direct observation of a section of the physical world contained in the apparatus of an experiment prepared as an analogue of the conditions assumed in the model. Such directly observed readings are called measured quantities. They include all observations on calibrated scales. It is customary to assign the same name and symbol to a measured quantity as to its theoretical analogue in the model. Thus the name deflection and the symbol d are assigned both to the successive directly observed pointer readings of an ammeter as a rheostat in a particular arrangement of conductors is turned, and to an analogue of this deflection in a mathematical formula devised to predict the behavior of a theoretical ammeter under hypotheses corresponding closely to the actual conditions. In addition to such analogous pairs of theoretical and measured quantities as are expressed in pointer readings, there may be many symbols and terms in every model which have no directly measurable analogues. Consider, for example, the theoretical model of an

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ammeter. This consists of a mathematical derivation, with its hypothesis, explanatory text, and diagrams, and a final formula expressing a functional relationship between the quantity called deflection and denoted by d and another quantity called current, symbolized by I. Both of these are defined in terms of the specific model of the ammeter and the more general and inclusive model known as electrical theory. For the theoretical deflection appearing in the model, there is an analogous, measured deflection directly observable on the physical instrument. For the current referred to in the model, however, there is no measurable analogue, and the symbol I as well as the concept of current belong entirely to the model. Because of the direct functional relationship existing in the mathematical model between d and I, it is customary to number the divisions on the meter in terms of I and assume that current, not deflection, is measured. In order to assure an accurate correspondence between the theoretical and the observed deflection, a standard experiment is designated in which theoretical hypotheses and practical construction are as closely analogous as possible. It is in performing this experiment that observed deflections are substituted directly for theoretical ones. The meter is then said to be calibrated as a primary standard.

If a general theory gives correct predictions wherever measurable analogues have been provided, it is universally accepted as a reliable model for predicting events and observations. In time, and in the minds of many, such a model is often identified completely with the physical world, and its structures and mechanisms as well as all its symbols are endowed with a physical reality whether or not they have observable and measurable analogues. Thus, as a result of its remarkable success in making pointer-reading predictions, the terminology of electrical theory and of its special models has been transferred to the physical world of measurable quantities, and the terms charge, current, electric intensity, magnetic flux, etc., are used as though they all had a directly measurable, physical reality. Strictly speaking, however, they are only constituents of a model that has accurately established a large number of pointer-reading analogues, an achievement that another, perhaps quite different model might do as well or better. In fact, the history of physical science is an account of new and better models designed to

describe the ever-growing numbers and the increasing complexity of physical apparatus and to provide theoretical analogues for their pointer readings. It is, therefore, not wise to identify completely any model with the physical world that it seeks to describe. To do so is to deny that spirit of scientific research which takes pride not only in a good representation but in ever striving for one that is at the same time more comprehensive and complete, while yet structurally and symbolically simpler. Although the engineering tradesman often, and rightly, finds physical theory strangely nebulous, the engineering scientist has long learned that it is from these nebulae that the great stars of the engineering future condense, become real and practical.

The electron theory of H. A. Lorentz is a recasting of Maxwell's electromagnetic theory into a form that relates it more closely to the atomic theory. It is an attempt to coordinate two models of the physical world, each of which has been signally successful in establishing accurate theoretical analogues for large numbers of experimental measurements drawn from widely different branches of physics. Since the Maxwellian model failed just where the atomic model predicted that it would have to fail, a new representation had to be devised which, while continuing to provide all the successful analogues of the old one, would also coordinate among them a large number of those which had failed to agree with experiment. The Maxwell-Lorentz model skillfully accomplishes this end. But it, too, has its limitations. Indeed, it is already apparent that one of the great tasks of the future is a systematic coordination of the Maxwell-Lorentz electrodynamics with the general theory of gravitation. But that is a chapter in theoretical physics, not in applied electromagnetism. It is mentioned here merely to point out that the Maxwell-Lorentz model of electrodynamics, amazingly successful as it is in predicting the behavior, for example, of antennas and transmission circuits, must not be accepted as an ultimate picture of the physical world. It does not pretend to describe physical reality, but merely to exhibit as close an analogy with the physical world as possible wherever a direct correspondence can be achieved. The statement, that a theoretical quantity has been experimentally verified or determined, may be understood to mean either that a direct experimental analogue for that

quantity has been found, or that a value has been calculated from the direct experimental analogues of other quantities functionally related to the one verified. In every case, a deliberate transfer from the pointer readings of an experiment to the theoretical analogues of the model is implied. A clear understanding of the fundamental significance of this transfer is essential in scientific engineering. It explains the difference between theoretical and empirical formulas. It must be considered carefully whenever a theoretical formula is applied to a practical problem. Differences between the predictions of a generally proved theory and a particular experiment are in almost all cases due to the fact that the measured quantity is not actually an analogue of the theoretical quantity bearing the same name, so that a transfer of pointer readings is not justified. For example, if a theoretical formula for the impedance of an antenna is derived for a specified physical model and an impedance is measured experimentally for an antenna which differs considerably from the physical model used in the theory, an apparent agreement between the two sets of results constitutes no verification of the theory. Scientific engineering is not possible without knowledge of the physical models assumed in theoretical derivations and the actual conditions obtaining in the practical problems to be analyzed. No theoretical formula and no tabulation of experimental observations are of practical value without a complete and accurate specification of assumed or existing conditions. The principal problem confronting the electromagnetic engineer is not that of substituting numbers in handbook formulas. His task requires intelligent coordination rather than blind identification of theoretical and empirical results. Electromagnetic theory is for the electromagnetic engineer what the X ray is for the doctor. Often it can reveal only broad outlines, not photographic detail. But when skillfully used and intelligently interpreted, its models and analogues are invaluable.

CHAPTER I

THE MATHEMATICAL DESCRIPTION OF MATTER: DENSITY FUNCTIONS AND VECTOR OPERATORS

Electromagnetic engineering is concerned with solutions of mathematical equations which can be interpreted physically and so applied to practical problems in electrical communication. It deals with an elaborate symbolism that describes the physical models in terms of which all electrical phenomena are explained. It requires a mathematical shorthand to express as concisely as possible the interrelations between the symbols in accordance with fundamental postulates of physical science. This chapter is concerned with the vector-analytical description of the electrical properties of matter.

STATIONARY STATES: THE STATIC STATE

1. Electric Charge.—In the systematic formulation of an electrodynamical model to serve as the mathematical foundation of electromagnetic engineering, it is convenient to consider electric charge a basic concept that is not derivable from or expressible in terms of other concepts. This view is in accord with atomic theory which constructs its models of matter in terms of the four concepts: space (length L), time (T), mass (M), and electric charge (Q). Its picture of matter involves vast numbers of molecules characterized by mass and random velocity in space. The molecules are combinations of atoms; the atoms are complex structures consisting of electrons and protons (neutrons, positrons). The electron is not divisible, is associated with a mass m, and is invariably characterized by a definite quantity of negative electric charge $-e$. For most engineering purposes a simplified model of the nucleus is adequate: a mass M with a positive electric charge Ne in an atom containing N electrons outside the nucleus. The atom as a whole is neutral. In constructing the electrodynamical model, it is adequate to treat electrons and positive nuclei as charges and masses associated with points. It is not necessary to specify shape or volume.

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Two fundamental properties of electric charge are postulated. The first is its indestructibility. A basic characteristic of electric charge is that it can neither be created nor destroyed. If a charge disappears from one point, it must reappear at another. This postulate is called the *principle of conservation of electricity*. The second postulate is that of *mutual interaction at a distance*. This assumes attraction and repulsion between charges both as a characteristic of the charges themselves and of their relative motions. No attempt is made at this point to formulate these two postulates with precision. They will be invoked qualitatively in describing the physical models upon which the mathematical structure that constitutes the real theory of electromagnetism is based. An essential part of this theory is the formulation of the mathematical counterparts of these two qualitative principles assumed in the physical model.

The fundamental problem of engineering electrodynamics is that of incorporating millions of electrons and positive nuclei in a mathematical model from which pointer readings associated, for example, with antennas and transmission circuits may be predicted in terms of their theoretical analogues. It is formally possible to set up this problem in terms of the individual position and motion of each of the fabulous numbers of electric charges contained in the atomic model of matter. But this formulation is mathematically so complicated that it is of no practical value for deriving experimentally verifiable quantities. Clearly, *a fundamental prerequisite of any model is a reasonable simplicity and transparency in its mathematical structure, even at the expense of consistency in the picture and of a degree of accuracy in its final predictions*. In the present case, as in others involving a very large number of similar units, great simplification results from a study of average rather than of individual behavior. This permits an approximate, over-all description in terms of continuous functions suitably defined throughout the body and so constructed that they assume at every point values characteristic of the average properties of a small region near the point. The number of such functions required and their complexity depend upon the number of classes of units present and their relative behavior, as well as upon the degree of approximation desired. For example, in representing the mass of a solid body in terms of its molecular model, a single, continuous func-

tion giving the average density D at every point is adequate to secure a good approximation. In this case there is only one class of units, since each molecule is characterized alike by mass. The corresponding electrical problem is considerably more complicated owing to the presence of two distinct classes of units, the positive nuclei and the electrons, and the possibility of widely dissimilar behavior of these because of their different structures and functions in the atom. In order to secure a reasonably good approximation in a large variety of problems, it is necessary to construct a mathematical model using several continuous functions called densities. These take account of the magnitude, the distribution, and the relative velocities of the charges. In certain problems requiring a higher degree of approximation, additional functions and a more complicated model may be constructed. These will not be described.

In order to define the continuous functions required for the mathematical model, it is convenient to construct and examine in detail separate atomic models with special structures. These are so chosen that a general case may be obtained by super-position or combination, although usually not without over-lapping. These special models are considered conveniently in two important groups or states called, respectively, the stationary and the nonstationary states. The former is subdivided into the static state and the steady state.

2. The Static State and the Atomic Model.—The static state is more correctly called the statistically stationary state because it does not involve charges at rest. On the contrary, all charges are assumed to be in motion in a most general and irregular way. A motion in which no regularity exists is called random. Each charge moves in its own unique way unlike that of any other. For any volume as a whole there must be no preferred directions, no common axes, nothing that would in any way permanently relate the average motion of one charge to that of another. Any effect due to the motion of one charge in a given direction or around a particular orbit is undone by the motion of other charges in the opposite direction somewhere in its neighborhood, or by the combined effect of the individual motions of many charges. All effects due to the individual random motions of the charges cancel over a time average taken over a period that is long compared with the time of atomic or molecular

events (such as rotations or collisions) but that may be very short from the point of view of an experimental observation. Such a time-average picture of the behavior of a large number of entities moving at random is called statistical. From the statistical point of view, a volume containing millions of charges moving at random is indistinguishable from the same volume containing the same charges with each fixed at an average rest position. Hence the charges may be spoken of as statistically at rest, and to each may be assigned a statistical rest position. The average over-all electrical properties of the statistically static model are the same as those of the dynamic one with random motion, and one may be substituted for the other wherever convenient. Any regularity in orientation or motion of the charges may be pictured in the dynamic model as superimposed upon the random motion; in the static model it is represented by a relative orientation or motion of the statistical rest positions of the charges.

It is possible to incorporate within the picture of random distribution a rather complicated model skillfully devised to represent the inner structure of the 92 elements. It assigns a definite and different number of electrons to the atom of each element and divides the electrons so assigned into groups according to a comprehensive scheme. This defines a set of so-called "energy levels," or shells, of which each is limited to a specified maximum number of electrons. Each positive nucleus has associated with it a certain number of electrons in an arrangement characteristic of a particular element. In the case of a molecule consisting of several closely bound atoms, the electronic distribution peculiar to each atom is complicated by an overlapping or sharing of shells. All the electrons in an atom or in a molecule are freely movable to the extent that each individual electron can exchange places with any other and in this way associate itself successively with different shells or nuclei belonging to different atoms or molecules throughout a body or region. However, the number of electrons belonging to each shell, nucleus, or group of nuclei in a molecule is a constant, time-average characteristic of each element or compound.

In each atom two classes of electrons may be distinguished solely with reference to the type of shell to which they belong. One class is associated with completed or full shells, the other

with shells that are only partly filled.   In most atoms there is only one incomplete shell, called the outermost one.   But in some cases (as for copper and silver) the difference between the energy levels of two shells is so small, as determined by the mathematical scheme used to define these, that the outermost one may already contain some electrons while the next one still has empty spaces.   The electrons in partly filled shells are called valence electrons.   It is assumed that closed shells in atoms and closely linked configurations in molecules are characterized by strong intra-atomic and intramolecular constitutive forces that act to maintain them over a time average.   Such forces are not presumed to act on valence electrons of atoms unless they are a part of a molecular configuration.   Electrons that are subject to strong constitutive forces are called closed-shell electrons or *bound charges;* valence electrons are called *free charges.*   The term bound charge does not mean that random exchanges may not take place freely.   It does mean that a definite and character-istic number of electrons is bound to each atom or molecule not only under random conditions but also under the action of strong external forces tending to disturb a random distribution.   The term free charge refers to a charge that may leave the atom or molecule with which it is statistically associated without having another charge take its place.   A motion of free electrons may occur from one section of a body to another leaving an excess of positive charge behind and bringing with it an excess of nega-tive charge.   But such a transfer of free charge can take place only under the action of external influences that disturb the normal random conditions.

The number of free electrons characteristic of an atomic configuration is a fundamentally significant property.   Atomic or molecular models in which there are very few or none will be called closed-shell or bound-charge models.   Other names are nonconductors, *dielectrics,* or insulators.   Models in which there is an abundance of free electrons will be called free-charge models or *conductors.*   Intermediate cases are half conductors or imperfect dielectrics.   Some of their macroscopic properties may be deduced by a suitable combination of the properties of the two extremes.   The free- and the bound-charge models are necessarily statistically identical so long as purely random con-ditions prevail.   But they behave quite differently under the

action of external forces which seek to establish preferred directions in the motions or relative positions of the charges or in the orientation of the atoms or molecules.

3. Volume Density of Charge.—The electrical properties of a region (or body) in which a random distribution of charge prevails may be described approximately by a *continuous function which assigns to every point in the region a number characteristic of the average condition of total charge in the neighborhood of the point.* Any function that assigns a *scalar*[1] to every point in a region in which it is defined is called a *scalar point function.* It might be supposed that such a function could be defined with the aid of the usual limit process of the calculus. An attempt at definition might be made by dividing a region containing many millions of electric charges into small elements and defining the desired function to be the limit approached by the ratio of the total charge in the element to the volume of the element as this latter is allowed to approach zero. In mathematical language, this definition has the form

$$\rho = \lim_{\Delta\tau\to 0} \frac{\sum\limits_{j=1}^{n} e_j}{\Delta\tau} \qquad (1a)$$

Since every charge is endowed with mass, a corresponding scalar point function to describe the distribution of mass is

$$D = \lim_{\Delta\tau\to 0} \frac{\sum\limits_{j=1}^{n} m_j}{\Delta\tau} \qquad (1b)$$

In $(1a)$, e_j is the algebraic magnitude of any one of the n charges in the volume element $\Delta\tau$. In $(1b)$, m_j is the mass associated with the same charge. Unfortunately, functions constructed in this way are discontinuous because the charges and masses are associated with discrete points. They have values different from zero only at points characterized by charge and mass. For example, at a point locating an electron $\rho = -\infty$; $D = \infty$. At a point locating a positive nucleus $\rho = \infty$; $D = \infty$. At all intermediate points both ρ and D vanish. This is precisely the

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[1] A **scalar** is a quantity characterized completely by a single number with appropriate units. It has magnitude only.

representation in terms of individual charges which has already been rejected because of its great mathematical complexity.

In order to obtain functions that will represent the average distribution of charge and of mass in a small element of volume, it is necessary to subdivide a larger region into small elements. But these may not be allowed to approach zero as a limit. The questions arise: How large must each element of volume be in order that the discontinuous effect of individual, more or less widely separated, charges and masses may be avoided? How small must each element be so that significant variations in charge and mass, which may characterize the region as a whole, are not obscured? Is it, in fact, possible to select volume elements that are at the same time sufficiently large and sufficiently small? The answers to the first two questions are easily given. Each volume element must be large enough to contain so many charges that statistical conditions prevail in it. But it must also be very small compared with physically measurable magnitudes. The answer to the last question, whether these two restrictions on the size of volume elements can be fulfilled simultaneously, depends upon the inner structure of the postulated atomic model, in particular upon the average distance between charges. Fortunately, atomic theory requires this distance to be so small that it is possible to construct volume cells that are large enough to contain many millions of charges, and that are yet extremely minute compared with laboratory magnitudes. In fact, the mean distance between charges in the model is assumed to be so short that a volume cell which is only as thick as this mean distance may still be made large enough to be statistically regular as a whole without approaching directly measurable magnitudes in length or breadth.

Let the region or body for which the continuous functions are to be constructed be subdivided into volume cells of which $\Delta \tau_i$ is a typical one. Its greatest dimension d_i must satisfy the inequality

$$\text{Mean distance between charges} << d_i$$
$$<< \text{laboratory magnitude} \quad (2)$$

Let there be n charges— positive, negative, or both—in the cell. Then the total charge c_i in the volume $\Delta \tau_i$ is the algebraic sum of the individual charges. It is

$$e_i = \sum_{j=1}^{n} e_j \tag{3a}$$

The total mass is

$$m_i = \sum_{j=1}^{n} m_j \tag{3b}$$

Here e_j is the charge, m_j the mass of an electron or of a positive nucleus. Appropriate magnitudes and algebraic signs are to be substituted in each case. Evidently the total charge in the volume element vanishes if it contains either no charges or equal amounts of positive and negative charge. On the other hand, the total mass vanishes only if $\Delta\tau_i$ contains no charges and, hence, no units of mass.

Two scalars are defined at the center of each volume cell according to the formulas

$$\rho_i = \frac{e_i}{\Delta\tau_i} \tag{4a}$$

$$D_i = \frac{m_i}{\Delta\tau_i} \tag{4b}$$

and giving, respectively, the average charge and mass per unit volume in the cell $\Delta\tau_i$. Corresponding scalars are defined at the center of every volume element throughout the entire body. Using all these scalars as a reference frame, two continuous scalar point functions may be constructed that will, by definition, assume, respectively, the values ρ_i and D_i at the center of each cell, while smoothly and continuously connecting between them at all intermediate points. The continuous functions constructed in this way are said to be *interpolated* from the discrete values ρ_i and D_i. They are called, respectively, the *volume density of charge* (denoted by ρ) and the *volume density of mass* (denoted by D). The former has the dimensions of charge divided by volume; the latter, of mass divided by volume

$$\rho \approx \frac{Q}{L^3} \frac{\text{coulombs}}{\text{cubic meters}} \tag{5a}$$

$$D \approx \frac{M}{L^3} \frac{\text{kilograms}}{\text{cubic meters}} \tag{5b}$$

From the very manner in which they are constructed, it is clear that such interpolated functions can adequately represent

the average densities of charge and of mass in a body only if this possesses considerable uniformity of structure. The mean distribution of charges may vary only slowly from volume cell to volume cell. In fact, an adequate representation of the properties of a body or region in terms of volume densities of charge and mass as defined above requires that these functions be so slowly varying through the body as to be sensibly constant over distances that are large compared with the dimensions of a volume cell. Such a condition of very slow variation must be assumed to prevail whenever the density functions are used.

4. Surface Density of Charge.—In constructing the volume density of charge ρ by interpolation from the scalars ρ_i defined at the centers of the volume cells $\Delta\tau_i$, it was assumed that these cells were all sensibly alike throughout the region in which ρ was defined. In the interior of a region this assumption merely implies a reasonably slow variation in characteristic structure in passing from cell to cell. On the surface of a region, or on the boundary between two dissimilar regions, however, this uniformity of structure does not exist, because there the volume cells are not completely surrounded by other similar ones. At a surface or boundary, all cells are asymmetrically placed because one side is necessarily exposed to surroundings that are entirely different from those experienced by the other sides. It must be expected, therefore, that at a surface or boundary the electrical properties of a region cannot in general be represented correctly by the same continuous function used to describe the interior. In particular, the volume density of charge ρ cannot be required to be as slowly varying as is demanded for the interior and at the same time to represent correctly the rapid change that may occur near the surface. This difficulty may be overcome by treating separately a layer of very thin surface cells, of which $\Delta\tau_s$ is a typical one. The thickness δ of this layer of cells is very small compared with the dimensions d_s of each cell measured parallel to the surface. It was stated in the preceding section that on the basis of the atomic theory these tangential dimensions may be so large that each surface cell contains enough charges to be statistically regular and yet so small that they are of a different order of magnitude compared with physical dimensions. It is, therefore, possible to choose the tangential dimensions d_s of the surface cells to be of the same order of magnitude as the

dimensions d_4 of the interior cells and still have both surface and volume cells satisfy the limiting· conditions for statistical regularity on the one hand, and physical smallness on the other. Because the surface cells may be like the volume cells in the tangential plane, it is reasonable to suppose that for such properties as depend only upon conditions in this plane, and not upon those along a perpendicular to a boundary, each surface cell will behave just like a correspondingly thin slice of an adjacent volume cell. It follows that a surface layer of cells need be used only when an asymmetry along the *normal* to the surface leads to significant and different effects from those determined from conditions of charge that are characteristic of the interior. Since the volume of each surface cell is negligible compared with that of the adjacent interior cell, *surface cells may be disregarded in describing tangential effects.*

In order to show the significance of thin surface layers of charge and their relation to the volume density of charge as already defined, consider the following example. Suppose that a closed region containing positive and negative charges is at first under the action of no external influence. The static condition of complete statistical equilibrium prevails, and every volume and surface element is electrically neutral because it contains on the average an equal amount of positive and negative charge. This is true both for the free-charge and the bound-charge models since a random distribution prevails in each. .If an unsymmetrical external force, for example, due to a positively charged body placed near its left end acts on the region, electrons in it experience an unbalanced force of attraction toward the left, whereas positive nuclei are repelled toward the right. In the free-charge model, mobile free electrons drift toward the left to make the left surface negatively charged and leave the right-hand surface positive. The drift extends through the entire region and does not stop until the average force acting on each charge due to the external charges and the two oppositely charged surface layers is again zero. A new condition of statistical equilibrium is established in this way with a greater electron density at the surface on the left and a correspondingly diminished electron density at the surface on the right. In the interior, each volume element remains neutral.

In the bound-charge model, conditions are different. All electrons are in closed shells, and although individually free to

move and exchange places with each other they are under the influence of strong intra-atomic forces that seek to maintain a distribution which is random from the point of view of a volume cell, but which on the average provides each nucleus with a characteristic number of electrons properly distributed in shells. Hence, even under the action of quite strong external forces, there can be no mean drift of electrons one way or the other. Over a time average the same number must always move away from an attracting external influence as toward it. Consequently, the statistical rest positions of the electrons can at most be shifted slightly to produce a distorted outer shell of electrons in each atom or molecule. If an external force is applied outside the left surface of a region to attract electrons as in the preceding case, each atom or molecule exhibits a slight distortion-orientation in the form of a shift of the statistical rest positions of the electrons relative to those of the positive nuclei. The mean rest position of the total negative charge associated with each atom or molecule is in this way displaced so that it no longer coincides exactly with that of the positive center. The net effect is a small shift of the electrons associated with each atom toward the left. This does not constitute a general drift in that direction throughout the charged region as in the free-charge model, but only an infinitesimal distortion and orientation within each atom. The final effect, however, is again to make the left-hand surface of the entire region negative, the right-hand surface positive, while the interior remains neutral.

From these simple illustrations it is clear that similar distributions of charge may exist in both models. In particular, charges may be so distributed that a volume density of charge would have to have the value zero throughout the interior, rise continuously but very steeply to a positive value at one surface, and descend to an equal negative value on the opposite surface. In general and under the identical external circumstances, the distribution, while similar in the two models, would not be the same because the conditions determining the final equilibrium are quite different. In the free-charge model, equilibrium is reached when enough negative charge has accumulated on the left-hand surface (by effectively taking it from the right) to neutralize in the interior of the region the action of the external positive charge. In the bound-charge model, equilibrium is reached when each atom has been distorted and oriented in

such a way that stronger internal forces are brought into play to balance the external influence. The amount of distortion, which determines the density of charge at the surface, depends upon the nature of the intra-atomic forces of constitution.

If such an equilibrium distribution of free or bound charge is to be described in terms of the volume density of charge ρ as defined above, difficulties are encountered. For a region that has been divided into similar volume cells at the surface and in the interior, the following observations may be made about the discrete densities of charge ρ_i defined for the individual cells. In the free-charge model ρ_i vanishes in each interior cell because it contains on the average equal magnitudes of positive and negative charge; ρ_i has a small negative value in the layer of cells along the left surface; a small positive value in the layer along the right surface. The values of ρ_i in the $\Delta\tau_i$ cells along the surfaces are very small because the surface layer constitutes only a minute part of the volume of these cells. The process of averaging over cells that are extremely thick compared with the surface layer of charges (the thickness of such a layer is of molecular magnitude) obscures the actually high concentration of charges in such a thin layer. The ρ_i's are average values defined at the centers of cubical volume cells. When ρ is interpolated, no account is taken of concentrations of charge in layers that are thin compared with the dimensions of the volume cells. In brief, a subdivision into $\Delta\tau_i$ cells of dimension d_i is too coarse near any surface or boundary where a thin layer of charge exists.

The same conclusion is reached for the bound-charge model. Since each volume cell is assumed to contain a large number of neutral atoms, a mere distortion of each atom by small shifts of the negative charges relative to the positive nucleus does not change the total charge in each volume element either in the interior or along the surface if boundaries are drawn so that they do not cut through atoms or molecules. Hence the volume density of charge is zero throughout the interior and along the surface.

In neither free-charge nor bound-charge model is a representation in terms of the volume density of charge ρ adequate to take account of surface conditions of charge. The reason in both cases is that a function interpolated from values averaged over thick volume cells cannot be sensitive to a thin surface distribu-

tion of charge that contributes little to the volume of these cells. What is needed is a separate treatment of the surface. This is easily accomplished by constructing a layer of extremely thin cells along each boundary, while the remainder of the volume is divided up into volume cells as before. The volume density of charge ρ, as already defined, continues to characterize the condition of charge in the interior, while a new function η, called the *surface density of charge*, is defined to describe the condition of charge along the surface.

The surface density of charge is defined in a manner entirely analogous to that followed in defining the volume density of charge. Let $\Delta\tau_s = \delta d_s^2$ be one of the thin surface cells each of thickness δ satisfying the inequality

$$\delta << d_s \doteq d_i \tag{1}$$

A scalar η_s is defined at the center of each surface cell by the relation

$$\eta_s = \frac{\sum_{j=1}^{n} c_j \delta}{\Delta\tau_s} = \delta\rho_s \tag{2}$$

A continuous scalar point function η is constructed for the entire surface by interpolation from the discrete values in (2). By definition, η has the values η_s at the centers of the individual surface cells; it connects these continuously and smoothly at all intermediate points. This definition of η is possible if the body represented is sufficiently uniform that η varies so slowly in tangential directions along the surface as to be sensibly constant over distances that are large compared with the lateral dimensions d_s of the cells. Since the thickness of the layer is of molecular magnitude, no condition of uniformity can or need be imposed in a direction normal to the surface. The dimensions of the surface density of charge are charge per unit area.

$$\eta \approx \frac{Q}{L^2} \frac{\text{coulombs}}{\text{square meters}} \tag{3}$$

A static distribution of charge in a region constructed according to the free-charge or the bound-charge model may be described in terms of the scalar functions ρ and η to a degree of approximation that is entirely adequate for all practical problems.

5. Alternative Modes of Representation.—Up to this point the statistically stationary state has been characterized in terms of the two continuous, slowly varying scalar point functions ρ and η. The surface density η was introduced because a representation in terms of the volume density ρ alone is inadequate to represent conditions at a surface or boundary resulting from asymmetry in the direction normal to the surface. The definition of η required a simple change in the mode of subdivision of the region into elementary cells. The new mode of subdivision and the definitions of ρ and η may be used for both free- and bound-charge models. But whereas in the free-charge model the surface conditions are *real surface effects* involving distinct layers of free charge that can be considered simply and naturally as separate from, and superimposed upon, volume phenomena, the same is not true of the bound-charge model. Here a condition ascribed specifically to the surface is actually a surface manifestation of a phenomenon existing throughout the interior. The appearance of a surface layer of charge is the result of a distortion and orientation *of all the atoms in the region*. And the charges on the surface are simply parts of the outermost layer of neutral but distorted atoms. From the physical point of view of the atomic model, the separate consideration of a thin surface sheet is not really appropriate or reasonable for the bound-charge model because the thin surface layer cuts off a part of the outer, closed shell of each atom. Hence, although a subdivision using surface cells and the separate definition of a surface density provides an adequate representation of the external properties of the bound-charge model from the mathematical point of view, an alternative more appropriate representation is desirable from the point of view of the physical model. According to this, the entire surface effect is fundamentally a part of a volume phenomenon, and it seems not merely plausible but logically necessary to provide an alternative representation entirely in terms of volume functions. Instead of changing the mode of subdivision and introducing separate surface cells and a separate surface density, an alternative procedure using the original subdivision into volume cells is required. The orientation-distortion effect in the interior, as well as its surface manifestation, must be represented by an additional function defined throughout the volume.

6. Polarization.—An alternative representation of the static state in terms of two volume functions instead of a volume and a surface function is designed specifically for the bound-charge model using a subdivision into volume cells only. The volume density of charge ρ is defined to describe the average condition of total charge throughout the region just as before. It does not take account of the average separation and orientation of the statistical rest positions of positive and negative charge, so that a new function must be constructed for this purpose. Consider a region containing only closely bound charges which are exposed to the action of an external agency that attracts negative and repels positive charge. Because there are no free charges in the region, there can be no general transfer of charge. Instead, the bound-charge groups associated with each atom or molecule are distorted and oriented in such a way that the statistical rest position of the entire negative charge in each group is moved away from coincidence with the rest position of the positive charge. Each bound group continues to be electrically neutral, but a statistical separation and orientation of its positive and negative charges take place. Such a group may be described in terms of a statistically stationary positive charge separated a distance d from a similar negative charge. A structure of this kind is called a *dipole*. Let a polar vector[1] $\mathbf{d}$ be drawn from the statistical center of the negative charge to that of the positive charge. Then the polar vector

$$\mathbf{p} = e\mathbf{d} \tag{1}$$

a measure of the statistical separation of the rest positions of the positive and negative charge and of their orientation in

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[1] A vector is a quantity characterized completely by a magnitude and a direction. It requires three numbers for its specification. Geometrically it may be represented by a line segment directed parallel to the vector and of length equal to its absolute magnitude. In Cartesian coordinates the vector $\mathbf{A}$ may be written in the form $\mathbf{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$. Here $\hat{x}$, $\hat{y}$, $\hat{z}$ are vectors of unit length directed along the positive x, y, z axes. Vectors are printed in Gothic type. The superscript $\wedge$ denotes a unit vector, i.e., a vector of unit length. The direction of a polar or ordinary vector differs from its opposite in a real physical sense. Thus, the polar vector $\mathbf{p}$ points from negative to positive charge where these two kinds of charge are physically different. A vector, called an axial vector, with different properties is defined later.

space. The polar vector p is called the average polarization of the bound group of charges in the atom or molecule; it is the polarization of a statistically equivalent dipole. The direction of the vector defines the axis of polarization; the magnitude of the vector is called the electric moment.

An entirely equivalent representation, which is more readily generalized to apply to a volume containing many charges, is illustrated in Fig. 6.1. An arbitrary origin is fixed at any convenient point near the rest positions of two equal and opposite charges. Let d_1 and d_2 be vectors drawn, respectively, from the origin to each of the two charges e_1 and e_2. The positive direc-
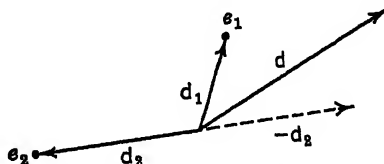
Fig. 6.1.—Two charges located by vectors drawn from an arbitrary origin.

tion of each vector is from the origin to the charge, regardless of the sign of this latter. The polarization of the dipole composed of the two charges is defined by

$$p = c_1 d_1 + e_2 d_2 \qquad (2)$$

Since c_1 and c_2 have been assumed to be equal in magnitude and opposite in sign, this relation may be written

$$p = e(d_1 - d_2) = ed \qquad (3)$$

It is, therefore, equivalent to (1). In (3) it has been assumed that e_2 is the negative charge.

The notation used in (3) may be applied to define the polarization of any number n of different charges in a volume element $\Delta\tau_i$. It is given by the algebraic vector sum

$$p_i = \sum_{j=1}^{n} e_j d_j \qquad (4)$$

The vectors d_j are drawn from an origin at the center of the volume element to the individual charges. In a region containing no free electrons, the $\Delta\tau_i$ elements may be so chosen that each one contains only complete atoms or molecules. With such a

subdivision the volume density of charge is zero throughout. The electrical properties of the region are then characterized entirely in terms of the polarization of the individual volume cells. Thus, if a vector P_i is defined at the center of each volume cell $\Delta\tau_i$ according to

$$P_i = \frac{p_i}{\Delta\tau_i} \qquad (5)$$

a vector function P which is continuous and slowly varying throughout the region may be constructed by interpolation from the discrete values P_i. If free charges are present in addition to the complete atoms, their distribution is described in terms of ρ. However, if ρ is not so slowly varying as to be sensibly constant over distances comparable with the dimensions of each $\Delta\tau_i$ cell, the nonuniformity of charge distribution contributes not only to ρ_i, but also to p_i defined at the center of the cell by (4). In order not to take this same variation into account twice, it must be subtracted from (5) if it is at all significant. Its value is easily determined as follows. Since ρ is a continuous function, the departure from a constant charge density at a point within the cell at a distance s from its center is given to a first approximation by $s(\partial\rho/\partial s)_i$. The derivative is evaluated at the center of the cell. The resulting contribution to the polarization of the entire cell referred to its center is obtained by multiplying this charge density by a vector s drawn from the center of the cell to the point and then integrating over $\Delta\tau_i$. Accordingly, if ρ is not sensibly constant, P_i must be defined as follows instead of by (5):

$$P_i = \frac{1}{\Delta\tau_i}\left\{ p_i - \int_{\Delta\tau_i} s\mathbf{s}\left(\frac{\partial\rho}{\partial s}\right)_i dr \right\} \qquad (6)$$

This function[1] evidently reduces to (5) if ρ is sensibly constant. A continuous function P may be interpolated from the discrete values in (5) or (6) as required. P is the vector volume density of polarization, or simply the *polarization vector*. A function, such as P, that assigns a vector to every point in a region is

[1] A more rigorous derivation and more detailed discussion is given in M. Mason and W. Weaver, "The Electromagnetic Field," pp. 19 and 61. The integral in (6) with a vector in the integrand is a shorthand for three integrals, one for each component.

called a vector point function. *Thus* P *assigns to each point a vector that is a measure of the direction and the average magnitude of the orientation-distortion of closely bound configurations of charge in a small region about that point.* Volume density of polarization has the dimensions of charge per unit area.

$$P \approx \frac{Q}{L^3} \frac{\text{coulombs}}{\text{square meters}}$$

7. Comparison of Two Representations.—In describing the static-state properties of a region in terms of volume and surface densities of charge ρ, η, each of these two functions is defined in an entirely separate part of the region in which it alone bears the full responsibility. On the other hand, in the alternative representation in terms of volume densities of charge ρ and of polarization P, both functions are defined in terms of the charges contained in the same volume elements. Both functions are defined throughout the whole region; to every point the function ρ assigns a scalar, the function P a vector. Since each point is characterized by two independently defined quantities, the question must arise to what extent the two functions overlap and to what extent the entire representation depends upon the mode of subdivision into volume cells. Furthermore, since the volume density of charge ρ is unable to take account of asymmetrical conditions in the form of thin layers of charge at surfaces and boundaries, the volume density of polarization must in one way or another describe these.

Let a region that is under the influence of an external force be examined. Suppose, first, that it is divided into volume elements in such a way that each cell contains only complete and therefore neutral bound-charge groups. The volume density of charge is then zero throughout. Suppose, further, that the region is so constructed that the polarization is everywhere fixed in direction, but increasing in magnitude uniformly in a direction parallel to the axis of polarization. This condition can be shown schematically by increasing the number of polarized units per unit volume from left to right. Figure 7.1 shows an *electrically one-dimensional* section of a volume so constructed. The volume density of polarization P is a uniformly increasing function in the direction from left to right. It completely

characterizes the electrical properties of the region since $\rho = 0$ and η is not defined.

Subdivision I: Representation entirely in terms
of the volume density of polarization P; $\rho = \eta = 0$
Note that doubly spaced dipoles are counted as two.

FIG. 7.1.—Schematic diagram to show two possible modes of subdivision into volume cells in the static state. For simplicity in constructing the diagram, polarization means horizontal polarization only. The diagram is electrically one dimensional and represents the distribution along a line from left to right. The separation of the charges up and down on the page is for convenience in construction. The center of each cell is a line from top to bottom midway between the side boundaries. The left side is at $x = 8$, the right side at $x = 24$. Subdivision I is characterized by $P_I = \frac{1}{2}x$; subdivision II by $\rho_{II} = -\operatorname{div} P_I = -\frac{1}{2}$; $\eta_{IIL} = (\hat{n}, P)_L = -2$; $\eta_{IIR} = (\hat{n}, P)_R = 6$.

Let the mode of subdivision be changed by separating a thin surface layer on each side with a boundary that cuts through the individual bound groups or atoms as shown in the figure in dotted lines. Each new volume element is found to be practically unpolarized but charged uniformly. Each surface element is

charged. Hence, ρ and η have nonvanishing values, while P is zero throughout.

If both of the two artificial modes of subdivision that have been described in turn are discarded, and no attempt is made to have the boundaries cut or not cut through bound groups of charges, statistical conditions may be assumed to prevail along and across the boundaries of the volume cells just as in their interior. There will be, on the average, some bound groups near the boundaries of the cells completely within a cell, some partly in and partly out so as to be cut by the boundary line. On the average there will be more partly in and partly out groups on the right-hand boundary of each cell than on the left, because the number of polarized groups increases toward the right throughout the region. Thus the characteristics of the two special modes of subdivision prevail simultaneously. Each volume cell may be expected to be both charged and polarized, so that both ρ and P have nonvanishing values. At the surface, η is likewise nonvanishing. The following questions must be answered: How can η, ρ, and P be combined to represent the magnitude, distribution, and orientation of charge throughout the volume and on all surfaces in a way that is unambiguous and independent of the mode of subdivision? Can this representation be extended to regions in which free charges as well as bound-charge groups are present? These questions are conveniently answered first for the interior, the description of which involves only ρ and P, and later for the surfaces that must be described in terms of η and P.

8. Essential Volume Characteristic of the Static State; Divergence of a Vector.—The relationship between ρ and P in the interior of a region may be determined by examining what happens to the densities describing the condition of charge in a typical volume cell (such as $\Delta\tau_2$ in Fig. 7.1) when the subdividing boundaries are shifted just enough to reduce the polarization in the cell to zero. Each dipole (or polarized atom or molecule) is assumed to be statistically fixed as shown one-dimensionally and schematically in Fig. 7.1. Using *subdivision* I, there are only complete, neutral dipoles in $\Delta\tau_2$ so that $\rho_2 = 0$. When the boundary is moved to form *subdivision* II, four negative charges are included in $\Delta\tau_2$ on the right, while only three negative charges are excluded on the left. The net result is an addition of one

negative charge.   In subdivision I no dipoles are cut by the boundaries of the cell.   In subdivision II three dipoles are cut by the left-hand boundary, four by the right-hand boundary. Therefore three polarization vectors

$$\mathbf{p} = e\mathbf{d} \tag{1}$$

pierce the left-hand boundary normally and pointing to the right, whereas four such vectors cross the right-hand boundary normally and also directed to the right.   In other words, there is an outwardly directed excess of one vector $\mathbf{p} = e\mathbf{d}$ pointing perpendicularly across the boundaries.   Correspondingly, there is an excess of one negative charge in the cell, or a deficit of one positive charge.   It follows from this highly simplified picture that there is a one-to-one correspondence between the number of negative charges appearing inside a volume cell in subdivision II and the excess of outwardly directed polarization vectors that pierce the cell walls normally in changing from I to II.

The volume function $\mathbf{P}$ measures the average density of polarization vectors due to individual dipoles or their equivalents in a small region about any point.   Accordingly, the component of $\mathbf{P}$ directed along the outward or external normal to a closed surface at any point, *viz.*,

$$P_n = (\mathbf{n},\mathbf{P})^* \tag{2}$$

is a measure of the average vector sum of the outwardly directed normal components of the elementary polarization vectors $\mathbf{p}$ that pierce a unit area of the surface on which P_n is defined.   Hence the surface integral†

$$\int_{\Sigma} (\mathbf{n},\mathbf{P})d\sigma \tag{3}$$

* The scalar product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined by

$$(\mathbf{A},\mathbf{B}) = AB \cos (A,B) = A_B \cdot B = A \cdot B_A.$$

In Cartesian coordinates

$$(\mathbf{A},\mathbf{B}) = (\hat{x}A_x + \hat{y}A_y + \hat{z}A_z, \hat{x}B_x + \hat{y}B_y + \hat{z}B_z)$$
$$= A_xB_x + A_yB_y + A_zB_z.$$

If one vector is a unit vector, such as $\mathbf{n}$, then $(\mathbf{n},\mathbf{P}) = 1 \cdot P \cos (n,P) = P_n$. The scalar product of two vectors is often called the dot product and written $\mathbf{A} \cdot \mathbf{B}$.

† Surface integrals of the *outwardly* directed normal component of a vector, in this case of $\mathbf{P}$, measure what is called the total outward normal

is a measure both of the number of elementary polarization vectors that pierce Σ normally and of the total positive charge that leaves (or the total negative charge that enters) the volume enclosed by the surface Σ ($\hat{n}$ = unit external normal) when the mode of subdivision is changed to make P vanish. Accordingly, the net outflow of positive charge per unit volume in the change in subdivision is given by

$$\left.\begin{array}{l}\text{+charge outflow or } -\text{charge} \\ \text{inflow per unit volume in} \\ \text{change from subdivision I to} \\ \text{subdivision II}\end{array}\right\} = \frac{\int_{\Sigma_i} (\hat{n},P_I)d\sigma}{\Delta\tau_i} \qquad (4)$$

Here Σ_i is the surface of $\Delta\tau_i$. The subscript I on P indicates that P_I is determined using subdivision I. In subdivision II $P_{II} = 0$. Alternatively,

$$\left.\begin{array}{l}\text{+charge inflow per unit} \\ \text{volume in change from} \\ \text{subdivision I to subdivi-} \\ \text{sion II}\end{array}\right\} = \frac{-\int_{\Sigma_i} (\hat{n},P_I)d\sigma}{\Delta\tau_i} \qquad (5)$$

An added positive charge per unit volume in $\Delta\tau_i$ may be expressed as a volume density of bound charge $(\rho_{bII})_i$ defined at the center of $\Delta\tau_i$. Thus

$$(\rho_{bII})_i = \frac{-\int_{\Sigma_i} (\hat{n},P_I)d\sigma}{\Delta\tau_i} \qquad (6)$$

A volume density of charge $(\rho_{bII})_i$ due to polarization might be defined for each volume element and a continuous volume function interpolated from these discrete values. Since P is such an interpolated continuous function defined at every point, this is not necessary and the ordinary limit process of the calculus may be used.

$$\rho_{bII} = \lim_{\Delta\tau \to 0} \frac{-\int (\hat{n},P_I)d\sigma}{\Delta\tau} \qquad (7)$$

The operation of taking the limit of the total outward normal flux

flux of a vector through the surface of integration. The integral (3) measures the total outward normal flux of the polarization vector through the closed surface Σ.

of a vector such as P *is called evaluating the divergence of the vector.*
It is assigned the symbol div P.   That is,[1]

$$\operatorname{div} \mathbf{P} \equiv \lim_{\Delta\tau \to 0} \frac{\int (\mathbf{\hat{n}}_1 \mathbf{P}) d\sigma}{\Delta\tau} \tag{8}$$

The divergence is a fundamental vector operator. With (8),
(7) becomes

$$\rho_{b\mathrm{II}} = -\operatorname{div} \mathbf{P}_\mathrm{I} \tag{9}$$

The electrical description of a bound-charge region may be

[1] The operator div may be expressed in any system of coordinates by
applying the defining formula (8) to an element of volume in the desired
system of coordinates. For example, in rectangular coordinates, a cubical
element of volume $\Delta\tau = \Delta x\, \Delta y\, \Delta z$ may be chosen with center at the origin
(or at any other point). The vector P at the origin has the value $\mathbf{P}_0$. The
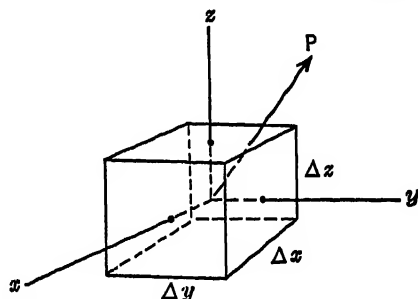
Fig. 8.1.   Cubical volume element with rectangular coordinate system.

average values of the components of P perpendicular to the x axis on front
and back faces of the cube in Fig. 8.1 are obtained by Taylor's formula.

$$(\mathbf{\hat{x}}_1 \mathbf{P})_{\Delta x/2} = (P_x)_0 + \left(\frac{\partial P_x}{\partial x}\right)_0 \frac{\Delta x}{2} + \cdots$$

$$(\mathbf{\hat{x}}_1 \mathbf{P})_{-\Delta x/2} = (P_x)_0 - \left(\frac{\partial P_x}{\partial x}\right)_0 \frac{\Delta x}{2} + \cdots$$

By writing y and z for x, the components on the sides, top, and bottom are
obtained.

div P =

$$\lim_{\Delta x\, \Delta y\, \Delta z \to 0} \left[\frac{\{(\mathbf{\hat{x}}_1 \mathbf{P})_{\Delta x/2} + (\mathbf{\hat{x}}_1 \mathbf{P})_{-\Delta x/2}\} \Delta y\, \Delta z + \{(\mathbf{\hat{y}}_1 \mathbf{P})_{\Delta y/2}}{+ (\mathbf{\hat{y}}_1 \mathbf{P})_{-\Delta y/2}\} \Delta z\, \Delta x + \{(\mathbf{\hat{z}}_1 \mathbf{P})_{\Delta z/2} + (\mathbf{\hat{z}}_1 \mathbf{P})_{-\Delta z/2}\} \Delta x\, \Delta y}{\Delta x\, \Delta y\, \Delta z} \right]$$

$$\operatorname{div} \mathbf{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \tag{8a}$$

tabulated as follows in the two extreme modes of subdivision:

Subdivision	I	II
$\rho_b =$	0	$-\text{div } P_I$
$P =$	P_I	0

$$\tag{10}$$

The *interior* of a region containing only distorted, bound-charge atoms or statistically equivalent simple dipoles may be described either in terms of P alone with $\rho = 0$ using subdivision I or in terms of ρ alone with $P = 0$ using subdivision II.

If the region contains free charges in addition to distorted closed-shell atoms or other equivalent dipoles, the continuous functions P and ρ are constructed as defined above. If subdivision I is selected, none of the boundaries of the $\Delta \tau_i$ cells cuts through an atom or a dipole, so that polarization contributes nothing to ρ. Hence, $\rho = \rho_f$, the volume density due to free charges only, and P has a nonvanishing value. If subdivision II is chosen, the boundary surfaces of the $\Delta \tau_i$ cells are by definition so placed that they cut through enough dipoles to make the polarization in each interior cell vanish. The volume density of charge due to the free charges is independent of small shifts in the bounding surfaces, so that ρ_f will be the same as before. However, ρ, if determined according to definition, includes not only ρ_f, but also ρ_b due to the charges actually present in each cell from parts of dipoles cut by the boundaries. Hence the volume density of charge interpolated from the discrete values ρ_i is

$$\rho = \rho_f + \rho_b \tag{11}$$

But ρ_{bII} is equal to $-\text{div } P_I$ so that $\rho_I - \text{div } P_1$ is equal to ρ_{II}. If an intermediate mode of subdivision III is used in which, for example, only one-half of the dipoles necessary to reduce the polarization to zero are cut by the boundaries, $\rho_{bIII} = \frac{1}{2}(\rho_{bII})$; $P_{III} = \frac{1}{2}(P_I)$; ρ_f is unchanged, and $\rho_{III} - \text{div } P_{III}$ has the same value as ρ_{II} or as $\rho_I - \text{div } P_I$. Thus

$$\bar{\rho} \equiv \rho - \text{div } P \tag{12}$$

called the *essential volume density of charge* characteristic of the interior of a region, *is independent of the mode of subdivision* of that region into volume cells. If subdivision I is used, P has a nonvanishing value characteristic of the $\Delta \tau_i$ elements

constructed and ρ measures only ρ_f, the volume density of free charge. If subdivision II is' used, P is zero and ρ measures a volume density equal to the sum of the volume densities of free charge ρ_f and of bound charge ρ_{bII}. If an intermediate subdivision is used, in particular if statistical conditions obtain along the cell boundaries, P has a value, smaller than P_I, that depends on the distribution of dipoles which are on the average partly in and partly out of the cells. ρ is then correspondingly larger than ρ_I. Note that $-\int(\hat{n},P)d\sigma$ has a nonvanishing value only if P is not constant. If P is constant the integral is everywhere zero and

$$\rho_{bII} = -\text{div } P_I = 0 \tag{13}$$

In order to describe the condition of charge in the interior of a body this may be divided into volume cells of correct size in any convenient way whatsoever. The discrete values ρ_i and P_i may be defined for the $\Delta\tau_i$ cells as constructed and the continuous functions ρ and P interpolated from them. The essential density $\rho - \text{div } P$ is then a scalar point function defined throughout the region, continuous and slowly varying, and independent of the mode of subdivision. It characterizes the condition of charge at all points in the interior.

9. Essential Surface Characteristic of the Static State.—In the preceding section it was shown that the function $\rho - \text{div } P$ characterizes the interior of a region containing atoms constructed according to either the free-charge or the bound-charge model in a way that is independent of the mode of subdivision of the region into volume cells. In particular, it was demonstrated that this function is invariant to a change in subdivision from mode I to mode II. Mode I is a subdivision into volume cells only with the dividing lines drawn so that no atoms or molecules are cut by cell boundaries. Mode II is a subdivision into both volume and surface cells with cell walls so placed that a thin layer of charges is treated separately around the surface and the average polarization of each cell in the interior is zero.

It has already been shown that a distribution of charge can be characterized completely using the second mode of subdivision in terms of a surface density of charge η and a volume density of charge ρ. The question now arises whether asymmetrical surface conditions (which are described in terms of η in sub-

division II) as well as distributions of charge in the interior can be represented completely by the volume functions ρ and P. Both of these functions are interpolated from discrete values defined at the centers of thick interior cells.   Moreover, since ρ is certainly unable to take account of thin layers of surface charge regardless of whether the surface condition is pictured as due to free charges or due to slices cut from polarized atoms or molecules, it follows that P, although also a slowly varying volume function, must be able to describe surface conditions if a representation in terms of ρ and P alone is at all possible.   This is evidently true only if the entire surface effect can be considered to be a part of an essentially volume phenomenon as in the bound-charge model in which the distortion and orientation of atoms in the outermost layer of volume cells is assumed to be the same as in the interior.   Accordingly, the slowly varying and continuous function P may be extrapolated to the edge of each volume cell along the surface.   This extrapolated value, in particular its normal component $(\hat{n}, P)$, is the characterization at the surface of a polarization effect throughout the interior. It must, therefore, replace η in describing surface conditions of charge using ρ and P alone insofar as the bound-charge model is concerned.

In order to see how the normal component of P in subdivision I (volume cells only) plays the part of η in subdivision II (volume and surface cells), it is instructive to examine what happens to a thin layer of charges along both sides of the bounding surface between two electrically dissimilar regions when a change is made from subdivision I to subdivision II.   Let each region be divided first into volume cells only according to scheme I with ρ_1 and P_1 defined on one side and ρ_2 and P_2 on the other side of the boundary.   With only neutral polarizable units assumed to be present, $\rho_1 = \rho_2 = 0$.   At the boundary, the normal components of the polarization vectors defined on each side are given by $(\hat{n}_1, P_1)$ and $(\hat{n}_2, P_2)$ with $\hat{n}_1$ and $\hat{n}_2$ outwardly directed unit normals referred to the volume indicated by the subscripts. This is illustrated in Fig. 9.1.

The change to subdivision II is made by pulling apart the imaginary envelopes of the two volume cells $\Delta \tau_i$, one on each side of the surface, to leave room between them for the two thin surface cells of combined thickness 2δ as shown in Fig. 9.1.

Graphically, this simply means that all charges in a thin layer on each side of the boundary are removed from the volume cells $\Delta\tau_i$, as these are shifted away from the surface, and left in the newly constructed surface cells. As in the analogous case described for the interior, the total outward normal flux of the vector **P** across a closed surface measures the total positive charge (due to cut polarized units) that is moved out from within the surface as the mode of subdivision is changed from I to II.
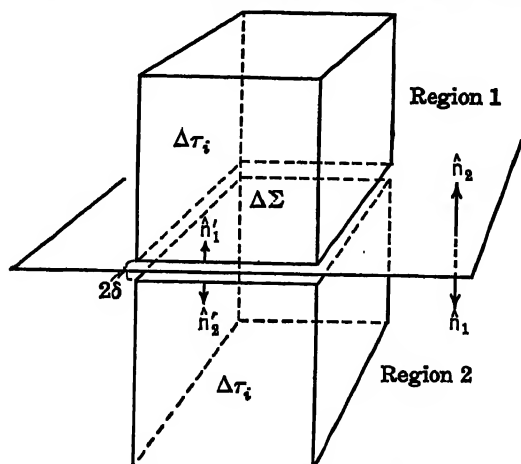
Fig. 9.1.—Volume and surface cells at a boundary.

The total positive charge that moves into the thin disk $2\delta\,\Delta\Sigma$ as this is formed is given by

$$- \int_{\Sigma} (\hat{n},\mathbf{P})d\sigma \tag{1}$$

Here $\hat{n}$ is the unit exterior normal to the surfaces enclosing the thin disk, and Σ is its entire superficial area. In changing the subdivision from I to II by moving the volume cells apart, the surfaces $\Delta\Sigma_{1,2}$, which are parallel to the boundary between the regions, are pierced by the charges that appear in the newly formed surface cells of combined volume $2\delta\,\Delta\Sigma$. The only significant contributions to this integral come from the two equal areas $\Delta\Sigma_1$ and $\Delta\Sigma_2$. The thin edges are not similarly pierced, and they are in any case negligible in size compared with $\Delta\Sigma_1$ or $\Delta\Sigma_2$. Thus, the integral

$$- \int_{\Sigma} (\hat{n},\mathbf{P})d\sigma = - \left\{ \int_{\Delta\Sigma_1} (\hat{n}_1',\mathbf{P}_1)d\sigma + \int_{\Delta\Sigma_2} (\hat{n}_2',\mathbf{P}_2)d\sigma \right\} \tag{2}$$

measures the total positive charge that was left in the two thin surface cells, each of volume $\delta \Delta \Sigma$, as these were formed by the moving apart of the volume cells $\Delta \tau_i$. Here $\hbar_1'$ points *into* region 1, $\hbar_2'$ *into* region 2, since both are external normals to the thin surface cells of combination volume $2\delta \Delta \Sigma$. The average positive charge per unit volume in these cells is obtained by dividing (2) by the volume $2\delta \Delta \Sigma$. Because P is a continuous function, it is permissible to pass to the limit and let $\Delta \Sigma$ approach zero to obtain the average density of bound charge in the surface layers due to contributions from cut, polarized units. With $\Delta \Sigma_1 = \Delta \Sigma_2 = \Delta \Sigma$, it is

$$(\rho_b)_{av} = \lim_{\Delta \Sigma \to 0} \left\{ \frac{\int_{\Delta \Sigma_1} (\hbar_1, P_1) d\sigma}{2\delta \Delta \Sigma_1} + \frac{\int_{\Delta \Sigma_2} (\hbar_2, P_2) d\sigma}{2\delta \Delta \Sigma_2} \right\} \tag{3}$$

Here $\hbar_1 (= -\hbar_1')$ and $\hbar_2 (= -\hbar_2')$ are normals directed outwardly with respect to the boundary surface between regions 1 and 2.

The average density of charge in the two layers on each side of the boundary is equal to one-half the sum of the individual densities defined in each region. Thus,

$$(\rho_b)_{av} = \tfrac{1}{2}(\rho_{1b} + \rho_{2b}) \tag{4}$$

A surface density of charge η may be defined as in (4.2)

$$\eta = \rho \delta \tag{5a}$$

Accordingly,

$$\delta(\rho_{1b} + \rho_{2b}) = \eta_{1b} + \eta_{2b} \tag{5b}$$

Upon multiplying (3) by the constant 2δ and using (4) and (5) the result is

$$\eta_{1b} + \eta_{2b} = \lim_{\Delta \Sigma \to 0} \left\{ \frac{\int_{\Delta \Sigma} (\hbar_1, P_1) d\sigma}{\Delta \Sigma} + \frac{\int_{\Delta \Sigma} (\hbar_2, P_2) d\sigma}{\Delta \Sigma} \right\} \tag{6}$$

The theorem of the mean for integrals[1] permits writing

$$\int_{\Delta \Sigma} (\hbar, P) d\sigma = (\hbar, \bar{P}) \int_{\Delta \Sigma} d\sigma = (\hbar, \bar{P}) \Delta \Sigma \tag{7}$$

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[1] This is a generalization to two variables of the first mean-value theorem for integrals. This states that if $f(x)$ is continuous in the interval (a,b), it is correct to write

$$\int_a^b f(x) dx = f(\xi) \int_a^b dx \qquad (a \leqslant \xi \leqslant b)$$

$f(\xi)$ is a mean value defined by this relation

where $(\hat{n},\bar{P})$ is a mean value at some point on $\Delta\Sigma$.   Hence,

$$\eta_{1b} + \eta_{2b} = \lim_{\Delta\Sigma\to 0} \left\{ \frac{(\hat{n}_1,\bar{P}_1)\Delta\Sigma}{\Delta\Sigma} + \frac{(\hat{n}_2,\bar{P}_2)\Delta\Sigma}{\Delta\Sigma} \right\} \qquad (8)$$

In passing to the limit, $\Delta\Sigma$ is shrunk down upon the point at which $(\hat{n},\bar{P})$ is defined.   It is then no longer necessary to indicate a mean value because the surface has zero area.   Let the following shorthand symbols be defined at every point along a surface of discontinuity:

$$\eta_b \equiv \eta_{1b} + \eta_{2b} \qquad (9a)$$
$$\text{surf. div } P \equiv (\hat{n},P) \equiv (\hat{n}_1,P_1) + (\hat{n}_2,P_2) \qquad (9b)$$

With this notation[1] (8) may be written in the following concise form with explicitly shown subscripts I and II to designate the mode of subdivision used in defining the densities:

$$\eta_{bII} = (\hat{n},P_I) \equiv \text{surf. div } P_I \qquad (10)$$

A value of the function η_b may be associated with every point along the surface separating regions 1 and 2.   It measures the total surface density of charge in both regions associated with that point using subdivision II.   It is important to bear in mind that η_b is a shorthand symbol for the sum of the two scalar functions η_{1b} and η_{2b} defined, respectively, in regions 1 and 2 at distances $\delta/2$ from each side of the boundary.

It is not difficult to see that the function $\eta + (\hat{n},P)$ is independent of the mode of subdivision.   If only volume cells are used, η is not defined at all and $(\hat{n},P)$ must describe the surface. If surface cells are used with a subdivision so devised that interior cells are all unpolarized, η describes the surface.   If surface cells are used, but statistical conditions prevail across all cell boundaries so that the number of bound charges required to reduce the polarization to zero throughout the interior is not necessarily cut off by cell walls, η includes bound charges cut off by the walls of the surface cells while $(\hat{n},\hat{P})$ adds to η the contribution to the surface effect due to whatever volume polarization has not been canceled in the subdivision.   It thus appears that the function $\eta + (\hat{n},P)$ is indeed independent of the mode of

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[1] It is to be noted that the two terms on the right in (6) are like the definition of the divergence (9.8) except that the surface element $\Delta\Sigma$ appears in place of the volume element $\Delta\tau$.   From this correspondence comes the name surface divergence.

subdivision into cells.   It is a scalar point function that may be associated with every point on the boundary between two regions.   It is called the essential surface density or surface characteristic.   For the present it applies only to bodies containing no free charge.   Let it be denoted by the symbol $\bar{\eta}$. Thus,

$$\bar{\eta} \equiv \eta + (\hat{n}, P) \tag{11}$$

Throughout the above discussion it was assumed that the body or region was constructed according to the bound-charge model.   This was necessary because all surface effects were looked upon as parts of a volume phenomenon in order that they could be described in terms of the normal component of the volume function P in the form $(\hat{n}, P)$ whenever a subdivision into volume cells only was used.   When free charges are present, a distinct surface layer of charges may exist which is definitely not part of a volume phenomenon.   Consequently, a description in terms of $(\hat{n}, P)$ alone is apparently not possible, so that

$$\bar{\eta} = \eta + (\hat{n}, P)$$

seems to be independent of the mode of subdivision only in the complete absence of free charges.   If free charges are present, η includes a contribution η_f due to the free charges and a contribution η_b due to the bound charges contained in the surface cells.   If the mode of subdivision is changed to volume cells only, η_b is represented by an equivalent value of $(\hat{n}, P)$ whereas neither ρ nor P can take account of the free surface charge represented by η_f.

Before concluding that $\eta + (\hat{n}, P)$ is useful (because of its independence of the mode of subdivision) only for the bound-charge model, the following must be considered.   Given a pair of functions ρ_I and P_I, it is certainly mathematically possible to construct two different functions ρ_{II} and η_{II}, which satisfy the pair of equations

$$\rho_{II} = \rho_I - \operatorname{div} P_I \tag{12}$$
$$\eta_{II} = (n, P_I)$$

The required values of ρ_{II} and η_{II} can be determined by solving these equations.   In terms of physical models, the electrostatic properties of a body A are described completely by ρ_{II} and η_{II} with a subdivision into volume and surface cells, while the

corresponding properties of a body B are described by ρ_2 and P_2 with a subdivision into volume cells only.   As a consequence of (12), the two bodies are electrostatically identical.   Yet body A may contain an abundance of free charge in a surface layer, while body B can have no free surface charge.   It thus appears that every free-charge body described by ρ_{f1} and η_{f1} has a bound-charge counterpart described by ρ_2 and P_2 that is indistinguishable from it in every way which can be described mathematically in terms of two continuous functions.

Two bodies that are electrostatically indistinguishable may be substituted for each other at will in all static-state conditions. Consequently, if every free-charge model is replaced mathematically by its bound-charge counterpart (whether this has a physical meaning or not), the function $\psi = \eta + (\hat{n}, P)$ may be used to describe mathematically the surface conditions of both bound- and free-charge models.

This entire formulation may be criticized by arguing that the actual construction of the functions $\psi = \eta + (\hat{n}, P)$ and $\beta = \rho - \operatorname{div} P$ is hopelessly intricate from the physical point of view.   It involves sorting out the free charges and constructing ρ_f and η_f to describe their distribution; defining new functions ρ, η, and P to characterize the bound-charge counterpart of the free-charge distribution; and finally combining the functions so obtained with the corresponding functions constructed for the bound charges actually present.   Such a criticism, however, overlooks the purpose of the formulation in terms of density functions.   It is because it is impossible physically to construct any one of the functions ρ, η, and P as defined in terms of the distribution of countless millions of charges that the continuous functions ρ, η, and P have been introduced.   It has been shown that the principal statistical properties of charge and its distribution can be described in a general way within a degree of approximation by the two functions β and ψ.   By building these functions into the mathematical model of electromagnetism and relating them to experimental analogues wherever possible, conclusions may be drawn regarding their structure in a variety of cases.   It is then possible in any particular instance to interpret this structure in terms of a bound-charge model, a free-charge counterpart, or a combination of the two using any convenient and appropriate mode of subdivision.   A choice

can be made to depend entirely upon physical reasons, because the mathematical specification of a given set of functions ρ, η, P does not demand a particular interpretation or mode of subdivision.

10. Surface Density of Polarization.—The mathematical representation of surface distributions of charge by means of the essential surface density $\bar{\eta}$ does not attain the same degree of approximation that is achieved in the description of interior conditions in terms of the essential volume characteristic $\bar{\rho}$. This follows directly from the fact that two volume functions but only a single surface function have been provided. Thus, in the interior ρ describes the statistical distribution of discrete charges and P takes account of the distribution of polarized units corresponding statistically to simple dipoles. $\bar{\eta}$, on the other hand, describes only the statistical distribution of charges in a thin layer along the surface. The term $(\hat{\Lambda},P)$, which contributes to η, must not be mistaken to be a separate surface function. Like $\bar{\eta}$ it is able to describe only a surface distribution of simple charges, not a *surface distribution of dipoles*. Actually, none of the three functions ρ, η or P is suited to represent a distinctly surface layer of simply polarized atoms or molecules as illustrated in Fig. 10.1.

Surface double — — — — — — — — — — — — —
layer of thick- + + + + + + + + + + + + +
ness δ Interior of body consists entirely of
neutral atoms or molecules

FIG. 10.1.—Surface polarization or double layer.

Such a thin surface double layer of charge is entirely plausible from the point of view of the atomic model. Certainly the outermost layer of atoms at a surface is subject to asymmetrical forces that are not so much a result of the presence of separate charged bodies in the vicinity as of the essential one-sidedness of the intra-atomic forces themselves. The electron shells of the outermost atoms at a surface experience the repulsion of similar shells due to neighboring atoms only on three sides. A distortion of the outer shells of electrons on the exposed side in the form of a shift relative to the positive nuclei might certainly be expected. Such a shift would naturally occur only in a direction normal to the boundary, and its effect would

be precisely that of a double layer of charge of opposite sign or of a single layer of dipoles.   If this effect exists, it is an intrinsic property of the atomic structure of the model in question.   A surface polarization due to the action of external forces, while possible, is certainly negligible compared with an intrinsic surface polarization.   A variation in forces due to relatively distant, externally situated charged bodies across the thickness δ of a surface cell must be small compared with the strong intra-atomic forces which are brought into play in establishing precisely that equilibrium which results in the intrinsic surface polarization.

The mathematical representation of a surface double layer, as pictured in the physical model, requires an additional density function k that can be defined in terms of P in a way quite analogous to the definition of η in terms of ρ.   Thus the average density of polarization P_s at the center of a thin surface cell $\Delta \tau_s$ of thickness δ and of volume $\delta \Delta \Sigma_s$ is

$$P_s = \frac{P_s}{\delta \Delta \Sigma_s} \tag{1}$$

A value k_s can be defined at the center of each surface cell according to

$$k_s = \delta P_s = \frac{\delta P_s}{\Delta \tau_s} = \frac{P_s}{\Delta \Sigma_s} \tag{2}$$

A continuous, slowly varying vector point function k can then be interpolated from the discrete values k_s and in this way defined throughout the surface shell.   Since the surface asymmetry is entirely in a direction perpendicular to the boundary, the polarization vector k must be directed everywhere normal to the surface.   Any polarization effect tangent to the surface differs in no essential way from the interior polarization which is completely characterized by the volume function P.

The surface function k plays no very significant part in the mathematical model of electromagnetism.   In fact, its nature is such that it is difficult to devise experiments to involve it in an essential way.   It is included here for completeness.

11. The Static State and Its Mathematical Model; Summary. The physical model of matter in the static state is constructed to conform with a picture of atoms and molecules built of posi-

tively and negatively charged units that are statistically at
rest. Two special models are used. In the free-charge model
an appreciable proportion of the charges experience no restrain-
ing constitutive forces over a time average to prevent them from
moving freely under the action of a suitable external influence.
In the bound-charge model all the charges are constrained by
strong forces of constitution to remain in definite and character-
istic statistical groups, which can be distorted and oriented by
external influences, but from which no charges can be removed
over a time average. Models in which only a relatively small
proportion of free electrons exist can be interpreted in terms of
the combined properties of the extremes. It is not assumed
that these models are necessarily counterparts of structures
actually existing in nature. It is merely affirmed that close
experimental analogues can be found for theoretical properties
derived from these distinct types and from combinations of
them. As an analytically convenient substitute for the intricate
mechanism represented by a volume containing countless millions
of atoms, two scalar point functions are defined throughout a
charged region in terms of its average statistical properties. The
atomic picture of discrete charges, each at a time-average loca-
tion, is replaced by a mathematical model consisting of continu-
ous functions defined at every point in a region or body. One
of these functions characterizes the surface; the other the interior.
The two functions $\bar{\eta} = \eta + (\hat{n}.P)$ and $\bar{\rho} = \rho - \text{div } P$ are called
the essential electrical characteristics or densities of charge.
They are expressed in terms of one vector and two scalar point
functions. The scalar η is defined only in a thin surface layer;
the scalar ρ and the vector P are defined throughout the volume.
Mathematically, any two of the three functions are sufficient to
represent uniquely the two essential scalar characteristics ρ and $\bar{\eta}$
so that a given region may be characterized by a volume and a
surface density of charge or by volume densities of charge and
of polarization if in each case an appropriate mode of subdivision
is used. In the first case, using ρ and η, surface and volume
cells are required with the dividing boundaries of the cells so
drawn that each cell is unpolarized. In the second case, only
volume cells are needed and the dividing boundaries are so
placed that no single atom is apportioned partly to one cell,
partly to another. In any given atomic model one, or the other,

or both of these special modes of subdivision may be quite unrealizable in a physically sensible way. For example, if a body is postulated which is composed entirely of atoms with closed electron shells, a subdivision into volume cells only and a representation in terms of ρ and P alone are physically reasonable, whereas a separate treatment of a surface layer and a subdivision of the interior that will cut through enough atoms to leave each cell unpolarized are not physically sensible. Exactly the reverse is true if a body is postulated which is composed of atoms with an abundance of free electrons. In this case there is no polarization due to distortion or orientation of atoms throughout the volume, so that a representation in terms of P is physically unreasonable. In both cases, however, it is possible to use an alternative model, not constructed as postulated, but which nevertheless exhibits exactly the same external electrostatic properties, and which is mathematically expressed in terms of a different pair of continuous functions. In other words, a body composed entirely of bound-charge atoms can be represented (insofar as its electrostatic properties are concerned) by a suitably constructed free-charge model using η and ρ. Or a region containing only free charges can be represented by an electrostatic counterpart composed entirely of bound-charge groups and characterized by ρ and P. This does not identify the bound-charge model with the free-charge one, nor does it affirm that they have the same structural characteristics. It does state that a given distribution of bound charges has its electrostatic free-charge counterpart, and vice versa. For practical purposes, either the given distribution or its counterpart may be used since they are electrostatically indistinguishable. A body containing bound groups and a few free electrons may be described in terms of either model, looking upon it physically on the one hand as an almost bound-charge model, or on the other hand as an almost free-charge model.

Although η, ρ, and P (and also k) were defined in terms of special physical models, the representation in terms of $\bar{\eta}$, $\bar{\rho}$ (and k) transcends in its generality any special case. In fact, the static state is characterized primarily by a mathematical model consisting of two (or three) functions and the restrictions of continuity and slow variation imposed upon them. Any physical model whatsoever, the properties of which can be

described in terms of such functions, is as satisfactory as any other from the analytical point of view. The theory of the static state and subsequently of its periodic variation in time is based primarily upon the characterization in terms of these mathematical functions and only secondarily (and actually not necessarily) upon the atomic models that were used in defining and describing them. A future discovery or observation which suggests a different kind of physical model from that described, but which leads by one way or another to a representation in terms of mathematical functions with the properties of $\bar{\eta}$ and $\bar{p}$ (and k), will not change the mathematical picture of the static state as here defined. Any such change in the model must not be mistaken for a change in the experimentally observable physical world. For neither the physical nor the mathematical model is more than a tool for deriving theoretical quantities that have direct and, hence, predictable experimental analogues. And neither the method nor the particular model that is used in deriving these fundamentally significant quantities is necessarily of particular or permanent importance in the pointer-reading description of the physical world.

STATIONARY STATES: THE STEADY STATE

12. The Steady State and the Atomic Model.—The statistical representation of the static state led to the definition of average or statistical rest positions for the positive and negative charges contained in a body or region. It was specifically emphasized that the individual charges might move in irregular ways about these rest positions, but that their motion was of such a nature that the time-average properties of the dynamic model of a body were identical with those of the statistically static model.

The steady state is a generalization of the static state. It is characterized by a steady drift or circulation of electric charges relative to the statistically stationary rest positions that characterize the static condition. In terms of the dynamic picture, a steady average flow may be imagined superimposed upon the random motions of the charges. This does not mean that at any instant all the charges are moving in the direction of the drift or circulation. It simply means that over a time average[1]

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[1] Taken over times that are long compared with atomic events, but short from the experimental point of view.

more charges of one sign move in the direction of the drift or circulation than in any other.   In a region containing only one kind of charge, free-charge atoms or positive and negative ions, a steady drift might consist of an uninterrupted flow at constant velocity of one kind of charge in a definite direction, or of two kinds of charge in opposite directions relative to statistical rest positions fixed in the region.   Such a drift is called a *convection current*.   A special form of convection current is a steady drift of free electrons relative to statistically stationary nuclei.   Such a drift is called a *conduction current*.   Another possible kind of nonrandom motion of charges is a microscopic circulation of electrons about their respective nuclei.   If the individual axes of rotation in the different atoms are oriented at random, the net effect of such intra-atomic rotation (insofar as a volume cell containing millions of them is concerned) is precisely that of random motion.   On the other hand, if more of the elementary whirls have axes parallel to one definite direction than to any other, then a nonrandom condition prevails.   A steady microscopic circulation of electrons about parallel axes is called a *magnetization current*.

In characterizing the steady state in terms of continuous functions, a method entirely analogous to that used in the static state is followed.   Corresponding to the volume density of static charge ρ, a volume density of drifting charge or of convection current i is defined.   In order to take into account possible asymmetrical conditions at the surface of a region, a surface density of drifting charge or of convection current l is constructed in close analogy with the surface density of charge η.   A nonrandom surface flow of charges may be interpreted in two ways.   Corresponding to the static accumulation of free charge at a surface, a motion of free charge along a surface may be imagined.   Similarly, corresponding to the static orientation and distortion of bound-charge groups to give an effective static surface charge, it is possible to picture an orientation and alignment of elementary current whirls or circulations throughout a volume to produce an effective moving surface charge.   In order to describe this volume effect in a physically more reasonable manner, a volume density of magnetization M is introduced in a way entirely analogous to the definition in the static state of the volume density of polarization P.   Finally, by suitably combin-

ing the three steady-state functions in a manner resembling that used in combining the three static-state functions, a single surface function and a single volume function may be defined which are independent of the mode of subdivision, and which are, then, the essential characteristics of the steady state.

13. Volume Density of Convection Current.—Suppose there is a continuous stream of electric charges through a typical volume element $\Delta\tau_k$ subject to the same size restrictions as the elements used in the static state. Such a stream may take the form of what is called a convection current in which all the charges in $\Delta\tau_k$ at any instant are a part of the drift; or it may be the special form of a convection current, called a conduction current, in which only the free electrons are moving in a steady stream while the positive charges are statistically at rest. A function that is to characterize a current of either kind must take account of the magnitude of the average moving charge in each volume element and of its average velocity. If a typical charge e_j is moving with a mean drift velocity v_j relative to a fixed reference frame in a region, the "quantity of moving charge" is defined to be $e_j v_j$. The quantity of moving charge in the volume element $\Delta\tau_k$, in which there are n charges, is the vector sum or the resultant of the individual quantities of moving charge. It is $\sum_{j=1}^{n} e_j v_j$.

Let the average quantity of moving charge in a volume cell be called its volume density of drifting charge or *volume density of convection* or *conduction current;* let it be denoted by i_k. For the cell $\Delta\tau_k$ it is

$$i_k = \frac{\sum_{j=1}^{n} e_j v_j}{\Delta\tau_k} \tag{1}$$

This is a vector defined at the center of the small volume cell; it is a measure of the average drift of electric charges through the cell both in magnitude and direction. A similar vector may be defined at the center of each volume cell throughout a body or region. By interpolating from these discrete vectors, a continuous *vector* point function i may be constructed which, by definition, assumes the value i_k at the center of each volume cell $\Delta\tau_k$ and which smoothly connects these at all intermediate points.

A somewhat different, though equivalent, definition of the volume density of current is in terms of the volume density ρ' of that part of the charge that is engaged in nonrandom motion with mean drift velocity u. In the definition of i formulated above, no specific account was taken of charges engaged in nonrandom motion as distinct from charges that are statistically at rest. It is, however, clear that if a charge e is statistically at rest so that its rest position is fixed, its drift velocity v is zero, and it contributes nothing to the volume density of current. If those charges in $\Delta\tau_k$ which are *not* statistically at rest are denoted by primes, it follows that

$$ i_k = \frac{\sum_{j=1}^{n} e_j v_j}{\Delta\tau_k} = \frac{\sum_{j=1}^{n'} e_j' v_j}{\Delta\tau_k} \tag{2} $$

The average vector velocity of moving charge in $\Delta\tau_k$ is

$$ u_k = \frac{\sum_{j=1}^{n'} e_j' v_j}{\sum_{j=1}^{n'} |e_j'|} \qquad \left(\sum_{j=1}^{n'} |e_j'| \neq 0 \right) \tag{3} $$

The volume density of charge ρ_k' due to the *magnitude* of all charges engaged in drift motion in $\Delta\tau_k$ is defined in a way analogous to the definition of volume density of charge, *i.e.*,

$$ \rho_k' = \frac{\sum_{j=1}^{n'} |e_j'|}{\Delta\tau_k} \tag{4} $$

With these two values substituted above,

$$ i_k = u_k \rho_k' \tag{5} $$

From discrete values of i_k, u_k, and ρ_k' defined at the center of every volume element, a continuous scalar point function ρ', a continuous vector point function u, and a continuous vector point function i are defined. These continuous functions are related by the expression

$$ i = u\rho' \tag{6} $$

which defines the vector volume density of convection or conduction current i in terms of the mean vector velocity of drifting charge u and its volume density ρ'. All three functions are

ing the three steady-state functions in a manner resembling that used in combining the three static-state functions, a single surface function and a single volume function may be defined which are independent of the mode of subdivision, and which are, then, the essential characteristics of the steady state.

13. Volume Density of Convection Current.—Suppose there is a continuous stream of electric charges through a typical volume element $\Delta\tau_k$ subject to the same size restrictions as the elements used in the static state. Such a stream may take the form of what is called a convection current in which all the charges in $\Delta\tau_k$ at any instant are a part of the drift; or it may be the special form of a convection current, called a conduction current, in which only the free electrons are moving in a steady stream while the positive charges are statistically at rest. A function that is to characterize a current of either kind must take account of the magnitude of the average moving charge in each volume element and of its average velocity. If a typical charge e_j is moving with a mean drift velocity v_j relative to a fixed reference frame in a region, the "quantity of moving charge" is defined to be $e_j v_j$. The quantity of moving charge in the volume element $\Delta\tau_k$, in which there are n charges, is the vector sum or the resultant of the individual quantities of moving charge. It is $\sum\limits_{j=1}^{n} e_j v_j$. Let the average quantity of moving charge in a volume cell be called its volume density of drifting charge or *volume density of convection* or *conduction current;* let it be denoted by i_k. For the cell $\Delta\tau_k$ it is

$$i_k = \frac{\sum\limits_{j=1}^{n} e_j v_j}{\Delta\tau_k} \tag{1}$$

This is a vector defined at the center of the small volume cell; it is a measure of the average drift of electric charges through the cell both in magnitude and direction. A similar vector may be defined at the center of each volume cell throughout a body or region. By interpolating from these discrete vectors, a continuous *vector* point function i may be constructed which, by definition, assumes the value i_k at the center of each volume cell $\Delta\tau_k$ and which smoothly connects these at all intermediate points.

A somewhat different, though equivalent, definition of the volume density of current is in terms of the volume density ρ' of that part of the charge that is engaged in nonrandom motion with mean drift velocity u. In the definition of i formulated above, no specific account was taken of charges engaged in nonrandom motion as distinct from charges that are statistically at rest. It is, however, clear that if a charge e is statistically at rest so that its rest position is fixed, its drift velocity v is zero, and it contributes nothing to the volume density of current. If those charges in $\Delta\tau_k$ which are *not* statistically at rest are denoted by primes, it follows that

$$i_k = \frac{\sum_{j=1}^{n} e_j v_j}{\Delta\tau_k} = \frac{\sum_{j=1}^{n'} e'_j v_j}{\Delta\tau_k} \tag{2}$$

The average vector velocity of moving charge in $\Delta\tau_k$ is

$$u_k = \frac{\sum_{j=1}^{n'} e'_j v_j}{\sum_{j=1}^{n'} |e'_j|} \qquad \left(\sum_{j=1}^{n'} |e'_j| \neq 0\right) \tag{3}$$

The volume density of charge ρ'_k due to the *magnitude* of all charges engaged in drift motion in $\Delta\tau_k$ is defined in a way analogous to the definition of volume density of charge, *i.e.*,

$$\rho'_k = \frac{\sum_{j=1}^{n'} |e'_j|}{\Delta\tau_k} \tag{4}$$

With these two values substituted above,

$$i_k = u_k \rho'_k \tag{5}$$

From discrete values of i_k, u_k, and ρ'_k defined at the center of every volume element, a continuous scalar point function ρ', a continuous vector point function u, and a continuous vector point function i are defined. These continuous functions are related by the expression

$$i = u\rho' \tag{6}$$

which defines the vector volume density of convection or conduction current i in terms of the mean vector velocity of drifting charge u and its volume density ρ'. All three functions are

continuous and slowly varying. The volume density of current differs from zero only when the density of charge in drift motion ρ' is not zero. If all the charges in the region are moving with the mean velocity u, then $\rho' \equiv \rho$ and i gives the convection current density. If only electrons are moving with a mean drift velocity u, ρ' is the volume density of charge only for electrons in nonrandom motion, and i gives the conduction current density. The volume density of current has the dimensions of charge per unit time per unit area. The coulomb per second has the name ampere.

$$i \approx \frac{Q}{L^2 T} \frac{\text{amperes}}{\text{square meters}} \qquad (7)$$

14. Surface Density of Convection Current.—As a result of asymmetrical conditions at the surface of a region or on the boundary between two dissimilar regions, the volume density of charge ρ is inadequate as a means of representing surface distributions of statistically stationary charge. In the same way and for similar reasons the continuous slowly varying volume density of current i is too insensitive to represent surface distributions of charge moving as a steady drift. Here, as before, the division of the region into volume cells alone is too coarse a mode of subdivision in the immediate vicinity of surfaces and boundaries to give proper weight to the rapid variations in current density that may exist there. This may be illustrated in terms of two relatively simple models which are analogous to the free-charge and the bound-charge models in the static state. They are called the free drift model and the spin model. In the drift model there may be a thin layer of free electrons which is moving with a steady drift velocity along the surface while the charges in the interior are all statistically stationary so that the volume density of current i as interpolated from the i_k of the volume cells $\Delta \tau_k$ is zero throughout the interior. For the volume cells along the surface, the i_k have nonvanishing values, but they are very small because the charges moving in a thin layer along the surface are averaged over the entire volume of the cell of which they occupy only a minute part. It is, therefore, necessary to treat a thin layer of cells along the surfaces separately.

In the spin model the electrons form small intra-atomic whirls rotating in the same sense about mutually parallel axes. The net

drift of charge through any volume element $\Delta \tau_k$ which contains a great many such microscopic whirls is zero for the element as a whole provided the cell boundaries nowhere cut the paths of the individual whirls. Thus the volume density of current i as defined above is zero in every volume cell $\Delta \tau_k$, whether these are in the interior or along the surface. However, wherever the whirls are tangent to a surface or boundary, the net effect is that of a continuous sheet of current flowing in a very thin layer in one direction. Thus, a surface current can be constructed out of properly aligned elementary whirls existing throughout the volume. The surface sheet of current due to such interior whirls is no more taken into account by the function i than is the surface sheet of charge due to distorted and oriented atoms represented by ρ in the analogous static case. Here, again, the difficulty may be overcome by treating a thin layer separately in terms of a specially constructed surface function.

A surface density of convection current may be defined much as was η by separating a very thin superficial layer of thickness δ and dividing this up into surface cells $\Delta \tau_s$. A continuous two-dimensional vector point function is then interpolated from the discrete values

$$I_s = \frac{\delta \sum_{j=1}^{n} c_j v_j}{\Delta \tau_s} = \frac{\sum_{j=1}^{n} c_j v_j}{\Delta \Sigma_s} \tag{1}$$

defined at the centers of the thin surface cells $\Delta \tau_s$. The velocities v_j are all tangent to the surface or boundary. Alternatively, I may be defined in terms of the surface density of the part of the charges (primed) that are engaged in nonrandom motion with a mean drift velocity u. That is,

$$I = u\eta' = \delta u \rho' = \delta i \tag{2}$$

Here u is constructed by interpolation from the u_s defined at the centers of the surface cells by an expression like (12.3); η' is interpolated from discrete values defined by

$$\eta_s' = \frac{\delta \sum_{j=1}^{n} |c_j'|}{\Delta \tau_s} \tag{3}$$

| has the dimensions of charge per unit time per unit length

$$l \approx \frac{Q}{LT} \frac{\text{amperes}}{\text{meters}} \tag{4}$$

It is necessary to require that the region to be represented is sufficiently uniform so that the two-dimensional vector point function | is so slowly varying as to be sensibly constant over lateral areas that are large compared with the largest dimension of the $\Delta\tau_s$ cells.

A steady-state distribution of charge moving in a region constructed according to either of the two steady-state models may be characterized in terms of the two continuous, slowly varying vector point functions i and |.

15. Alternative Modes of Representation.—Just as the static state was at first defined only in terms of surface and volume densities of charge using a special mode of subdivision appropriate to these functions, so the steady state has, up to this point, been represented entirely by surface and volume densities of drifting charge requiring the same special mode of subdivision into separate surface and volume cells. As in the static state, this mode of subdivision is physically reasonable for the free drift model in which a separate and distinctly surface flow of free charges exists.   On the other hand, the division into thin surface cells is not appropriate physically for the spin model because the boundaries of elementary cells are carefully and deliberately arranged to cut interior whirls (consisting of oriented groups of circulating electrons) so that in their steady circulation the charges continually pass from one cell into another and back. From the purely physical point of view, it is certainly artificial to call such parts of whirls at the surface a surface sheet of moving charge and to say that there is no circulation in the interior because of a clever allotment of parts of individual whirls to different volume cells in such a way that the net average circulation in each vanishes.   This is illustrated roughly and schematically in Fig. 15.1. From the mathematical point of view it is immaterial whether the subdivision used has a physical meaning or not, so long as it leads to a correct representation of the desired average effects.   Nevertheless, a mathematical formulation that is closely adjusted to the physical model is often a valuable one.   Hence, as in the static case, an alterna-

tive representation is introduced that is physically appropriate for the spin model and that describes a volume effect in terms of a volume function. Instead of separating the surface layer and defining a special density for it, only volume cells are used and a new volume density is defined throughout the region to take

SUBDIVISION I: Volume cells only.
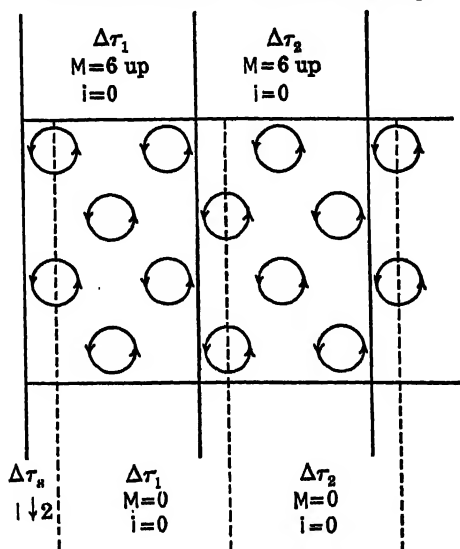
SUBDIVISION II: Surface and volume cells.

Fig. 15.1.—Surface sheet of current caused by current whirls.

account of the elementary electronic whirls both in the interior and at the surface.

16. Magnetization.—In the static state the volume density of polarization P represents both in magnitude and in direction the average distortion-orientation of bound-charge groups. A simple model of a bound-charge atom consists of a closed shell of electrons associated with a positive nucleus. Under the action of suitable external forces a separation of the average relative rest positions of the positive nucleus and the electrons may occur; the statistically equivalent representation of such a distorted atom is a dipole consisting of a positive charge separated a short distance from an equal negative charge. The distribution and orientation of statistical dipoles along a direction specified by the external force is called polarization.

In the steady state a model consisting of electrons rotating about an axis through an atom is an elementary magnet.   Elementary magnets are produced by forces that cause the charges in an atom to change their random orbits and circulate about a common axis.   The orientation of such elementary magnets along parallel axes with a common direction of rotation is called magnetization.   Such a circulation-orientation may be superimposed upon a random distribution.   In order to characterize the magnetization of a volume element, it is necessary to specify both the axis
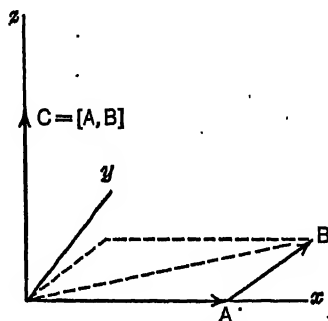
Fig. 16.1.—Vectors illustrating the vector product.
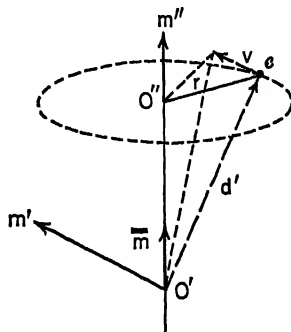
Fig. 16.2.—Charge rotating about an axis referred to origins on the axis.

and the direction of rotation, the magnitude and velocity of the whirling charge, and its distance from the axis.

Consider first a single charge c moving in a circle of radius r with a velocity v that is constant in magnitude as shown in Fig. 16.2.   The rotation of the charge referred to an origin O'' at the center of the circle is completely described in terms of a so-called axial vector at O'' pointing along the axis of rotation (the z axis) in the direction of advance of a right-hand screw rotating in the same sense as the charge.   Let the magnitude of this vector be $m'' = \frac{1}{2} rev = c \times$ area of triangle formed by r and v.   The axial vector itself may be defined in terms of one-half the vector product[1] of the polar vector r from O'' to c, and the

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[1] The vector product of two polar vectors A and B is itself an axial vector C defined by $[A,B] = C$; $C = AB \sin (A,B)$.   The axial vector C is defined to be normal to the plane of A and B and so directed that A, B, C form a right-handed system (Fig. 16.1).   This is a system such that when A turns into B by the shortest arc, which is at the same time the direction of rotation of a right-hand screw, C points in the direction of advance of the screw.   In

polar vector ev defining the quantity of moving charge.

$$\mathbf{m}'' = \tfrac{1}{2}[\mathbf{r}, ev] \tag{2}$$

Since the direction of $\mathbf{m}''$ does not change as the charge rotates, it specifies both an instantaneous and a time-average value.

If the origin is at a point O' on the axis through the center of the circle but not at the center, the magnitude m'' can be expressed in terms of the distance d' from the new origin to the rotating charge.

$$m'' = \tfrac{1}{2}d'ev \sin (z,d') = \tfrac{1}{2} rev \tag{3}$$

This is the magnitude of the projection of the vector

$$\mathbf{m}' = \tfrac{1}{2}[\mathbf{d}', ev] \tag{4}$$

along the axis. That is,

$$m'' = m' \sin (z,d') \tag{5}$$

as shown in Fig. 16.2. The vector $\mathbf{m}'$ rotates around the z axis, remaining always perpendicular to the triangle formed upon $\mathbf{d}'$

Cartesian coordinates

$$[\mathbf{A},\mathbf{B}] = \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x)$$

since $[\hat{x},\hat{x}] = 0$; $[\hat{x},\hat{y}] = 1$. It may be written in the simple determinant form

$$[\mathbf{A},\mathbf{B}] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1}$$

The magnitude C is equal to the area of the parallelogram formed on $\mathbf{A}$ and $\mathbf{B}$. The vector product is often called the cross product and written $\mathbf{A} \times \mathbf{B}$.

An axial vector, also called a rotor, differs from an ordinary or polar vector in a fundamental way. In a polar vector the direction of the vector and its opposite differ in a physical sense as explained in defining the polar vector $\mathbf{p}$ in Sec. 6. On the other hand, in an axial vector such as $\mathbf{m}''$ in (2) above, which is defined in terms of the vector product of two polar vectors, the direction of the vector and its opposite are equivalent since the geometrical and physical data are symmetrical with respect to a plane perpendicular to the vector. The direction of an axial vector results from an *arbitrary* choice between these two physically equivalent directions, a choice that is usually characterized by reference to the direction of advance and rotation of a right-hand screw or its equivalent. Analytically, polar and axial vectors differ as follows. If the positive directions of the axes of a coordinate system are reversed, the components of a polar vector change their sign, while those of an axial vector do not.

and v. The time-average value of m′ is m″. That is,

$$\overline{\overline{m}}' = \tfrac{1}{2}[\overline{\overline{d}}',\overline{ev}] = \tfrac{1}{2}[r,ev] \tag{6}$$

The superscript double bar denotes a time average. If interest lies in a time-average value, the location of the origin along the axis of rotation is immaterial.

Fig. 16.3.—Charge rotating about an axis referred to an origin off the axis.

If the origin is at any arbitrary point O (Fig. 16.3) relative to a single rotating charge, the vector

$$m = \tfrac{1}{2}[d,ev] \tag{7}$$

changes both its length and its direction as the charge rotates. The vector d is drawn from O to the charge. It may be expressed in terms of the vector s which is a perpendicular dropped from O onto the axis of rotation at O', and the vector d' from O' to the charge.

$$d = s + d' \tag{8}$$

Substitution in (7) gives

$$m = \tfrac{1}{2}[s,ev] + \tfrac{1}{2}[d',ev] \tag{9}$$

Since s is fixed in space, the time average value of $[s,ev]$ must vanish. Hence, with (6)

$$\overline{m} = \tfrac{1}{2}[\overline{d}',\overline{ev}] = \tfrac{1}{2}[r,ev] \tag{10}$$

The time-average value of m *as defined in* (7) *is independent of the location of the reference origin.*

The time-average circulation of each of a large number of statistically distributed charges in an element of volume $\Delta\tau_i$ may be referred to the center of $\Delta\tau_i$. The resultant time-average vector $\overline{m}_i$ is

$$\overline{\mathbf{m}}_i = \sum_{j=1}^{n} \overline{\mathbf{m}}_j = \sum_{j=1}^{n} \frac{1}{2} [\overline{\mathbf{d}_j, e_j \mathbf{v}_j}] \tag{11}$$

However, in a statistical distribution in which millions of circulating charges are in random phases in their respective orbits, the vector sum of the instantaneous values $\mathbf{m}_j$ is equal to the vector sum of the time-average values $\overline{\mathbf{m}}_j$. Hence it is possible to define the instantaneous or time-average value of $\mathbf{m}_i$ for a volume element $\Delta\tau_i$ in which by definition statistical conditions prevail without taking time averages.

$$\mathbf{m}_i = \sum_{j=1}^{n} \mathbf{m}_j = \sum_{j=1}^{n} \frac{1}{2} [\mathbf{d}_j, e_j \mathbf{v}_i] \tag{12}$$

Here the vectors $\mathbf{d}_j$ are drawn from the center of $\Delta\tau_i$ to the charges e_j moving with velocities $\mathbf{v}_j$ at any one instant. The vector $\mathbf{m}_i$ is called the vector magnetization of the volume element. Its direction is the axis of magnetization; its magnitude is the magnetic moment.

The vector volume density of magnetization of the circulating charges in $\Delta\tau_i$ is defined by

$$\mathbf{M}_i = \frac{\mathbf{m}_i}{\Delta\tau_i} \tag{13}$$

If a body or region is divided into volume cells $\Delta\tau_i$ and a vector $\mathbf{M}_i$ is defined at the center of each cell to describe the average circulation-orientation of the electric charge, it is possible to interpolate from the discrete values $\mathbf{M}_i$ a continuous vector point function $\mathbf{M}$ defined throughout the volume. This function is the *volume density of magnetization*. It assigns an axial vector to every point in the region or body in which it is defined, so that the direction of the vector is along the axis of magnetization in the positive sense according to the right-hand screw convention, and the magnitude of the vector measures the average circulation of charge about the axis in a small region near the point. Its dimensions are charge per unit length per unit time.

$$M \approx \frac{Q}{LT} \frac{\text{amperes}}{\text{meters}} \tag{14}$$

and v. The time-average value of m′ is m″. That is,

$$\overline{\overline{m}}' = \tfrac{1}{2}\overline{[d',ev]} = \tfrac{1}{2}[r,ev] \qquad (6)$$

The superscript double bar denotes a time average. If interest lies in a time-average value, the location of the origin along the axis of rotation is immaterial.

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If a body or region is divided into volume cells $\Delta\tau_i$ and a vector $\mathbf{M}_i$ is defined at the center of each cell to describe the average circulation-orientation of the electric charge, it is possible to interpolate from the discrete values $\mathbf{M}_i$ a continuous vector point function $\mathbf{M}$ defined throughout the volume. This function is the *volume density of magnetization*. It assigns an axial vector to every point in the region or body in which it is defined, so that the direction of the vector is along the axis of magnetization in the positive sense according to the right-hand screw convention, and the magnitude of the vector measures the average circulation of charge about the axis in a small region near the point. Its dimensions are charge per unit length per unit time.

$$M \approx \frac{Q}{LT} \frac{\text{amperes}}{\text{meters}} \tag{14}$$

As in all preceding definitions of continuous densities, the body or region must be assumed to be sufficiently uniform that each $\Delta\tau_i$ cell differs only slightly from its neighbors, and the continuous, interpolated function M varies only very slowly over distances that are large compared with the dimensions of the volume cells. The definition of M implies in particular that the volume density of drifting charge i has an extremely small variation from cell to cell.[1]

The function M characterizes the steady-state circulation-orientation of electric charge in a way analogous to the characterization of the static-state distortion-orientation by the function P.

17. Comparison of Two Representations.—In describing the steady-state properties of a region in terms of a volume and a surface density of drifting charge and, alternatively, in terms of two volume densities, one of drifting charge and one of magnetization, a situation arises that is analogous to that encountered in the static cases. In the mode of representation using surface and volume functions, the region is treated in two parts in each of which one function alone carries the burden of representation. In the alternative mode using volume functions only, both functions are defined in the entire region using the same volume cells. Since the volume density of drifting charge i (just as the volume density of charge ρ) is not sufficiently fine-grained to take account of surface conditions, it follows that the volume density of magnetization M must describe surface conditions of moving charge (much as P describes surface conditions of charge).

In examining the parts played by the three functions i, I, and M and their dependence upon the mode of subdivision, it is possible to proceed by complete analogy with the static state and its three functions ρ, η, P. If a given region containing spin atoms only is divided into volume cells in such a way that each cell contains only complete current whirls or elementary magnets, the volume density of current i is zero. There is no drift of charge across the boundaries of any volume element. Let the magnitude of the magnetization (or circulation-orientation) in the region increase uniformly in a direction normal to

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[1] The mathematical basis for this restriction is given in M. Mason and W. Weaver, "The Electromagnetic Field," p. 193.

the fixed axis of magnetization. This condition is shown schematically and one-dimensionally in Fig. 17.1 where the number of elementary whirls increases steadily from left to right. The axis of magnetization is normal to the paper, pointing upward. Hence the volume density of magnetization M is

Subdivision I: Representation entirely in terms of the volume density of magnetization, M; $i = l = 0$

Subdivision II: Representation in terms of both volume and surface densities of moving charge, i and l; $M \doteq 0$
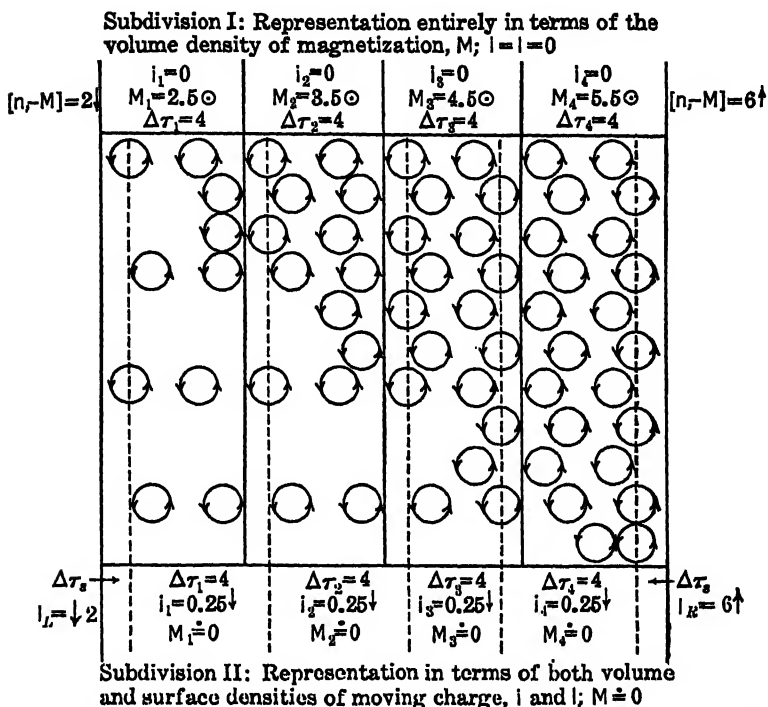
FIG. 17.1.—Electrically one-dimensional schematic diagram to show two possible modes of subdivision. The symbol $\odot$ means directed vertically upward from the plane of the paper. The magnitude of $\sum_{j=1}^{n} \frac{1}{2}[d_j, c_j v_j]$ is taken to be one for each whirl. If the left side is at $x = 8$, the right side at $x = 24$, subdivision I is characterized by $M_I = \frac{1}{2}x2$; subdivision II by $i_{II} = \text{curl } M_I = -\frac{1}{2}\hat{y}$; $l_{IIL} = [\hat{n}, -M_I]_L = -2\hat{y}$; $l_{IIR} = [\hat{n}, -M_I]_R = 6\hat{y}$.

directed upward and M increases uniformly from left to right. Since i is zero everywhere, and l is not defined at all, M must characterize completely the steady-state properties of the region both in the interior and at the surface. By following around the boundaries of a cell and noting the direction of each half-current whirl nearest the boundary in every row, these will all be

found directed to produce a counterclockwise circulation around the interior of each $\Delta\tau_i$ cell.

Now let the subdivision be changed as indicated in the figure so that some of the elementary magnets near the surface are cut in two by the cell boundaries. The interior is then subdivided by the dotted lines in such a way that the magnetization in each volume cell is approximately zero. This can be verified in Fig. 17.1 by again following around the edge of each volume cell and noting the direction of the current in each half whirl *nearest* the edge in *every* row, regardless of whether it is to the right or to the left of the solid boundaries of subdivision I. It will be found that the directions of the half whirls now alternate, so that the net circulation around each volume element is very small. This can also be verified using (16.13) if it is noted that the center of each cell is a vertical line since the figure is one-dimensional. In any case, for a volume element containing millions of elementary magnets, M can be made practically zero. On the other hand, it will be found that each volume cell as a whole contains an extra half whirl downward which is not compensated by an upward half. This means that there is a net drift of current downward throughout the interior, so that the volume density of current has a nonvanishing value. Thus, in the second mode of subdivision the interior is characterized entirely by i, whereas in the first mode it was characterized by M. On the other hand, in the first mode of subdivision there were no surface cells, and hence no surface density I was defined. Properties peculiar to the surface thus had to be included in the description given by M alone. In the second subdivision there are surface cells, and a surface current is seen to flow down on the left surface and a somewhat larger one up on the right. Hence I has a nonvanishing value in subdivision II and the region with spin-model atoms is characterized, just as in the free-drift model, by I and i.

If neither of the artificial modes of subdivision is used and no precautions are taken to have the cell walls cut or not cut through a certain number of atomic whirls on the average, statistical conditions prevail along these walls as well as in the interior of the cells. Accordingly, there are always some atoms near the boundary completely in one volume cell, while others are cut by the boundary and are partly in one, partly in an

adjacent cell. In the case considered in conjunction with Fig. 17.1, the number of elementary whirls increases from left to right, so that on the average more of them are cut by the right than by the left boundary of each cell. The peculiarities of each of the two modes of subdivision schematically illustrated in Fig. 17.1 exist simultaneously in the general picture, and all three functions i, l, and M have nonvanishing values. The situation is analogous to the static state in which all three functions ρ, η, and P had nonvanishing values for unrestricted subdivisions. It is to be expected by analogy that a suitable combination of i and M and of l and M will permit the definition of essential volume and surface characteristics that are independent of the mode of subdivision.

18. Essential Volume Characteristic of the Steady State; Curl of a Vector.—A volume density of moving charge that is independent of the mode of subdivision may be obtained by a method analogous to that used in defining the function $\bar{\rho} = \rho - \text{div } P$ in the static state. That is, the representation of the distribution of moving charge in a typical volume cell is investigated when the cell walls are shifted from the one to the other of the two modes of subdivision illustrated in Fig. 17.1. With subdivision I each volume cell, *c.g.*, $\Delta\tau_2$, contains only complete elementary magnets or whirls—none is cut by the boundaries. In moving the boundaries to subdivision II, in which $M = 0$, three half whirls are cut on the left and removed from the cell, while four half whirls are cut on the right and added to the cell. The net result is an addition to the cell of one half whirl representing a downwardly directed current of a certain magnitude. A similar result is observed in each volume cell. Hence, the volume density of current i$_k$, as defined in the usual way, has a nonvanishing value if this schematic picture is replaced by a distribution involving millions of current whirls. A continuous volume density of moving charge i can be interpolated for the entire region. It is clear from the highly schematic model that there is a direct correspondence between the number of elementary magnetization vectors $m = \frac{1}{2}[d,ev]$, or current whirls cut by cell boundaries (in changing from subdivision I to subdivision II), and the volume density of current appearing in each volume cell in subdivision II. To include a *part* of a steady-state current whirl in a volume cell is exactly equivalent to adding an

element of current flowing parallel to that boundary which cuts
the whirl.   (In the static state the inclusion of a part of a dipole
cut by a boundary was in effect the same as adding a definite
amount of electric charge.)

   It is necessary to define a quantity that is a measure of the
number and orientation of the current whirls cut by a boundary
and simultaneously of the equivalent current sheet due to these
cut whirls on each side of the boundary.   Such a quantity must
resemble $(\hbar,P)$ in the static state which is a measure as well
of the number of dipoles cut as of the amount of charge added
to a volume cell in a change in subdivision.   The vector M
is a continuous function which measures the average density
and direction of current whirls or of elementary magnetization
vectors in a small region about any point.   Since it is parallel
to the axis of rotation of the circulating charges, and a nonrandom
axially directed drift is included in the function i, M must be
perpendicular to the plane of the motion of charges to be repre-
sented by M.   Charges moving parallel to the bounding surface
of any volume cell must be moving perpendicular to the unit
external normal $\hbar$ to that surface.   It follows that the circulating
charges that are a part of a current whirl cut by a cell boundary
in changing from subdivision I to II, and moving parallel to a
boundary, must be perpendicular both to M and to an external
normal $\hbar$ to the surface.   This does not require M to be perpen-
dicular to $\hbar$.   A vector that has a direction perpendicular to M
and $\hbar$ and that is at the same time proportional to M (which
measures the average vector sum of all the elementary magneti-
zation vectors $m = \frac{1}{2}[d,ev]$, as defined for each spin atom or
elementary whirl per unit volume) is the vector product

$$[\hbar,M] \tag{1a}$$

Since $\hbar$ is a unit vector, the magnitude of (1a) is

$$M \sin (n,M) = M \cos (s,M) \tag{1b}$$

Here s is a vector tangent to the surface.   In other words,
$[\hbar,M]$ has a magnitude that is equal to the component of M
parallel to the surface along which the circulation of charge
takes place.   An examination of Fig. 18.1 (which is drawn for
the simplest case in which M is itself parallel to the surface)
shows that in order to preserve the right-hand-screw convention

as applied to current whirls of positive charge it is necessary to use a vector equal in magnitude but opposite in direction to $[\hat{n}, M]$. Such a vector is

$$[\hat{n}, -M] \tag{2}$$

Figure 18.2 shows the orientation of the vectors and the normal $\hat{n}$ for the general case in which $\hat{n}$ and M are not perpendicular to each other.

The magnitude of the vector $[\hat{n}, -M]$ is a measure of the average vector sum of the *tangential components* of the elementary
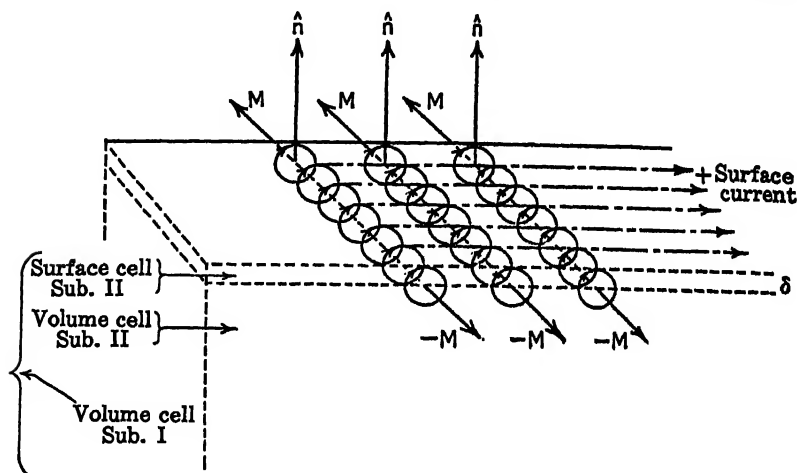
Fig. 18.1.   Surface current due to magnetization whirls.

magnetization vectors that pierce or lie in the plane of a unit area of the surface for which $[\hat{n}, -M]$ is defined.   The direction of the vector $[\hat{n}, -M]$ is the same as the direction of motion v along the surface of the *positive* charge in the elementary half whirls (characterized by the magnetization vectors) on the side of the surface for which $\hat{n}$ is an external normal.   This is illustrated in Fig. 18.2 in which one current whirl is shown.

In order to determine the significance of the vector $[\hat{n}, -M]$ at a surface enclosing a volume cell, let the change in subdivision from I (volume cells only — no whirls cut) to II (surface and volume cells such that $M = 0$) be examined in Fig. 18.1. If the boundaries of the volume cells are moved *down* half the diameter of a current whirl, the positive current sheet (composed

of half whirls) which is *removed* from this cell across each unit of its surface as the boundary is shifted down is measured by [ň, −M]. This is equivalent to moving an identical, oppositely directed current sheet *into* the newly formed cell. That is, the sheet of *positive* current that *appears in* the volume cell per unit of surface in the change in subdivision is measured by −[ň, −M] or [ň,M]. Since M is a continuous function, the vector [ň,M] is defined at every point along the boundary of each volume cell
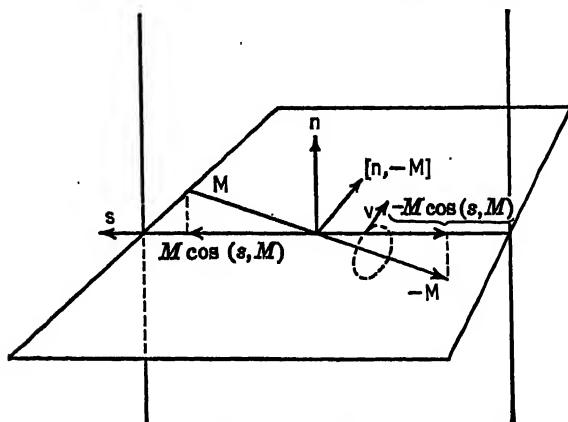
FIG. 18.2.—Orientation of magnetization vector and current whirl at a surface.

in subdivision I. By integrating over the entire surface Σ of a volume cell

$$\int_{\Sigma(\text{volume cell})} [\hat{n},M_I]d\sigma \tag{3}$$

a measure is obtained of the average number of positive half-current whirls that are cut and effectively *added* to the cell in the change in subdivision, and of the mean direction of their circulations as specified by their respective magnetization vectors. This integral is, also, a measure of the average positive current that is transferred across the boundaries of a volume cell as these are shifted to change from mode I to mode II. In (2), ň is external to the cell, and M is necessarily the value in subdivision I, since M = 0 in subdivision II. If this integral vanishes, the same number of current whirls is cut on each side, so that there are no half whirls downward *added* on one side that are not compensated by half whirls downward *lost* on the other side as illustrated in Fig. 15.1. If the integral does not

vanish, the compensation is incomplete and a current is observed as in Fig. 17.1. The current per unit volume appearing in $\Delta\tau_k$ in subdivision II is

$$\frac{\int_{\Sigma} [\hat{n}, M_I] d\sigma}{\Delta\tau_k} \tag{4}$$

In order to verify that (4) measures the positive current density, let a very simple case be studied in detail. Consider
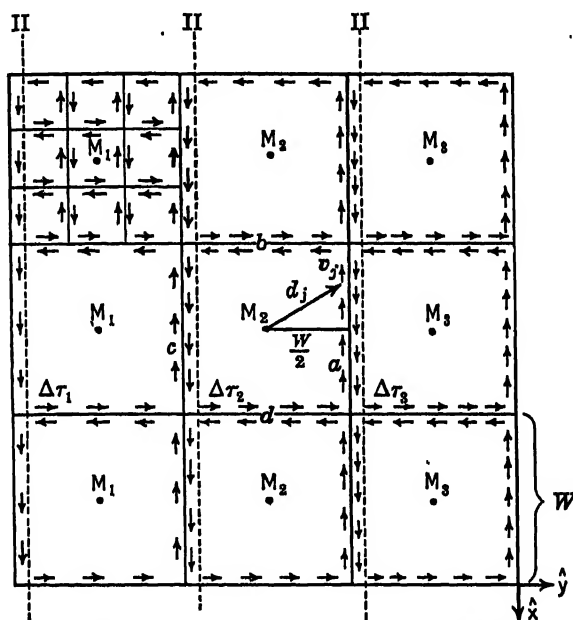
Fig. 18.3.—Magnetization and current along boundaries of volume cells. M is directed vertically upward from the paper; its magnitude increases uniformly from left to right. Greater detail of the typical circulation in each cell is shown in the upper left-hand corner. The boundaries for subdivision II are shown in broken lines.

a region in which M is constant in direction, vertically upward, but in which it increases uniformly toward the right. A top view of nine cubical volume elements of such a region is shown in Fig. 18.3. The average value of M_k at the center of each cell as calculated using (16.6) is equivalent to a circulation around the cell boundaries as shown. In $\Delta\tau_1$, $3N$ charges are moving along each wall of area W^2; in $\Delta\tau_2$, $4N$ charges; in $\Delta\tau_3$, $5N$ charges.

Since the direction of M_k is known, it is merely necessary to calculate its magnitude at the center of each cell using (16.13).

$$M_k = \frac{\displaystyle\sum_{\Delta\tau_k} \frac{1}{2} d_j e_j v_j \sin (d_j, v_j)}{W^3} = \frac{\displaystyle\sum_{\Delta\tau_k} e_j v_j}{4 W^2} \tag{5}$$

The second equality in (5) follows because

$$d_j \sin (d_j, v_j) = \frac{W}{2}$$

However, $\dfrac{1}{4} \displaystyle\sum_{\Delta\tau_k} e_j v_j$ is the total current I_{ks} along each of the four sides in $\Delta\tau_k$ multiplied by the length of this side.

$$WI_{ks} = \frac{1}{4} \sum_{\Delta\tau_k} e_j v_j \tag{6}$$

Therefore

$$M_k = \frac{I_{ks}}{W} \tag{7}$$

The magnitude of M on the boundary a, between $\Delta\tau_2$ and $\Delta\tau_3$, is obtained by interpolation—in this case simple averaging—to be

$$M_a = \tfrac{1}{2}(M_2 + M_3) \tag{8a}$$

Similarly on the boundary c between $\Delta\tau_1$ and $\Delta\tau_2$

$$M_c = \tfrac{1}{2}(M_1 + M_2) \tag{8b}$$

On the boundaries b and d the value of M varies continuously. However,

$$M_b = M_d \tag{9}$$

at opposite points.

One can now form (3). Since the magnitude of $[\hat{n}, M]$ on the two faces that are perpendicular to M is zero because $\hat{n}$ and M are parallel

$$\int_{\Sigma_k} \frac{[\hat{n}, M] d\sigma}{\Delta\tau} = \frac{W^2}{W^3} \{ [\hat{n}_a, M_a] + [\hat{n}_c, M_c] \}$$

$$+ \frac{1}{W^3} \int_{z = W^2} \{ [\hat{n}_b, M_b] + [\hat{n}_d, M_d] \} d\sigma \tag{10}$$

Here the $\hat{n}$'s are unit external normals to $\Delta\tau_2$ on the sides indicated by the subscripts. Clearly

$$\hat{n}_c = -\hat{n}_a; \qquad \hat{n}_d = -\hat{n}_b \qquad (11)$$

With (11) and (9)

$$\frac{\int_{\Sigma_2}[\hat{n},\mathsf{M}]d\sigma}{\Delta\tau_2} = \frac{1}{W}(M_a - M_c)\hat{z} \qquad (12)$$

Here $\hat{z}$ is a unit vector pointing toward the bottom of the page as shown in Fig. 18.3. It specifies the direction of $[\hat{n}_a,\mathsf{M}_a]$ using the right-hand-screw convention. With (8a,b)

$$(M_a - M_c) = \tfrac{1}{2}(M_3 - M_1) \qquad (13a)$$

However,

$$M_2 = \tfrac{1}{2}(M_1 + M_3) \qquad (13b)$$

Hence

$$(M_a - M_c) = M_3 - M_2 \qquad (13c)$$

Accordingly with (13c) and (7)

$$\frac{\int_{\Sigma_2}[\hat{n},\mathsf{M}]d\sigma}{\Delta\tau_2} = \hat{z}\,\frac{(I_{3a} - I_{2a})}{W^2} \qquad (14a)$$

The expression on the right is the difference between the sheet of current I_{3a} directed down along side a in $\Delta\tau_3$ and I_{2a} flowing up along a in $\Delta\tau_2$. It measures the net flow in the direction $\hat{z}$ along both sides of the boundary a between $\Delta\tau_3$ and $\Delta\tau_2$. If the mode of subdivision is changed to II as indicated by the dotted lines in Fig. 18.3, this net flow appears in $\Delta\tau_2$. Thus

$$\hat{z}\,\frac{(I_{3a} - I_{2a})}{W^2} = i_{m2} \qquad (14b)$$

This simple example shows that (4) has the magnitude and the direction as well as the dimensions of a volume density of current.

The quantity

$$(i_{m\mathrm{II}})_k = \frac{\int[\hat{n},\mathsf{M}_\mathrm{I}]d\sigma}{\Delta\tau_k} \qquad (15)$$

is the mean density of magnetization current in the cell $\Delta\tau_k$. If such a value is determined for each cell, a continuous volume density may be interpolated in the usual way. Since M is

already such an interpolated function and is continuous through-out the region, it is permissible to pass to the limit directly and let the volume approach zero. The volume density of current that appears in the interior of a region as a result of changing the mode of subdivision from I to II is

$$i_{mII} = \lim_{\Delta\tau\to0} \frac{\int [\hat{n}, M_I] d\sigma}{\Delta\tau} \qquad (16)$$

An important shorthand symbol for defining the operation indi-cated on the right in (16) is the *curl* of a vector. It is defined by[1]

$$\text{curl } M = \lim_{\Delta\tau\to0} \frac{\int [\hat{n}, M] d\sigma}{\Delta\tau} \qquad (17)$$

and is itself a polar vector. The volume density of magnetiza-

[1] The symbol rot, an abbreviation for rotation, is also used in place of curl. The operator curl may be expressed in any system of coordinates by apply-ing the defining formula (17) to an element of volume in the desired system of coordinates. For example, in rectangular coordinates, a cubical element of volume $\Delta\tau = \Delta x \, \Delta y \, \Delta z$ may be chosen with center at the origin or, more generally, with center at any point x, y, z as shown in Fig. 18.4. At this
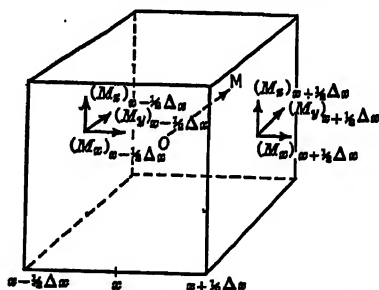
FIG. 18.4.—Cubical element of volume showing the magnetization M at the center O of the cube, and its rectangular components at the centers of the two sides.

point the vector M has a specified magnitude and direction. In rectangular coordinates it is $M = \hat{x}M_x + \hat{y}M_y + \hat{z}M_z$. On the right-hand face of the cube in Fig. 18.4, $\hat{n} = \hat{x}$; $[\hat{x},\hat{x}] = 0$; $[\hat{x},\hat{y}] = \hat{z}$; $[\hat{x},\hat{z}] = -\hat{y}$. On the left face, $\hat{n} = -\hat{x}$, and the above vector products all have the negative sign. Hence, on the right-hand face,

$$[\hat{n}, M] = \{\hat{z}M_y - \hat{y}M_z\}_{x+\Delta x/2}$$

On the left-hand face,

$$[\hat{n}, M] = \{-\hat{z}M_y + \hat{y}M_z\}_{x-\Delta x/2}$$

The average values of the components of M at the centers of the two faces

tion current i_m is a continuous vector point function defined throughout the body or region.

By a mere change in subdivision from I to II the description of the volume cells containing spin atoms only is altered as follows:

Subdivision	I	II	
	(volume cells only; no cut spin atoms)	(volume and surface cells; enough spin atoms cut to make $M_{II} = 0$)	(18)
$i =$	0	$i_m = \text{curl } M_I$	
$M =$	M_I	0	

are obtained from the value M at the center of the cube by Taylor's theorem

$$(M_y)_{x+\Delta x/2} = (M_y)_x + \left(\frac{\partial M_y}{\partial x}\right)_x \left(\frac{\Delta x}{2}\right) + \cdots$$

$$(M_y)_{x-\Delta x/2} = (M_y)_x - \left(\frac{\partial M_y}{\partial x}\right)_x \left(\frac{\Delta x}{2}\right) + \cdots$$

Corresponding expressions may be written for M_z on the two faces. If these values are substituted above and similar expressions are obtained for top and bottom, front and rear of the cube, the following is readily written:

$$\text{curl } M = \lim_{\Delta\tau\to 0} \left\{ \frac{[\hat{n},M]_{x+\Delta x/2}\Delta y\,\Delta z + [\hat{n},M]_{x-\Delta x/2}\Delta y\,\Delta z + [\hat{n},M]_{y+\Delta y/2}\Delta z\,\Delta x}{\Delta x\,\Delta y\,\Delta z} \right.$$
$$\left. + [n,M]_{y-\Delta y/2}\Delta z\,\Delta x + [\hat{n},M]_{z+\Delta z/2}\Delta x\,\Delta y + [n,M]_{z-\Delta z/2}\Delta x\,\Delta y \right\}$$

or

$$\text{curl } M = \lim_{\Delta\tau\to 0} \left\{ \frac{\left(\hat{z}\frac{\partial M_y}{\partial x} - \hat{y}\frac{\partial M_z}{\partial x}\right)\Delta x\,\Delta y\,\Delta z + \left(\hat{x}\frac{\partial M_z}{\partial y} - \hat{z}\frac{\partial M_x}{\partial y}\right)\Delta x\,\Delta y\,\Delta z + \left(\hat{y}\frac{\partial M_x}{\partial z} - \hat{x}\frac{\partial M_y}{\partial z}\right)}{\Delta x\,\Delta y\,\Delta z} \right\}$$

This finally leads to

$$\text{curl } M = \hat{x}\left(\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right) + \hat{y}\left(\frac{\partial M_x}{\partial z} - \frac{\partial M_z}{\partial x}\right) + \hat{z}\left(\frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}\right)$$

This may be written in the following convenient determinant form:

$$\text{curl } M = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M_x & M_y & M_z \end{vmatrix} \tag{19}$$

The evaluation of the curl in curvilinear coordinates may be carried out in the same way, but care must be exercised with elements of volume that do not have parallel sides.

The *interior* of a region containing only spin-model atoms (describable by means of elementary magnets or whirls) may be characterized either in terms of M alone using subdivision I, or in terms of i alone using subdivision II. In the former case it is said to be magnetized.

If a region contains both free-drift- and spin-model atoms, a representation of the interior using subdivision I (which does not cut through any of the elementary whirls) shows that each volume element is characterized by a volume density of magnetization M due to current whirls and by a volume density of current i_f due to the moving free charges. A volume density of magnetization current i_m due to the elementary magnets vanishes in subdivision I, since none of the whirls is cut by the cell boundaries. Hence, $i = i_f$. When the same region is subdivided according to mode II, the interior is characterized by $M = 0$ and by $i = i_f + i_m$. Since i_f is slowly varying, it is also independent of small shifts in the location of cell boundaries, so that the same function is defined using either mode of subdivision. It has been shown that i_m in subdivision II is the same in magnitude and direction as curl M in subdivision I. Consequently, $i +$ curl M has the same value in either mode of subdivision. If any intermediate subdivision is chosen in which the cell boundaries cut through *some* current whirls, but not through enough to reduce the magnetization to zero, or if, in terms of the dynamic model, there is a statistical distribution along the boundaries so that there are always some whirls cut, but not necessarily enough to make M vanish, both M and i_m have nonvanishing values. But curl M has a value precisely equal to the addition that would be made to i if a subdivision were chosen to reduce M to zero. Thus $i +$ curl M is a vector point function that is independent of the mode of subdivision. It is the essential volume density of the steady state, analogous to the essential volume density of the static state $\rho -$ div P. Let it be denoted by

$$\mathbf{I} \equiv \mathbf{i} + \text{curl } M \equiv \mathbf{i} - \text{curl } (-M) \tag{20}$$

Here the expression on the extreme right is written so that the sign agrees with that in $\rho -$ div P. It appears that $-M$ and not M is analogous to P.

19. Alternative Definition of the Curl of a Vector.—An alternative and often convenient definition of the curl of a vector is expressed in terms of the component of the curl in a definite direction or, in particular, in a direction normal to a given surface. Consider, for example, the component of curl M normal to the surface ΔS which cuts across a volume element $\Delta\tau$ dividing it into two parts. For simplicity, let the volume element $\Delta\tau$ be a small cube of side w (a volume element of any shape may be used), and let ΔS be a square drawn parallel to the front face, as
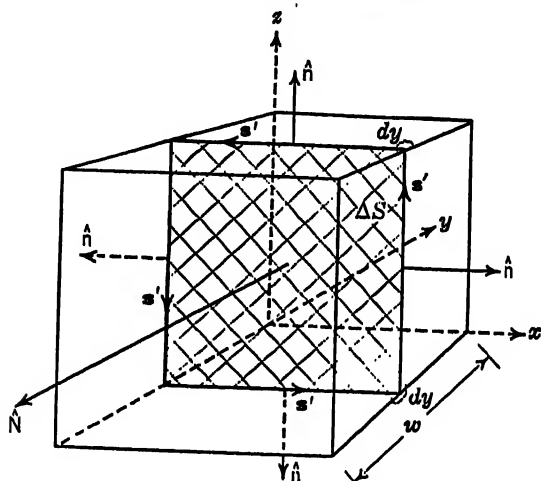
FIG. 19.1.—Cubical element to illustrate the alternative definition of the curl.

in Fig. 19.1. It is bounded by lines up and down on the two sides and across the top and the bottom faces. Let $\hat{N}$ be a unit normal to ΔS pointing out toward the front face from ΔS. The component of curl M [as defined by (18.17)] normal to ΔS is

$$(\hat{N}, \text{curl } M) = \lim_{\Delta\tau\to 0} \frac{\int (\hat{N},[\hat{n},M])d\sigma}{\Delta\tau} \tag{1}$$

The double scalar and vector product may be rearranged as follows:

$$(\hat{N},[\hat{n},M]) = (M,[\hat{N},\hat{n}])* \tag{2}$$

* This double-product relation is quickly proved if it is recalled that [$\hat{n}$,M] is a vector normal to the plane containing $\hat{n}$ and M and equal in magnitude to the area of the parallelogram formed on the vectors $\hat{n}$ and M. Hence ($\hat{N}$,[$\hat{n}$,M]) is equal to the volume of the parallelepiped formed on the

$\hat{N}$ always points in the same direction; $\hat{n}$ is an external normal to the surface of the cube, and hence points in different directions as $d\sigma$ moves from one face of the cube to another, in the course of the integration.   The vector product

$$\hat{s}' = [\hat{N}, \hat{n}] \tag{3}$$

defines a unit vector $\hat{s}'$ which is normal to both $\hat{N}$ and $\hat{n}$.   Since $\hat{N}$ is fixed, $\hat{s}'$ lies in the plane ΔS.   In order to remain perpendicular to $\hat{n}$ and satisfy the right-hand-screw relation of the vector product, $\hat{s}'$ points along the bounding lines of ΔS in a counterclockwise sense as indicated in the figure.   The substitution of (3) in (1) gives

$$(\hat{N}, \text{curl } M) = \lim_{\Delta r \to 0} \frac{\int_{z} (M, \hat{s}') d\sigma}{\Delta r} \tag{4}$$

The integral in (4) includes no contributions from the front and back faces on which $\hat{s}'$ vanishes because $\hat{N}$ and $\hat{n}$ are in the same or in opposite directions.   Integration over the side, top, and bottom faces is equivalent to a sum of integrals

$$\oint_{s} (M, ds') \tag{5}$$

around the edges of each of an infinite family of parallel surfaces ΔS parallel to the xz-plane.   Let $d\sigma = ds' dy$, then

$$(\hat{N}, \text{curl } M) = \lim_{\Delta r \to 0} \frac{\int_{z} (M, ds') dy}{\Delta r} \tag{6}$$

Since M is continuous, the theorem of the mean for integrals may be applied to the y integration.   That is,

$$(\hat{N}, \text{curl } M) = \lim_{\Delta r \to 0} \frac{\overline{\oint_{s} (M, ds')} \int dy}{\Delta r} \tag{7}$$

Here the barred integral is a mean value for some contour s around one of the family of parallel surfaces ΔS.   The integral

---

three vectors $\hat{N}$, $\hat{n}$, M.   The products $(M, [\hat{n}, \hat{N}])$ and $(\hat{n}, [M, \hat{N}])$ are easily interpreted to be equal to the same volume.   Hence

$$(\hat{N}, [\hat{n}, M]) = (M, [\hat{N}, \hat{n}]) = (\hat{n}, [M, \hat{N}])$$

$\int dy$ gives simply the thickness w of the cube; also $\Delta \tau = w \, \Delta S$. Hence (7) reduces to

$$(\hat{\mathsf{N}}, \text{curl } \mathsf{M}) = \lim_{\Delta S \to 0} \frac{\oint_s (\mathsf{M}, d\mathsf{s}')}{\Delta S} \tag{8}$$

The bar indicating a mean value is omitted, since in the limit $\Delta \tau$ approaches zero on the surface ΔS around which the integral $\mathscr{S}(\mathsf{M}, d\mathsf{s}')$ has a mean value. The contour s traces the boundary of that surface. The positive direction of s' around the contour obeys the right-hand-screw convention with respect to the normal $\hat{\mathsf{N}}$. The definition (8) of the component of the curl of a vector normal to a plane is often simpler to use and to visualize as an operation than (18.17). For example, in rectangular coordinates

$$\text{curl } \mathsf{M} = \mathit{x} \lim_{\Delta y \, \Delta z \to 0} \left(\frac{\mathscr{S}(\mathsf{M}, d\mathsf{s})}{\Delta y \, \Delta z} \right) + \mathit{y} \lim_{\Delta z \, \Delta x \to 0} \left(\frac{\mathscr{S}(\mathsf{M}, d\mathsf{s})}{\Delta z \, \Delta x} \right)$$
$$+ \mathit{z} \lim_{\Delta y \, \Delta x \to 0} \left(\frac{\mathscr{S}(\mathsf{M}, d\mathsf{s})}{\Delta y \, \Delta x} \right) \tag{9}$$

20. Essential Surface Characteristic of the Steady State.—The complete characterization of the steady state requires a surface function that is independent of the mode of subdivision of a region into volume and surface cells. In the static state such a function was found to be $\eta + (\hat{\mathsf{n}}, \mathsf{P})$. In the steady state the analogous function must be a combination of i and M, so that the present situation with respect to M is analogous to that discussed for P in defining the surface characteristic of the static state. It has been shown (18.1b) that the vector product $[\hat{\mathsf{n}}, -\mathsf{M}]$ is a measure of the direction and magnitude of positive charges moving parallel to the surface to which $\hat{\mathsf{n}}$ is an external normal. At the boundary between two dissimilar regions $[\hat{\mathsf{n}}, -\mathsf{M}]$ describes in subdivision I what i represents in subdivision II as can be verified directly by examining what happens to a thin layer containing parts of current whirls along both sides of a bounding surface between two regions when a change is made in the method of subdivision in each region from mode I to mode II. Consider first both regions divided according to mode I. Each contains only volume cells, and both i and M may have nonvanishing values if both whirling and freely drifting charges are present. The value of i is due entirely to the drift of free

charges, the value of M entirely to the circulation in spin atoms. Suppose, for the moment, that no free charges are present, so that $i = 0$; let M_1 be defined in region 1 on one side of the boundary, while M_2 is defined in region 2 on the other side. At the boundary the quantities $[\hat{n}_1, -M_1]$ and $[\hat{n}_2, -M_2]$ characterize the positive circulation on each side parallel to the boundary with normals outwardly directed with respect to the regions. Consider two cubical volume cells $\Delta\tau$ one on each side of the boundary surface with adjacent faces $\Delta\Sigma_1$ and $\Delta\Sigma_2$. Let the mode of subdivision of the two regions be changed by pulling apart the two volume cells $\Delta\tau_1$ and $\Delta\tau_2$ as in Fig. 9.1 to form two thin surface cells, of equal volumes $\delta\Delta\Sigma_1$ and $\delta\Delta\Sigma_2$, one on each side of the boundary between the dissimilar regions. Each of the newly formed thin surface cells contains parts of current whirls still partly in $\Delta\tau_1$ and $\Delta\tau_2$. The number of current whirls that are cut and partly *added* to a given cell in the form of a positive current in changing the mode of subdivision is obtained by integrating the vector $[\hat{n}, M]$ over the enclosing surfaces as explained in conjunction with Fig. 18.2. If this integral is evaluated over the surface of the thin disk of thickness 2δ, significant contributions are obtained only from the large flat surfaces $\Delta\Sigma_1$ and $\Delta\Sigma_2$ because only these surfaces cut through the current whirls when the change in subdivision is made. Thus

$$\int_{\Sigma} [\hat{n}, M] d\sigma = \int_{\Delta\Sigma_1} [\hat{n}_1', M_1] d\sigma + \int_{\Delta\Sigma_2} [\hat{n}_2', M_2] d\sigma \tag{1}$$

where $\hat{n}_1'$ points *into* region 1, $\hat{n}_2'$ *into* region 2, since both must be *external* to the volume contained in the disk, $2\delta\Delta\Sigma$. Also $\Delta\Sigma_1 = \Delta\Sigma_2$.

The average volume density of moving charge due to contributions from cut magnetized units (current whirls) in these cells is

$$(i_m)_{av} = \tfrac{1}{2}(i_{1m} + i_{2m}) \tag{2}$$

If (1) is divided by the volume $2\delta \Delta\Sigma$ and $\Delta\Sigma$ is allowed to approach zero in the limit (since M is a continuous function this is permissible) the following continuous volume density of moving charge is defined:

$$\frac{1}{2}(i_{1m} + i_{2m}) = -\lim_{2\delta \Delta\Sigma \to 0} \left\{ \frac{\int_{\Delta\Sigma_1} [\hat{n}_1, M_1] d\sigma}{2\delta \Delta\Sigma_1} + \frac{\int_{\Delta\Sigma_2} [\hat{n}_2, M_2] d\sigma}{2\delta \Delta\Sigma_2} \right\} \tag{3}$$

Here n_1 ($= -\mathsf{n}_1'$) and n_2 ($= -\mathsf{n}_2'$) are normals directed outwardly with respect to the boundary surface between the regions 1 and 2. From (14.2), $i_1\delta$ is by definition the surface density of moving charge l_1. Similarly, l_2 is defined along the surface of region 2 by $i_2\delta$. Thus

$$l_{1m} + l_{2m} = i_{1m}\delta + i_{2m}\delta \tag{4}$$

If (3) is multiplied by 2δ and combined with (4),

$$l_{1m} + l_{2m} = - \lim_{\Delta\Sigma \to 0} \left\{ \frac{\int_{\Delta\Sigma_1} [\mathsf{n}_1, \mathsf{M}_1] d\sigma}{\Delta\Sigma_1} + \frac{\int_{\Delta\Sigma_2} [\mathsf{n}_2, \mathsf{M}_2] d\sigma}{\Delta\Sigma_2} \right\} \tag{5}$$

By applying the theorem of the mean for integrals to each integral in (5)

$$l_{1m} + l_{2m} = - \lim_{\Delta\Sigma \to 0} \left\{ \overline{[\mathsf{n}_1, \mathsf{M}_1]} \frac{\int_{\Delta\Sigma_1} d\sigma}{\Delta\Sigma_1} + \overline{[\mathsf{n}_2, \mathsf{M}_2]} \frac{\int_{\Delta\Sigma_2} d\sigma}{\Delta\Sigma_2} \right\} \tag{6}$$

Here $\overline{[\mathsf{n}_1, \mathsf{M}_1]}$ and $\overline{[\mathsf{n}_2, \mathsf{M}_2]}$ are, respectively, mean values at points on the surfaces $\Delta\Sigma_1$ and $\Delta\Sigma_2$. In passing to the limit $\Delta\Sigma_1$ and $\Delta\Sigma_2$ approach zero at the points where the mean values are defined so that

$$l_{1m} + l_{2m} = -[\mathsf{n}_1, \mathsf{M}_1] - [\mathsf{n}_2, \mathsf{M}_2] \tag{7}$$

Let the following shorthand notation be introduced:

$$l_m = l_{1m} + l_{2m} \tag{8a}$$
$$\text{surf. curl } \mathsf{M} = [\mathsf{n}, \mathsf{M}] = [\mathsf{n}_1, \mathsf{M}_1] + [\mathsf{n}_2, \mathsf{M}_2] \tag{8b}$$

With this symbolism and with the mode of subdivision used in defining the densities explicitly indicated by the subscripts I and II, (7) becomes

$$l_{m\text{II}} = - [\mathsf{n}, \mathsf{M}_\text{I}] = -\text{surf. curl } \mathsf{M}_\text{I} \tag{9}$$

Just as for $\eta_{m\text{II}}$ in the analogous case in the static state (9.10), a value of $l_{m\text{II}}$ may be associated with every point along the surface separating the two regions. It is a measure of the total surface density of moving charge in both regions associated with that point using subdivision II.

It is easy to see that the function $l - [\mathsf{n}, \mathsf{M}]$ is independent of the mode of subdivision at least for a region containing no free charges by applying the same reasoning used in Sec. 9 in

the analogous static-state case.   This function is assigned the symbol | and is named the essential surface density of the steady state.   It is

$$\bar{I} \equiv I - [n,M] \equiv I + [n,-M] \tag{10}$$

The expression on the extreme right is written in a form to bring out the analogy with the static state function $\bar{\eta} = \eta + (n,P)$. It again appears that $-M$ rather than M is the analogue of P.

Because $[n,M]$ can describe surface effects only if these are surface manifestations of a volume phenomenon, it cannot be used to characterize a sheet of drifting free charge.   On the other hand, a description in terms of i_I and M_I can always be made mathematically equivalent to a description in terms of i_{II} and I_{II} if the following conditions are fulfilled.

$$i_{II} = i_I + \text{curl } M_I$$
$$I_{II} = -[n,M_I] \tag{11}$$

Since every free-drift model may be replaced by a mathematically equivalent spin model, the function

$$\bar{I} = I - [n,M] \tag{12}$$

may be used to describe mathematically the surface conditions of both spin- and free-charge models.   By building this function and the volume function i into the mathematical model of electromagnetism, it is possible to interpret them at will or for physical reasons in terms of a spin model, a free-drift model, or a combination of these.

21. Surface Density of Magnetization.—In order to complete the mathematical description of the steady state, an additional surface function to represent a surface distribution of current whirls might be defined.   A surface magnetization is the steady-state analogue of a surface polarization or double layer in the static state.   Mathematically, it is a simple matter to construct a steady-state function analogous to k in the static state. Whereas the function k plays only a very minor part in the coordination of the mathematical theory with experimental observations, no experimental effects have been observed that require a theoretical analogue that depends upon a surface magnetization.   Furthermore, it is difficult to conceive of

conditions in terms of the atomic model that are sufficiently complex to produce surface magnetization. Accordingly, no surface density of magnetization is defined.

22. Formal Analogy between the Steady State and the Static State.—The physical and mathematical models of the steady state exhibit a complete parallelism in form, in development, and in interpretation with corresponding models in the static state. Atomic models describe two kinds of motion of the charges: a steady drift of the free charges which are statistically at rest in the static state, and a circulation-orientation of charge. The latter is only formally a steady-state analogue of the distortion-orientation of charge in the bound-charge static model. Actually, the two effects are parallel only in that both are concerned with an orientation. The distortion in the static state and the circulation in the steady state are quite unrelated, as are the external forces that produce them. In its mathematical model of continuous functions, the steady state is closely analogous to the static state. Both are characterized in terms of two continuous functions defined, one throughout the interior, the other on the surface of a region containing charges. In addition, the scalar product appears in the static state where the vector product occurs in the steady state; the divergence of a vector appears in the static state where the curl of the vector appears in the steady state. This formal analogy between the two states is maintained throughout the mathematical development.

For convenient reference, the table on page 72 is constructed to summarize the definitions of important quantities in the two states and to demonstrate the analogy between them.

THE NONSTATIONARY STATE

23. Essential Volume Characteristic of the Unsteady State; Equation of Continuity for Electric Charge.—The static and steady states are both stationary states because the density functions that characterize conditions of statistical rest and of steady motion of electric charge at every point in a region are *independent of the time*. In the nonstationary state the same density functions may be used to describe *instantaneous* distributions of charge and current, but they vary in time at every point. Moreover, as a consequence of the postulate of conservation of

Quantity	Static state	Steady state
Condition of charge	Dynamic: random motion	Dynamic: steady drift or circulation superimposed on random motion
	Static: statistically at rest	Static: steady drift or circulation relative to statistical rest positions
Atomic models	Free charge Bound charge	Free drift Spin
Elementary models	Dipole; $p = ed$ p defines electric moment or magnitude of polarization; its direction is axis of polarization	Magnet; $m = \frac{1}{2}[d, ev]$ m defines magnetic moment or magnitude of magnetization; its direction is axis of magnetization
Density functions: Volume density of... Surface density of.... Volume density of...	Charge ρ; $(\rho_i = \Sigma e_i / \Delta \tau_i)$ Charge η; $(\eta_s = \Sigma \delta e_j / \Delta \tau_s)$ Polarization P; $(P_i = \Sigma e_j d_j / \Delta \tau_i$ $\qquad = \Sigma p_j / \Delta \tau_i)$	Drifting charge, i; $(i_k = \Sigma e_j v_j / \Delta \tau_k)$ Drifting charge, I; $(I_s = \Sigma \delta e_j v_j / \Delta \tau_s)$ Magnetization, $-M$; $(M_i = \Sigma \frac{1}{2}[d_j, e_j v_j] / \Delta \tau_i$ $\qquad = \Sigma m_j / \Delta \tau_i)$
Essential volume characteristic of......... Essential surface characteristic of.........	Charge $\bar{\rho} = \rho - \text{div } P$ Charge $\bar{\eta} = \eta + (\mathfrak{n}, P)$	Moving charge, $\bar{I} = i - \text{curl}(-M)$ Moving charge, $\bar{I} = I + [\mathfrak{n}, -M]$
Operations............	Scalar or dot product, () Divergence, div	Vector or cross product, [ ] curl

electric charge, definite relationships exist between the time rates of change of the densities of charge $\bar{\rho}$, $\bar{\eta}$ and the densities of moving charge $\bar{I}$, $\bar{I}$.

In order to determine the essential characteristics of the nonstationary state from the corresponding stationary quantities, consider a region containing electric charges for which all the

four volume densities ρ, P, i, and M are defined in the usual way. The electrical properties of the region are described in terms of the essential densities

$$\bar{\rho} = \rho - \operatorname{div} \mathsf{P} \tag{1a}$$
$$\bar{\imath} = i + \operatorname{curl} \mathsf{M} \tag{1b}$$

which are independent of the mode of subdivision into volume cells. In the static and steady states they are by definition constant in time.

Let an unsteady state exist in which the charges in the region move in a nonrandom way so that ρ and P are functions of the time. M and i may be constant or variable in time. Since $\partial\rho/\partial t$ and $\partial\mathsf{P}/\partial t$ are nonvanishing, the time rate of change of $\bar{\rho}$ in (1a) is

$$\frac{\partial \bar{\rho}}{\partial t} = \frac{\partial \rho}{\partial t} - \frac{\partial}{\partial t} (\operatorname{div} \mathsf{P}) \tag{2a}$$

Because time and space variables are independent, differentiations with respect to them may be interchanged.

$$\frac{\partial \bar{\rho}}{\partial t} = \frac{\partial \rho}{\partial t} - \operatorname{div} \frac{\partial \mathsf{P}}{\partial t} \tag{2b}$$

In terms of the atomic picture a variation in time of ρ means that the total charge contained in a volume cell $\Delta\tau$ increases or decreases or changes periodically as time passes. Since it has been postulated as a fundamental attribute of electricity that electric charge is conserved, so that no charges can be created or destroyed in $\Delta\tau$, it follows that electric charges must flow inward or outward across the closed bounding surface of $\Delta\tau$ if $\partial\rho/\partial t$ differs from zero. Such a flow may consist of free charges moving from volume cell to volume cell, or of parts of periodically distorted closed shells vibrating across the bounding surfaces. No contributions to such a flow come from oriented spin groups because the total charge at every instant in a given volume is unchanged by a variation in time of the number of current whirls or of their axes of rotation. With ρ defined to measure the average total charge per unit volume, it is clear that $\partial\rho/\partial t$ must measure the time rate of change of the average total charge in each unit of volume. This can be related to the volume density of current i in the following way.

Let Σ be the enclosing surface of a typical volume cell $\Delta\tau$. The postulate of conservation requires the rate of increase of the positive charge $\rho\Delta\tau$ in $\Delta\tau$ to equal the net positive charge entering $\Delta\tau$ across Σ per unit time. (Negative charge leaving $\Delta\tau$ is mathematically equivalent to positive charge entering it.) At any instant the flow of positive charge across Σ into $\Delta\tau$ is given by the integral over Σ of the inward normal component of the volume density of moving charge $\mathbf{i}$. This integral is the negative of the total outward normal flux of the vector $\mathbf{i}$ across a closed surface. Thus

$$\frac{\partial}{\partial t}(\rho\,\Delta\tau) = -\int_\Sigma (\mathbf{\hat{n}},\mathbf{i})d\sigma \tag{3}$$

Here $\mathbf{\hat{n}}$ is an external normal to the surface Σ. Since $\Delta\tau$ is constant in time

$$\frac{\partial}{\partial t}(\rho\,\Delta\tau) = \Delta\tau\,\frac{\partial\rho}{\partial t} \tag{4}$$

Since both ρ and $\mathbf{i}$ are continuous functions defined throughout the region, the simple limit process of the calculus may be used to obtain

$$\frac{\partial\rho}{\partial t} = -\lim_{\Delta\tau\to 0}\frac{\int_\Sigma (\mathbf{\hat{n}},\mathbf{i})d\sigma}{\Delta\tau} \tag{5}$$

By definition of the divergence of a vector, (8.8),

$$\text{div }\mathbf{i} = \lim_{\Delta\tau\to 0}\frac{\int_\Sigma (\mathbf{\hat{n}},\mathbf{i})d\sigma}{\Delta\tau} \tag{6}$$

Accordingly, (5) is equivalent to the following differential equation:

$$\frac{\partial\rho}{\partial t} + \text{div }\mathbf{i} = 0 \tag{7}$$

This equation is called the *equation of continuity for electric charge*. It expresses the postulated principle of conservation of electricity in terms of the density functions of the mathematical model. It explicitly establishes a connection between the instantaneous steady-state volume density of drifting charge $\mathbf{i}$ and the instantaneous time rate of change of the volume density of charge ρ.

If the value of $\partial\rho/\partial t$ given by the equation of continuity (7) is inserted in (2b), the time rate of change of the essential density of charge becomes

$$\frac{\partial\bar{\rho}}{\partial t} = -\operatorname{div} \mathbf{i} - \operatorname{div} \frac{\partial\mathbf{P}}{\partial t} = -\operatorname{div}\left(\mathbf{i} + \frac{\partial\mathbf{P}}{\partial t}\right) \qquad (8)$$

or

$$\frac{\partial\bar{\rho}}{\partial t} + \operatorname{div}\left(\mathbf{i} + \frac{\partial\mathbf{P}}{\partial t}\right) = 0 \qquad (9)$$

This is a formally generalized equation of continuity which has, in effect, been obtained from (7) simply by adding and subtracting the term div $(\partial\mathbf{P}/\partial t)$. Its usefulness lies in the fact that the characteristic of charge $\bar{\rho}$ is independent of the mode of subdivision into volume cells, whereas ρ is not. It follows at once that $(\mathbf{i} + \partial\mathbf{P}/\partial t)$ must characterize the moving charge in a way that is also independent of the mode of subdivision. This is true provided there are no spin atoms, so that $\mathbf{M} = 0$. It has already been shown for the steady state that if $\mathbf{M}$ differs from zero so that circulations of charge exist, $\mathbf{i} + \operatorname{curl}\mathbf{M}$, and not $\mathbf{i}$ alone, is the characteristic of moving charge that is independent of the mode of subdivision into volume cells. In the general unsteady state in which $\mathbf{M}$ does not vanish, the essential characteristic of moving charge that is invariant of the mode of subdivision may be obtained by noting that

$$\operatorname{div} \operatorname{curl} \mathbf{M} = 0^* \qquad (10)$$

so that curl $\mathbf{M}$ may be added to $(\mathbf{i} + \partial\mathbf{P}/\partial t)$ in (9) without disturbing the balance of the equation.

$$\frac{\partial\bar{\rho}}{\partial t} + \operatorname{div}\left(\mathbf{i} + \operatorname{curl}\mathbf{M} + \frac{\partial\mathbf{P}}{\partial t}\right) = 0 \qquad (11)$$

* This is a general vector identity. It is easily proved in rectangular coordinates.

$$\operatorname{div} \operatorname{curl} \mathbf{M} = \operatorname{div}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M_x & M_y & M_z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M_y & M_z \end{vmatrix} + \frac{\partial}{\partial y}\begin{vmatrix} \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} \\ M_z & M_x \end{vmatrix} + \frac{\partial}{\partial z}\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ M_x & M_y \end{vmatrix} = 0$$

The *essential density of moving charge* is

$$\overline{\rho_m v} = i + \operatorname{curl} M + \frac{\partial P}{\partial t} \qquad (12)$$

With (12), (11) becomes

$$\frac{\partial \bar{\rho}}{\partial t} + \operatorname{div} \overline{\rho_m v} = 0 \qquad (13)$$

This is a symbolically convenient, generalized equation of continuity that differs only formally from (7), from which it may be obtained by adding and subtracting a term in P and adding a term in M which is identically zero. It is useful for obtaining the essential density of moving charge $\overline{\rho_m v}$ in the nonstationary state. This is at every instant a continuous, slowly varying function of the space coordinates defined throughout the interior of a body or region. All its terms may be functions of the time. The velocity v is a continuous, slowly varying vector point function characterizing the average statistical velocity of moving charge including nonrandom motions of free charges and of closed shell configurations of either the polarized or magnetized kind. $\bar{\rho}_m$ is the volume density associated with v; it includes all charges engaged in nonrandom motion.

The physical significance in terms of the atomic model of associating the term $\partial P/\partial t$ with the convection or conduction current density i and the magnetization current density $i_m = \operatorname{curl} M$ is readily visualized. Superficially, it is plausible that a more or less periodic reversal in the direction of polarization of a distorted and oriented group of bound charges, such as the closed shell of an atom or molecule, is equivalent to an alternating current. The actual explanation for the appearance of the term $\partial P/\partial t$ in the essential characteristic of moving charge $\overline{\rho_m v}$ is precisely the same as the explanation for the appearance of $-\operatorname{div} P$ in the essential characteristic of charge $\bar{\rho}$. The contributions to $\partial \bar{\rho}/\partial t$ by the two terms $-\operatorname{div} i$ and $-\operatorname{div}(\partial P/\partial t)$, just as the contributions to $\bar{\rho}$ by ρ and $-\operatorname{div} P$, depend upon the mode of subdivision of a body into volume cells. If the boundaries of the cells are so placed that they cut no closed groups even when these experience periodically reversing distortion-orientation effects along a definite axis (subdivision I), then $-\operatorname{div} P$ is the contribution to $\bar{\rho}$ of a nonuniform polarization, and $-\operatorname{div}(\partial P/\partial t)$ is the contribution to $\partial \bar{\rho}/\partial t$ of a time rate of change

of this polarization. This is equivalent to stating that $\partial P/\partial t$ accounts for the contribution to the characteristic of moving charge of periodically reversing distortion-orientations of bound groups. In this mode of subdivision, ρ and i measure, respectively, only the densities of free charge and of moving free charge, since all closed shells remain within the volume cells with which they are associated.

If the mode of subdivision is changed to mode II so that just enough closed shells are cut by the cell boundaries to make P everywhere zero, then div P and $\partial P/\partial t$ vanish. ρ and i, calculated in the usual way, differ from the values calculated in the first mode of subdivision precisely by the amounts, respectively, $\rho_{bII} = -\text{div } P_1$ and $i_{pII} = \partial P_1/\partial t$. In this case, ρ is a measure of both the average density of free charge ρ_f and of the average density ρ_b of the parts of the closed shells of atoms that are cut by and included within the boundaries of the cell. Correspondingly, $i = i_f + i_p$ measures in i_f convection or conduction drifts of charge across the boundaries of cells and in i_p the average motion of parts of vibrating closed shells across the same boundaries. The change in the mode of subdivision in the physical picture corresponds to a mere shift in the responsibility of representation in the mathematical model from one set of functions to another. The quantity $i_p = \partial P/\partial t$ is a volume density of polarization current.

For all modes of subdivision satisfying the restrictions on the size of individual cells, the two continuous functions

$$\bar{\rho} = \rho - \text{div } P \qquad (14a)$$

$$\overline{\rho_m v} = i + \text{curl } M + \frac{\partial P}{\partial t} \qquad (14b)$$

completely characterize the instantaneous conditions of charge and moving charge in the interior of a body or region to the degree of approximation here attempted.

24. Essential Surface Characteristic of the Unsteady State; Surface Equation of Continuity for Electric Charge.— The equation of continuity, expressing the fundamental postulate of conservation of electric charge, is equally true along a surface, or more generally along the boundary between two electrically dissimilar regions as in the interior. Its form is obtained by applying

(23.7) to a thin surface layer on each side of the boundary between regions 1 and 2.

Consider a small rectangular surface element $\Delta \tau_s$ of thickness δ on each side of the boundary and of area of base and top $\Delta \Sigma$. The combined volume of the double element is $\Delta \tau_s = 2\delta\,\Delta\Sigma$. Using the fundamental definition (8.8) of the divergence of a vector, (23.7) may be written in the form

$$\frac{\partial \rho}{\partial t} = \frac{-\lim\limits_{2\delta\,\Delta\Sigma \to 0} \int_{\Sigma} (\hbar, i)d\sigma}{2\delta\,\Delta\Sigma} \tag{1}$$

where $\hbar$ is an external normal to the enclosing surface Σ of the element. Let Σ be considered in three parts. Σ_δ is the combined area in both regions 1 and 2 of the four narrow edges of the entire element of thickness 2δ; $\Delta\Sigma_1$ is the area of the top in region 1; $\Delta\Sigma_2$ is the area of the base in region 2. The surface integral in (1) can be written as the sum of three integrals over these three parts of the closed surface of the cell.

$$\frac{\partial \rho}{\partial t} = -\lim_{2\delta\,\Delta\Sigma \to 0} \left\{ \frac{\int_{\Sigma_\delta} (\hbar, i)d\sigma}{2\delta\,\Delta\Sigma} + \frac{\int_{\Delta\Sigma_1} (\hbar_1', i_1)d\sigma}{2\delta\,\Delta\Sigma_1} + \frac{\int_{\Delta\Sigma_2} (\hbar_2', i_2)d\sigma}{2\delta\,\Delta\Sigma_2} \right\} \tag{2}$$

Here $\hbar_1'$ points out of $\Delta\tau_s$ from $\Delta\Sigma_1$ into region 1; $\hbar_2'$ points across $\Delta\Sigma_2$ into region 2.

In (2), ρ is the average volume density of charge in $2\delta\,\Delta\Sigma$. In terms of the volume densities in each region,

$$\rho = \tfrac{1}{2}(\rho_1 + \rho_2) \tag{3}$$

Similarly i, in the integral over the narrow surface Σ_δ, is the average value in the two regions.

$$i = \tfrac{1}{2}(i_1 + i_2) \tag{4}$$

Using (3) and (4) in (2) and multiplying through by 2δ gives

$$\frac{\partial}{\partial t}(\delta\rho_1 + \delta\rho_2) = -\lim_{2\delta\,\Delta\Sigma \to 0} \left\{ \frac{\int_{\Sigma_\delta} (\hbar, \delta i_1 + \delta i_2)d\sigma}{2\delta\,\Delta\Sigma} \right.$$
$$\left. + \frac{\int_{\Delta\Sigma_1} (\hbar_1', i_1)d\sigma}{\Delta\Sigma_1} + \frac{\int_{\Delta\Sigma_2} (\hbar_2', i_2)d\sigma}{\Delta\Sigma_2} \right\} \tag{5}$$

The surface density of charge (4.2) is

$$\eta = \delta\rho \tag{6}$$

The surface density of current (14.2) is

$$\mathsf{I} = \delta\mathsf{i} \tag{7}$$

From (6)

$$\delta\rho_1 + \delta\rho_2 = \eta_1 + \eta_2 \equiv \eta \tag{8}$$

Here η is the total surface density of charge associated with the boundary surface between the two regions. Using (7)

$$(\mathsf{n},\delta\mathsf{i}_1 + \delta\mathsf{i}_2) = (\mathsf{n},\mathsf{I}_1 + \mathsf{I}_2) \equiv (\mathsf{n},\mathsf{I}) \tag{9}$$

Here I is the total surface density of current associated with the surface between the two regions. Since the normal in the first integral is directed across the narrow edges of the rectangular parallelepiped $\Delta\tau_s$, and hence is tangent to the bounding surface, it is clear that the current densities in (9) are parallel to the surface. With (8) and (9), (5) becomes

$$\frac{\partial\eta}{\partial t} = -\lim_{2\delta\,\Delta\Sigma\to 0}\left\{\frac{\int_{\Sigma\delta}(\mathsf{n},\mathsf{I})d\sigma}{2\delta\,\Delta\Sigma} + \frac{\int_{\Delta\Sigma_1}(\mathsf{n}_1',\mathsf{i}_1)d\sigma}{\Delta\Sigma_1} + \frac{\int_{\Delta\Sigma_2}(\mathsf{n}_2',\mathsf{i}_2)d\sigma}{\Delta\Sigma_2}\right\} \tag{10}$$

Here the first integral is the divergence of the two-dimensional vector I. The last two integrals may be transformed using the theorem of the mean.

$$\frac{\partial\eta}{\partial t} + \operatorname{div}\mathsf{I} = -\lim_{\Delta\Sigma\to 0}\left\{\overline{(\mathsf{n}_1',\mathsf{i}_1)}\frac{\int_{\Delta\Sigma_1}d\sigma}{\Delta\Sigma_1} + \overline{(\mathsf{n}_2',\mathsf{i}_2)}\frac{\int_{\Delta\Sigma_2}d\sigma}{\Delta\Sigma_2}\right\} \tag{11}$$

Upon performing the integration, passing to the limit, and introducing the exterior normals to the regions 1 and 2

$$\mathsf{n}_1 = -\mathsf{n}_1', \qquad \mathsf{n}_2 = -\mathsf{n}_2' \tag{12}$$

the following equation, which contains both surface and volume functions, is obtained:

$$\frac{\partial\eta}{\partial t} + \operatorname{div}\mathsf{I} - (\mathsf{n},\mathsf{i}) = 0 \tag{13}$$

The customary shorthand notation is used in (13).

$$(\mathsf{n},\mathsf{i}) = (\mathsf{n}_1,\mathsf{i}_1) + (\mathsf{n}_2,\mathsf{i}_2) \tag{14}$$

For a surface effect involving a drift of charge along the surface only, (13) becomes

$$\frac{\partial\eta}{\partial t} + \operatorname{div}\mathsf{I} = 0 \tag{15}$$

For a volume effect in which there is no motion of charge tangent to the boundary between two surfaces, (13) leads to the important boundary condition for volume density of current crossing a boundary

$$(\mathfrak{h},\mathfrak{i}) \equiv (\mathfrak{h}_1,\mathfrak{i}_1) + (\mathfrak{h}_2,\mathfrak{i}_2) = 0 \tag{16}$$

If the surface and volume effects are independent, (15) and (16) are simultaneously and independently true.

The mixed surface equation (13) corresponds to the volume equation (23.7) in that it is not written in terms of essential densities that are independent of the mode of subdivision of the region into volume and surface cells. In order to generalize (13) to correspond to the volume equation (23.13), and in this way determine the essential surface characteristic of the non-stationary state, the procedure used in deriving the volume characteristic (23.12) might be followed. A shorter, more artificial way will be used. It involves adding and subtracting $(\mathfrak{h},\partial P/\partial t)$, subtracting div $[\mathfrak{h},M]$, and subtracting $(\mathfrak{h}, \text{curl } M)$ in (13). It is readily shown that div $[\mathfrak{h},M]$ is identically zero with the aid of the definition of the divergence and the relation $[\mathfrak{h},\mathfrak{h}] = 0$, so that the balance of the equation is not disturbed by subtracting it. In a thin surface layer $(\mathfrak{h}, \text{curl } M)$ also vanishes, since in such a layer curl M has the value $[\mathfrak{h},M]$ and

$$(\mathfrak{h},[\mathfrak{h},M]) = (M,[\mathfrak{h},\mathfrak{h}]) = 0$$

Since (13) applies only to a surface or boundary layer, it is not changed by adding $-(\mathfrak{h}, \text{curl } M)$. The same conclusion is reached using the atomic picture in which the density of magnetization current $\mathfrak{i}_m = \text{curl } M$ is due to oriented circulations or whirls of charges. At a surface such a whirl is tangent to the boundary, and the normal component $(\mathfrak{h}, \text{curl } M)$ of a tangent vector is zero.

After performing the three steps described and suitably arranging and grouping the terms, (13) becomes

$$\frac{\partial}{\partial t}\{\eta + (\mathfrak{h},P)\} + \text{div }\{\mathfrak{l} - [\mathfrak{h},M]\} + \left((\mathfrak{h},\mathfrak{l}) + \text{curl } M + \frac{\partial P}{\partial t}\right) = 0 \tag{17}$$

or

$$\frac{\partial \bar{\eta}}{\partial t} + \text{div }\overline{\eta_m v} - (\mathfrak{h},\overline{\rho_m v}) = 0 \tag{18}$$

In (17) and (18) the following essential volume and surface densities have been used:

$$\bar{\rho} \equiv \rho - \text{div } \mathsf{P} \tag{19}$$

$$\overline{\rho_m \mathsf{v}} \equiv \mathsf{i} + \text{curl } \mathsf{M} + \frac{\partial \mathsf{P}}{\partial t} \tag{20}$$

$$\bar{\eta} \equiv \eta + (\mathsf{n}, \mathsf{P}) \tag{21}$$

$$\overline{\eta_m \mathsf{v}} \equiv \bar{\mathsf{I}} \equiv \mathsf{I} - [\mathsf{n}, \mathsf{M}] \tag{22}$$

It follows from (22) that the essential surface characteristic of moving charge in the unsteady state is the same in form a the corresponding steady-state function. The new notation $\overline{\eta_m \mathsf{v}}$ is introduced to correspond to the volume function $\overline{\rho_m \mathsf{v}}$. In the nonstationary state all terms in (22) are functions of the time. By simply requiring the invariance of all densities in time, they reduce to the functions previously defined for the stationary states.

25. Equation of Continuity for a Cylindrical Conductor.—An interesting and subsequently useful partial integration of the equation of continuity is readily carried out for a cylindrical region of length $2h$ and radius a. The following inequality is postulated:

$$a << h \tag{1}$$

The cylinder is oriented with its axis along the z axis of a cylindrical system of coordinates r, θ, z, as shown in Fig. 25.1. Complete rotational symmetry is assumed to prevail. The cylinder will be identified later with a current-carrying conductor for which it will be shown that

$$\rho = 0; \qquad \mathsf{P} = 0 \tag{2}$$

At this point it is sufficient to regard (2) as a special condition imposed upon the cylinder. Subject to (2) the cylindrical region is characterized by

$$\bar{\rho} \quad 0; \qquad \overline{\rho_m \mathsf{v}} \ \cdots \ \mathsf{i} + \text{curl } \mathsf{M} = \mathsf{I} \tag{3a}$$

$$\bar{\eta} \quad \eta_{I}; \qquad \overline{\eta_m \mathsf{v}} \ \cdots \ \mathsf{I} - [\mathsf{n}, \mathsf{M}] \tag{3b}$$

Since both volume and surface densities appear in (3a,b) it is necessary to use a representation in terms of volume and surface cells. The equations of continuity subject to (2) for the interior
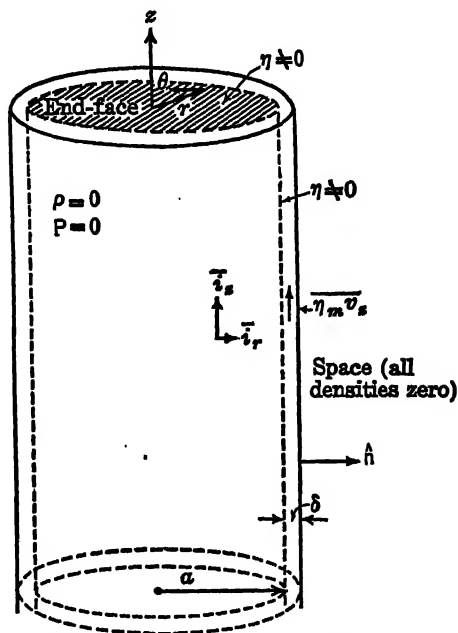
Fig. 25.1.—Section of a cylindrical conductor.

and for the surface are

$$\operatorname{div} \mathsf{I} = 0 \tag{4a}$$

$$\operatorname{div} \overline{\eta_m v} + \frac{\partial \eta_f}{\partial t} - (\mathsf{h}, \mathsf{I}) = 0 \tag{4b}$$

The current into the cylindrical surface layer of atomic thickness δ from the interior is the radial component $\bar{\imath}_r$ at

$$r = a - \delta$$

That is, in (4b)

$$(\mathsf{h}, \mathsf{I}) = (\bar{\imath}_r)_{r=a-\delta} \tag{5}$$

Because rotational symmetry exists, it may be assumed that

$$\overline{\eta_m v_\theta} = 0 \tag{6a}$$

$$\bar{\imath}_\theta = 0 \tag{6b}$$

Using (5) and (6a) in (4b), and noting that the divergence is two-dimensional involving only the variables θ and z,

$$(\bar{\imath}_r)_{r=a-\delta} = \frac{\partial \eta_f}{\partial t} + \frac{\partial}{\partial z} (\overline{\eta_m v_z}) \tag{7}$$

With (6b), (4a) is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{\imath}_r) + \frac{\partial\bar{\imath}_z}{\partial z} = 0 \tag{8}$$

A partial integration of (8) with respect to r gives

$$(r\bar{\imath}_r)_{r=a-\delta} = -\int_0^{a-\delta} r\frac{\partial\bar{\imath}_z}{\partial z}\,dr = -\frac{\partial}{\partial z}\int_0^{a-\delta} r\bar{\imath}_z dr \tag{9}$$

The total axial current is defined by

$$I_z \equiv \int_0^{a-\delta} 2\pi r\bar{\imath}_z dr + 2\pi a\overline{\eta_m v_z} \tag{10a}$$

This always reduces to a *total* current of free charge only,

$$I_z = \int_0^{a-\delta} 2\pi r i_{fz} dr + 2\pi a l_{fz} \tag{10b}$$

but the radial distribution of i_{fz} depends upon M, if this is not zero, as will be shown in Chapter V. That I_z in (10a) and (10b) are the same follows from the fact that the terms containing M in $\mathbf{\iota}$ and $\overline{\eta_m v}$ in (10a) cancel. This follows directly from

$$\int_0^{a-\delta}(\hat{z}, \text{curl } M)2\pi r\,dr = \oint M_\theta a\,d\theta = 2\pi a M_\theta* \tag{11}$$

and

$$-2\pi a(\hat{z},[\hat{r},M]) = -2\pi a M_\theta \tag{12}$$

or from simple arguments in terms of a physical model of current whirls. In (11), a is written on the right for $a - \delta$ because the extremely small distance δ is negligible compared with a.

If (9) is multiplied by 2π and (7) by $2\pi a$, the right sides of (7) and (9) may be equated after again writing a for $a - \delta$. Let the charge per unit length be

$$q = 2\pi a\eta_f \tag{13}$$

then

$$\frac{\partial I_z}{\partial z} + \frac{\partial q}{\partial t} = 0 \tag{14}$$

This is the equation of continuity for the cylindrical conductor. It could have been written down more quickly and without imposing (2) if the postulate of conservation of electric charge

*This depends on Stokes's theorem given in Sec. II.6.

had been applied directly to a length dz of the cylinder. This was not done in order to bring clearly into the foreground the parts played by the several densities in a cylindrical region restricted by (2) and (6a,b). Thus, it is clear from (8) that *even though a radial component of current does not appear in* (14), *such a component actually plays a fundamental part in maintaining a surface density of charge along the entire cylindrical surface of the conductor.*

Near the ends of the conductor, specifically at $z = \pm(h - \delta)$, the total axial current entering the surface layer of thickness δ of each end surface must satisfy the condition

$$(I_z)_{z=\pm(h-\delta)} = \frac{\partial Q_e}{\partial t} \tag{15}$$

with

$$Q_e = \int_0^{a-\delta} 2\pi r \eta_f dr \tag{16}$$

Here Q_e is the total surface charge on the end surface.

26. Representation of a Current-carrying Conductor in Terms of Volume Density of Polarization.—Although a cylindrical conductor carrying an alternating current is characterized by axial and radial currents and a surface density of free charge only as shown in Sec. 25, it is sometimes convenient to represent the entire distribution by a physically fictitious but mathematically equivalent model characterized by P alone. Since P can under no circumstances describe surface currents, it follows that a description in terms of P alone is possible only if $\overline{\eta_m v}$ vanishes. That is, both I and M must be zero. This is no serious restriction in most cases in which the use of an equivalent model is convenient. Let it be assumed, therefore, that the actual conductor is described completely by

$$\bar{\rho} = 0; \qquad \overline{\rho_m v} = i_f = \hat{r} i_r + \hat{z} i_z \tag{1a}$$

$$\bar{\eta} = \eta_f; \qquad \overline{\eta_m v} = 0 \tag{1b}$$

and seek to represent this in terms of a mathematically equivalent model using P alone. The essential densities as expressed in terms of P are

$$\bar{\rho} = -\operatorname{div} \mathsf{P}; \qquad \overline{\rho_m v} = \frac{\partial \mathsf{P}}{\partial t} \tag{2a}$$

$$\bar{\eta} = (\hat{n}, \mathsf{P}); \qquad \overline{\eta_m v} = 0 \tag{2b}$$

In order to make the representation $(2a,b)$ equivalent to $(1a,b)$ in the interior, it is necessary that

$$P_\theta = 0 \qquad \text{(rotational symmetry)} \tag{3a}$$

$$\text{div } \mathsf{P} = \frac{1}{r}\frac{\partial}{\partial r}(rP_r) + \frac{\partial P_z}{\partial z} = 0 \tag{3b}$$

$$\frac{\partial P_r}{\partial t} = i_r; \qquad \frac{\partial P_z}{\partial t} = i_z \tag{3c}$$

For equivalence on the cylindrical surface

$$(P_r)_{r=a} = \eta_f \qquad \text{(cylinder)} \tag{4}$$

For equivalence on the end surfaces

$$(P_z)_{z=\pm h} = \eta_f \qquad \text{(ends)} \tag{5}$$

It is readily verified that these conditions and both equations of continuity are satisfied if

$$\frac{\partial \mathsf{P}}{\partial t} = i_f \tag{6}$$

and

$$I_z = \frac{\partial}{\partial t}\int_0^a 2\pi r P_z dr = \frac{\partial p_z}{\partial t} \tag{7}$$

In (7)

$$p_z = \int_0^a 2\pi r P_z dr \tag{8}$$

is the axial polarization or electric moment per unit length. It may be regarded as the axial component of an equivalent dipole for a unit length.

It is, therefore, possible to represent a distribution of free charge and current confined to a cylinder by a mathematically equivalent, but usually physically unavailable model using a distribution of volume density of polarization.

In some instances it is mathematically convenient to analyze a conductor first in parts and then combine the results so determined to obtain a solution involving the entire conductor. In this case the condition (6) written for each part lying between arbitrarily drawn mathematical boundary surfaces is not sufficient to ensure an equivalent representation of each part treated separately in terms of P alone. In fact, such a representation is not possible. Where the current-carrying conductor is cut in

two by an imaginary boundary at an arbitrary cross section, the boundary condition is

$$(\mathfrak{h},i_f) = 0 \qquad \text{or} \qquad (i_{fz})_{\text{side } 1} = (i_{fz})_{\text{side } 2} \qquad (9)$$

clearly

$$\bar{\eta} = 0 \qquad (10)$$

at the imaginary boundary.

| If (6) is used to define an equivalent model for *each* part of the cylinder, one above and the other below the imaginary boundary, the condition (9) is fulfilled. On the other hand, it is necessary that

$$\bar{\eta} = (\mathfrak{h},\mathsf{P}) \qquad (11)$$

at the imaginary boundary. That is, the representation in terms of P involves the appearance of a surface density of charge at the boundary. The condition (11) requires that $\bar{\eta}$ at the top surface of the lower part must be equal and opposite to $\bar{\eta}$ on the bottom surface of the upper part, so that when both parts are considered together the two surface densities cancel and contribute nothing to the representation of the cylinder as a whole. On the other hand, if each half is considered separately, the effects due to the representation in terms of P alone will differ from those in terms of i_f and η_f by the effects due to a surface density $\bar{\eta} = P_s$ on the top surface of the lower part and a surface density $\bar{\eta} = -P_s$ on the lower surface of the upper part. These differences in individual effects due to the parts are necessarily significant; they cancel completely only when the effects of both parts are combined. *An equivalent representation of the individual parts in terms of* P *alone is not possible.* In order to satisfy (10), it is necessary to provide surface densities η on the two imaginary surfaces such that

$$\bar{\eta} = (\mathfrak{h},\mathsf{P}) + \eta = 0 \qquad (12)$$

$\eta = -P_s$ on the lower surface of the upper part, $\eta = P_s$ on the upper surface of the lower part. The two representations are then mathematically equivalent, and effects at outside points due to a distribution of P and η will be the same as those due to the actual distribution in terms of i_f and η_f *for each part and for the conductor as a whole.* A change in the representation of a current-carrying conductor considered in parts from a

distribution in terms of i_f and η_f (the latter on the cylindrical envelope) to an *equivalent* distribution in terms of P and η (the latter, only on the end surfaces at the mathematical boundaries of the parts) cannot alter the fact that each part considered separately is *not* an electrically complete and self-contained region so long as current crosses mathematical surfaces separating it from the rest of the conductor. Important consequences of this fact in the determination at outside points of the effects of a current-carrying conductor as a whole and in parts are considered later (Sec. IV.1).
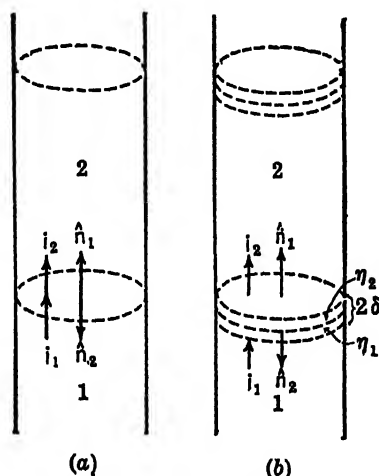
Fig. 26.1.—Cylindrical conductor divided by fictitious boundaries. (a) Boundaries with no fictitious charges; $i_1 = i_2$. (b) Boundaries with thin layers δ for fictitious surface charges; $\dfrac{\partial \eta_1}{\partial t} = i_{s1}$; $\dfrac{\partial \eta_2}{\partial t} = i_{s2}$; $\eta_2 = -\eta_1$.

It is possible to make each part of a conductor an electrically complete and self-contained region, without altering the representation of the conductor *as a whole* when the parts are combined, by adding fictitious distributions of surface charge at the equally fictitious boundaries between the parts in such a way that these charges cancel when the parts are put together. The boundary condition (9) when written at a fictitious surface between parts 1 and 2 of a conductor as shown in Fig. 26.1a is

$$(\hat{n}_1, i_1) + (\hat{n}_2, i_2) = 0 \qquad (13)$$

This is equivalent to

$$\cdot(\hbar_1, i_1) - \frac{\partial \eta_1}{\partial t} + (\hbar_2, i_2) - \frac{\partial \eta_2}{\partial t} = 0 \qquad (14)$$

if it is required that

$$\eta_1 = -\eta_2 \qquad (15)$$

and η_1 and η_2 are surface layers of charge in thin layers of thickness δ on each side of the boundary as shown in Fig. 26.1b. By requiring in addition that

$$(\hbar_1, i_1) - \frac{\partial \eta_1}{\partial t} = 0; \qquad (\hbar_2, i_2) - \frac{\partial \eta_2}{\partial t} = 0 \qquad (16)$$

the equation of continuity is obviously satisfied at each boundary for each part of the conductor. In terms of the physical model this means that the current $i_1 = i_2$ which actually crosses the boundary from part 1 to part 2 (Fig. 26.1a) is represented as a current i_1 directed into the surface layer and increasing the density η_1 of positive charge on the lower side of the boundary, while an equal current i_2 directed out from the upper surface layer reduces the positive or increases the negative surface charge η_2. *No charge* crosses the boundary between the two surface layers.

Exactly the same thing may be accomplished more simply in terms of P. It has been shown that the entire conductor can be represented in terms of an equivalent distribution of P alone. Since such a representation implies boundaries that cut no dipoles, it automatically assures electrical completeness for any part of a conductor *if no surface charges are added at the fictitious boundaries.* By *not* adding the surface charges to satisfy (12), the distribution in terms of P is equivalent for each *part* of the conductor to that required by (16), and the equation of continuity is satisfied for each part. However, the distribution is *not* equivalent to that actually obtaining insofar as each part of the conductor is concerned when treated separately. The significance of this fact in the description at outside points of the effects of current-carrying conductors as a whole and in parts is considered later (Sec. IV.1, 11; also in Vol. II).

27. Representation of a Current-carrying Loop in Terms of Volume Density of Magnetization.—A circular loop of wire lies

in the r, θ plane of a cylindrical system of coordinates r, θ, z as shown in Fig. 27.1. It is characterized completely by a volume density of current $i_f = \theta i_{f\theta}$. The radius a of the conductor is small compared with the mean radius b of the loop. The total current I_θ at a cross section of the conductor is

$$I_\theta = \int_{z=\pi a^2} i_{f\theta} d\sigma \tag{1}$$

The problem is to replace this physical model with its mathematical representation in terms of i by a physically fictitious but mathematically equivalent model using only the volume density of magnetization M. This can be done in several ways. The one to be described is particularly simple and subsequently

FIG. 27.1.—Circular loop of wire carry-   FIG. 27.2.—Thin, flat hoop carrying
ing current.   current.
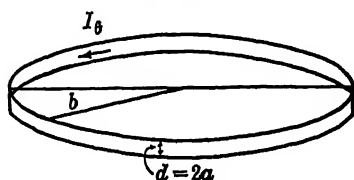
convenient. Unlike the representation of the current-carrying conductor in terms of P in the preceding section in which only the electrical description of the cylinder in terms of density functions was changed, the characterization of the loop that is given below involves a change not only in the density functions used, but also in the region in which they are defined. This implies that the mathematical equivalence will be valid only in determining effects due to the current in the loop at points *outside the regions where the density functions are defined both in the actual and the fictitious model.*

In order to simplify the representation it is much more convenient to replace the wire of radius a carrying the total current I_θ by an extremely thin flat hoop of thickness δ and width $d = 2a$, as shown in Fig. 27.2. The mean radius of the hoop is unchanged, and it carries the same total current assumed to be distributed uniformly in a cross section. Thus,

$$I_\theta = dl_{f\theta} \tag{2}$$

where $l_{f\theta}$ is the free-charge current per unit width. The thick-

ness δ of the hoop may be assumed to be of the same magnitude as the thickness of surface cells used in defining surface densities. (If a more rigorous procedure is desired, the conductor can be imagined cut up into a large number of hoops of various widths between zero and $2a$ and carrying different densities of current. The following analysis is then applied to each hoop and the final representations combined.)

For calculating effects due to the current I_θ at points outside the volume $\pi b^2 d$ which the loop encloses on the side, the current-carrying hoop may be replaced by a solid, magnetized disk of radius b and thickness $d = 2a$ placed to fill exactly the volume $\pi b^2 d$ surrounded by the hoop (Fig. 27.3). Let this disk be characterized by a volume density of magnetization $\mathsf{M} = \hat{z} M_s$
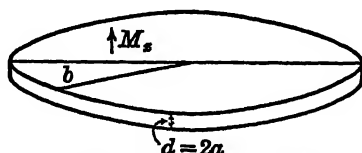
Fig. 27.3.—Magnetized disk.

which has the same magnitude at any instant at all points in the disk. At outside points the effect of this disk may be computed either from the magnetization $\mathsf{M_I}$ using a mode of subdivision I into volume cells only or in terms of a subdivision II using surface and volume cells so chosen that M is reduced to zero in the interior. The surface density of current $\mathsf{I_{II}}$ in this case is

$$\mathsf{I_{II}} = -[\hat{n}, \mathsf{M_I}] \tag{3}$$

Specifically, if $\mathsf{M_I} = \hat{z} M_{sI}$,

$$\mathsf{I_{II}} = -[\hat{r}, \hat{z}] M_{sI} = \hat{\theta} M_{sI} \tag{4}$$

Thus the magnetized disk may be described either in terms of M_{sI} or in terms of $l_{\theta II} = M_{sI}$ depending upon the choice of subdivision.

Let M_{sI} in the mathematical model be so chosen that the surface density of current $l_{\theta II}$ (due to the current whirls throughout the interior of the magnetized disk) is equal to $l_{f\theta}$ in the hoop. That is, let

$$M_{sI} = l_{\theta II} = l_{f\theta} = \frac{I_\theta}{d} \tag{5}$$

With (5) the magnetized disk is exactly equivalent at all outside points to a uniform current of free charges circulating around its periphery. The magnetization of the entire disk so defined is

$$m = \tfrac{1}{2} M_s \pi b^2 d = \tfrac{1}{2} I_\theta \pi b^2 \tag{6}$$

Accordingly, a loop of wire carrying a current I_θ around a circle of mean radius b may be replaced for all purposes of calculation at outside points by a circular disk of radius b and thickness equal to the diameter d of the wire. The disk must be characterized by a volume density of magnetization M given by (5) or by a magnetization given by (6).

For simplicity the above discussion was carried out for a circular loop of wire. Actually, the shape of the loop is of no importance so long as the enclosed area has no dimension that is not large compared with the radius of the wire carrying the current around its perimeter. For any singly connected loop of any shape with mean enclosed area S lying in a plane perpendicular to the z axis and carrying a current I of uniform amplitude around its perimeter, a mathematically equivalent disk of the same shape must be characterized by a volume density of magnetization

$$M = \tfrac{1}{2} \frac{I}{d} \tag{7}$$

or by a magnetization

$$m = \tfrac{1}{2} S I \tag{8}$$

A mathematical model described in terms of M that is constructed to be equivalent to a current in a closed loop is often called a *magnetic shell*. For mathematical reasons it is convenient for some purposes to replace a loop antenna by an equivalent magnetic shell. In principle, this involves only replacing circulating free charges by a surface layer of circulating bound charges, and then using a representation in terms of volume cells only.

CHAPTER II

MATHEMATICAL DESCRIPTION OF SPACE AND OF SIMPLE MEDIA

The description of the average statistical electrical properties of matter in simple and reasonably exact mathematical form is the subject of Chapter I. Along with the definition of the four essential density functions in general form, the fundamental postulate of conservation of electric charge was expressed mathematically in the equation of continuity. In this second chapter the density functions defined for separate bodies and regions are interrelated, and the repulsion-attraction postulate is formulated.

THE FIELD EQUATIONS

1. Physical Model of Space.—The six density functions introduced to characterize the average electrical properties of matter are defined in Chapter I in terms of schematic models consisting of rather vaguely described, indestructible positive and negative charges of electricity bound together more or less firmly. No attempt was made to assign any real physical significance to the picture, nor to the densities defined to represent them mathematically. What was actually accomplished was to replace a physical model consisting of a given volume filled with electric charges in motion by a mathematical volume containing only scalars and vectors, *i.e.*, numbers and directions.

A region in which a scalar point function is defined is called a *scalar field;* a region in which a vector point function is defined is called a *vector field.* Using this terminology the charge-filled bodies and regions of the physical picture have been replaced in the mathematical model by superimposed scalar and vector fields. Such fields have the geometrical outline of the physical model that they represent because the numbers and directions, which alone characterize every point both in the interior and along the surface, reduce to zero or change abruptly as a surface

or a boundary is crossed. The boundaries of the physical model of matter are the mathematical envelopes where the scalar and vector density fields are discontinuous.

In order to predict and coordinate logically the pointer readings that can be obtained by performing operations on arrangements of actual bodies, two steps have been taken. The first step was to construct a composite physical picture to provide a structure in terms of which the external behavior of bodies and regions can be made plausible to the mind. This physical picture is the atomic model of matter. The second step was to describe a mathematical scheme for representing significant average characteristic properties of the physical model symbolically. This mathematical scheme is the field of superimposed vector and scalar densities.

Since electrically charged bodies act on each other even when separated in a vacuum, a third step is required, *viz.*, a fitting together of the models of individual and different bodies both when in contact with each other and when separated in space. What is needed is a model of the space between two or more bodies. But a physical model cannot be built out of mere emptiness. Consequently physicists, notably Faraday and Maxwell, decided to fill it. They devised a model of space which, instead of being empty, was built out of a medium called the electromagnetic ether. This ether was assumed to fill the space between all bodies. In order to provide a plausible mechanism in this composite model for explaining effects analogous to those observed for actual bodies in a vacuum, the ether model had to be assigned a variety of properties which were represented by suitable mathematical functions and functional relations. The entire scheme proved amazingly fruitful in predicting theoretical effects for which exact experimental analogues were found. This was true in spite of the fact that the properties which had to be assigned to the ether medium were not at all plausible or self-consistent, since they combined the properties of a super steel with those of emptiness. In fact, it had to be admitted that it was impossible to coordinate them in any *logical* scheme. In other words, although the mathematical model was successful, the physical model upon which it had been based bordered on the absurd. In a logical scheme there is no alternative but to rededicate the ether model

from the practice to the history of science.   There it may continue to stand as a monument to that industry which will stop at nothing, not even the logically impossible, to discover or to invent a method for coordinating observable natural phenomena. Such a discarding of models that have outlived their usefulness is characteristic of the trial-and-error method of natural science. Every generation of physicists constructs models of many kinds of which the following generation accepts only a limited few which have proved themselves successful both in providing accurate analogues of experimentally measured quantities and in coordinating these clearly and logically.   At times, two quite different models may serve equally well.   But eventually one is usually found to prevail, not because it is right, but because it is both more convenient and more logically constructed. *After all, models are constructed for convenience in thinking and recording, not as photographic images of nature.*

Since the ether model of space confuses rather than clarifies the modern mind, it is best discarded as no longer generally useful.   And with it must go the strains and the stresses, the displacements and the distortions in terms of which it pictured a mechanism for explaining observable electrical effects.   But what remains to take its place, to describe the effects that continue to be observed?   By means of what physical model can they be represented?   In principle it may be possible to devise a general mechanical model in terms of which electromagnetic phenomena may be visualized.   But why encumber the mind with a comprehensive model, which will certainly be exceedingly intricate, when it can get along without it?   Is it not simpler to correlate a mathematical model of matter directly with a corresponding mathematical model of space without the aid of an additional physical model of something that our senses observe as emptiness?   Certainly nothing fundamental is lost if such a physical model is omitted.   This statement must not be interpreted as suggesting a general scientific policy that pictures and schematic diagrams of all sorts are to be discarded, or even that they should be looked upon with suspicion.   Every graphical or mechanical representation that aids the mind in visualizing and understanding intricate mathematical formulas and relations may be used legitimately wherever convenient. But it is one thing to illustrate diverse mathematical functions

by diagrams, curves, or analogies, and quite another to construct a comprehensive and logical physical model. A miscellaneous collection of individually simple and useful graphs, pictures, and mechanical analogies no more constitutes a sensible physical model than does an assortment of sentences yield a meaningful paragraph.

The mathematical description of space will be introduced without a physical model and, for the present, without geometrical or mechanical aids.

2. Electromagnetic Field and Maxwell-Lorentz Equations. The fundamental purpose that a mathematical model of space must serve in the larger model of electromagnetism is to interconnect the density fields of matter. From the mathematical point of view, space consists of nothing more intricate than a coordinate system that assigns three numbers or coordinates to every point in order to relate it to an arbitrarily selected origin. In certain regions, definitely located by these coordinates, the scalar and vector fields of the continuous densities characterizing matter have nonvanishing values. These regions define the geometrical positions of mathematical bodies. At all other points, *i.e.*, in empty space, the density fields are zero. In order to interconnect scattered density fields, the mathematical model is extended to include space. This is accomplished by assigning two vectors to every point in space including that which is empty and that which contains regions where the density fields are nonvanishing. The electrical structure of mathematical space is described in terms of two vector fields. To one of the two vector point functions, the *electric vector*, is assigned the symbol E; to the other, the *magnetic vector*, the symbol B. In a region in which E has a value at every point, an E field or an electric field is said to exist. In a region in which B has a value at every point a B field or a magnetic field exists. The superposition of the two fields is called the electromagnetic field. Thus, the electromathematical structure of all space is completely identified with the electromagnetic field. The definition of each of the two vectors E and B involves a numerical, experimentally determined, proportionality constant with appropriate dimensions. These are the *fundamental electric constant* ϵ_0 and the *fundamental magnetic constant* ν_0. They are the factors with the aid of which numerical coordination is achieved between

the mathematical model of electromagnetism and the world of experimental pointer readings. As such, they play the role of universal constants.[1] Like all the density fields, the electromagnetic field is a purely mathematical construct for which no *direct* experimental analogues are presumed to exist. It differs from the density fields in that it is not based upon a physical model. No attempt is made to describe a mechanism to serve as a picture for the electromagnetic field in the way the atomic model serves the density fields. The electromagnetic field is thus a purely mathematical extension of the mathematical model of matter, not of its physical model.

The definition of the vectors E and B in terms of the continuous densities (which characterize the space occupied by matter) depends upon a fundamental theorem in vector analysis which is accepted without proof.

THEOREM:[2] A vector field is uniquely determined if its divergence and curl are specified, and if the normal component of the field is known over a closed surface, or if the vector vanishes as $1/r^2$ at infinity.

The definition of the vectors E and B in terms of their respective divergences and curls is the second fundamental principle of electromagnetism. The first fundamental principle is the conservation of electric charge; it is mathematically expressed in the equation of continuity. The second principle (which contains the first) is expressed by the Maxwell-Lorentz field equations; these define the divergence and curl of the E and B vectors in terms of the density functions and the constants ϵ_0 and ν_0 as follows:

$$\epsilon_0 \operatorname{div} E = \rho \tag{1}$$
$$\operatorname{curl} E = -\dot{B} \tag{2}$$
$$\nu_0 \operatorname{curl} B = \overline{\rho_m v} + \epsilon_0 \dot{E} \tag{3}$$
$$\operatorname{div} B = 0 \tag{4}$$

The constants ϵ_0 and ν_0 may be written in the denominator on the right in (1) and (3). For most purposes it is more con-

[1] The electric constant is also called the "permittivity of free space" or the "dielectric constant of free space"; ν_0 is called the "reluctivity of free space." The reciprocal $\mu_0 = 1/\nu_0$ is called the "permeability of free space."

[2] See, for example, M. Mason and W. Weaver, "The Electromagnetic Field," p. 357.

venient to leave them on the left as shown.   The superscript dot
is written for $\partial/\partial t$ and

$$\overline{p} = \rho - \text{div } \mathsf{P} \tag{5}$$

$$\overline{\rho_m \mathsf{v}} = \mathsf{i} + \text{curl } \mathsf{M} + \dot{\mathsf{P}} \tag{6}$$

It is assumed that the region (or regions) that is characterized
by $\overline{p}$ is as a whole at rest relative to the observer.[1]   The
defining relations (1) to (4) describe the electromagnetic field
completely in terms of the essential volume characteristics of
the unsteady state.   It is important to note that the vectors
E and B as defined in terms of the *average* or interpolated density
functions $\overline{p}$ and $\overline{\rho_m \mathsf{v}}$ are themselves *average* values at points within
matter.   They do not define a "microscopic" or "local" field.

In the stationary states all time rates of change vanish,
all functions are constant in time, and the field equations assume
the completely symmetrical form

$$\epsilon_0 \text{ div } \mathsf{E} = \overline{p} \tag{7}$$

$$\text{curl } \mathsf{E} = 0 \tag{8}$$

$$\nu_0 \text{ curl } \mathsf{B} = \mathsf{i} \tag{9}$$

$$\text{div } \mathsf{B} = 0 \tag{10}$$

This definition of the stationary electromagnetic field makes it
possible to extend the analogy exhibited between the static and
steady states in the table on page 72.

Quantity	Static state	Steady state
Electromagnetic field	Electric vector E Electric constant ϵ_0	Magnetic vector B Magnetic constant ν_0
Operations	curl	div

The dimensions of the electric and magnetic vectors and
scalars are conveniently expressed in terms of the auxiliary
dimensional symbol V and three of the four dimensional symbols

---

[1] If the region or body characterized by $\overline{p}$ is in motion relative to the
observer, a generalization of $\overline{\rho_m \mathsf{v}}$ must be made to include, in addition to the
motion of charge relative to the region, as measured by $\overline{\rho_m \mathsf{v}}$, the motion of
the charged region relative to the observer.   This generalization is possible
in quite simple form for slowly moving regions; for rapidly moving ones, a
relativistic formulation is necessary.

already introduced. The dimensional equivalent of V in terms of Q, L, M, T, is given in Appendix V.13. The unit associated with the symbol V is the volt.

$$E \approx \frac{V}{L} \frac{\text{volts}}{\text{meters}} \tag{11}$$

With (11), (2) gives

$$B \approx \frac{VT}{L^2} \frac{\text{volt-seconds}}{\text{square meters}} \text{ or } \frac{\text{webers}}{\text{square meters}} \tag{12}$$

The volt-second is called the weber by many engineers. With (11) and (1) it follows that

$$\epsilon_0 \approx \left\{ \frac{Q}{VL} \right\} \frac{\text{coulombs}}{\text{volt-meters}} \text{ or } \frac{\text{farads}}{\text{meters}} \tag{13}$$

The coulomb per volt is called the farad. Similarly (3) and (12) lead to

$$\nu_0 \approx \frac{QL}{VT^2} \frac{\text{ampere-meters}}{\text{volt-seconds}} \text{ or } \frac{\text{meters}}{\text{henrys}} \tag{14}$$

$$\frac{1}{\nu_0} \equiv \mu_0 \approx \frac{VT^2}{QL} \frac{\text{henrys}}{\text{meters}} \tag{15}$$

The volt-second per ampere or weber per ampere is called the henry.

Numerical values of the universal constants ϵ_0 and ν_0 and of their reciprocals, as obtained from suitably designed standard experiments, are

Electric constant
Permittivity of space $\Big\}$ $\epsilon_0 = 8.854 \times 10^{-12}$

$$\doteq \frac{1}{36\pi} \times 10^{-9} \frac{\text{farads}}{\text{meters}} \tag{16}$$

Magnetic constant
Reluctivity of space $\Big\}$ $\nu_0 = 7.95 \times 10^5$

$$= \frac{1}{4\pi} \times 10^7 \frac{\text{meters}}{\text{henrys}} \tag{17}$$

Elastivity of space $\dfrac{1}{\epsilon_0} = 1.129 \times 10^{11} \doteq 36\pi \times 10^9 \dfrac{\text{meters}}{\text{farads}}$ (18)

Permeability of space $\mu_0 = \dfrac{1}{\nu_0} = 1.257 \times 10^{-6}$

$$= 4\pi \times 10^{-7} \frac{\text{henrys}}{\text{meters}} \tag{19}$$

3. Field Equations at a Surface; Boundary Conditions.—A boundary surface is either the mathematical envelope between a charged region and space, where the density fields associated with the region vanish, or it is the mathematical envelope between two electrically different regions in contact, where the density fields associated with the two change abruptly. Conditions at the boundary between a charged region and space are obtained from those for two charged regions in contact by writing zero for one set of density fields.

Since the electromagnetic vectors E and B are defined in terms of all the volume densities, they cannot represent more rapid fluctuations in electrical conditions than can the densities themselves. Therefore, discontinuities in E and B can exist only at a boundary where an abrupt change from one set of densities to another occurs. In a thin layer of atomic thickness δ on each side of a boundary are defined the essential surface densities of charge and moving charge $\bar{\eta} \equiv \eta + (n,P)$ and $\overline{\eta_m v} = | - [n,M]$. Since the vectors E and B are defined in the interior in terms of the essential volume densities of charge and moving charge, it is to be expected that their behavior at a boundary is determined by the corresponding surface densities.

Let the equation

$$\epsilon_0 \operatorname{div} E = \rho - \operatorname{div} P \tag{1}$$

be written for a boundary between regions 1 and 2 in which E_1, P_1, ρ_1 and E_2, P_2, ρ_2 are defined, respectively. The exterior unit normals to the regions are n_1 and n_2. From the definition (I.8.8) of the divergence of a vector, (1) may be written as follows for an element $\Delta\tau_s = 2\delta\,\Delta\Sigma$ between the two regions:

$$\lim_{2\delta\,\Delta\Sigma\to 0} \frac{\epsilon_0 \int_\Sigma (n',E)d\sigma}{2\delta\,\Delta\Sigma} = \rho - \lim_{2\delta\,\Delta\Sigma\to 0} \frac{\int_\Sigma (n',P)d\sigma}{2\delta\,\Delta\Sigma} \tag{2}$$

Here n' points *out* from the surface enclosing $\Delta\tau$.

The element of volume $2\delta\,\Delta\Sigma$ is the same thin disk, half in region 1 and half in region 2, used in defining the essential surface density of charge (Sec. 1.9); as before, ρ is the average density of charge in the double cell.

$$\rho = \tfrac{1}{2}(\rho_1 + \rho_2) \tag{3}$$

The two integrals in (2) are evaluated over the entire surface Σ

of the element of volume $2\delta\,\Delta\Sigma$. This involves integration over the two parallel surfaces $\Delta\Sigma_1$ and $\Delta\Sigma_2$ and over the four narrow surfaces of width 2δ which are perpendicular to the boundary. As has been stated before, surface effects that cannot be described by volume functions are due to rapid variations in distributions of charge in a direction *perpendicular* to the surface. Variations in distribution *parallel* to the surface do not differ from the ordinary slow variations in the interior and are, therefore, adequately represented by the volume functions. The integration over the narrow edges of width 2δ thus contributes nothing to a distinctly surface effect of which complete account is taken in the integration over the surfaces $\Delta\Sigma$. The integrals are all four of the familiar form

$$\lim_{\Delta\Sigma\to 0} \frac{\int_{\Delta\Sigma_1} (\hat{n}_1',P_1)d\sigma}{\Delta\Sigma} = \lim_{\Delta\Sigma\to 0} \frac{\overline{(\hat{n}_1',P_1)}\int d\sigma}{\Delta\Sigma} = (\hat{n}_1',P_1) \qquad (4)$$

The second integral in (4) is obtained using the theorem of the mean for integrals. The last step follows directly after allowing $\Delta\Sigma$ to approach zero. $(\hat{n}_1',P_1)$ is the value at the point where $\Delta\Sigma$ vanishes and at a distance δ from the boundary. Let (3) and four integrals of the form (4) (with appropriately changed subscripts and with E written for P in two) be substituted in (2), and let this be multiplied through by 2δ. Next, introduce the surface densities according to the definition

$$\eta_1 + \eta_2 = \rho_1\delta + \rho_2\delta \qquad (5)$$

The exterior unit normals $\hat{n}_1$ and $\hat{n}_2$ to the large regions 1 and 2 are related to the outward normals $\hat{n}_1'$ and $\hat{n}_2'$ to $2\delta\,\Delta\Sigma$ by $\hat{n}_1 = -\hat{n}_1'$; $\hat{n}_2 = -\hat{n}_2'$. With this change (2) finally becomes

$$\epsilon_0(\hat{n}_1,E_1) + \epsilon_0(\hat{n}_2,E_2) = -\{\eta_1 + \eta_2 + (\hat{n}_1,P_1) + (\hat{n}_2,P_2)\} \qquad (6)$$

In the usual shorthand notation (6) is written

$$\epsilon_0(\hat{n},E) = -\{\eta + (\hat{n},P)\} = -\bar{\eta} \qquad (7)$$

By the same reasoning the equation

$$\text{div } B = 0 \qquad (8)$$

has the following form at a boundary:

$$(\hat{n}_1,B_1) + (\hat{n}_2,B_2) \equiv (\hat{n},B) = 0 \qquad (9)$$

If the third field equation

$$\nu_0 \ \text{curl} \ \mathsf{B} = \mathsf{i} + \text{curl} \ \mathsf{M} + \dot{\mathsf{P}} + \epsilon_0 \dot{\mathsf{E}} \tag{10}$$

is expanded by introducing the definition for the curl (I.18.17) written for the small volume $\Delta\tau = 2\delta \, \Delta\Sigma$ at the boundary, it becomes

$$\lim_{2\delta \, \Delta\Sigma \to 0} \frac{\nu_0 \int_\Sigma [\hat{n}',\mathsf{B}] d\sigma}{2\delta \, \Delta\Sigma} = \mathsf{i} + \dot{\mathsf{P}} + \epsilon_0 \dot{\mathsf{E}} + \lim_{2\delta \, \Delta\Sigma \to 0} \frac{\int_\Sigma [\hat{n}',\mathsf{M}] d\sigma}{2\delta \, \Delta\Sigma} \tag{11}$$

If i_1 and i_2 are defined in the two regions, their average value in the combined surface cell is

$$\mathsf{i} = \tfrac{1}{2}(\mathsf{i}_1 + \mathsf{i}_2) \tag{12}$$

Each of the two integrals includes integration over the parallel faces $\Delta\Sigma_1$ and $\Delta\Sigma_2$ and over the narrow surfaces of width 2δ perpendicular to the boundary. Only integration over $\Delta\Sigma_1$ and $\Delta\Sigma_2$ contributes significantly to a distinctly surface effect caused by a rapid variation in the distribution of charge or moving charge as the boundary is approached from the interior. Care must be exercised not to be misled by the fact that the vectors $[\hat{n},\mathsf{B}]$ and $[\hat{n},\mathsf{M}]$ are actually *parallel* to the boundary on $\Delta\Sigma_1$ and $\Delta\Sigma_2$. A rapid variation in these vectors due to asymmetries at the boundary must, nevertheless, occur in a direction *perpendicular* to the boundary. That is, a vector parallel to the boundary may change its direction or its magnitude very rapidly as the boundary is approached, while yet remaining parallel to it. Across the narrow surfaces of width 2δ, $[\hat{n},\mathsf{B}]$ and $[\hat{n},\mathsf{M}]$ can experience only the slow variation characteristic of the interior, and represented by curl B and curl M. A typical one of four integrals evaluated over the surfaces $\Delta\Sigma$ is

$$\lim_{\Delta\Sigma_1 \to 0} \frac{\int_{\Delta\Sigma_1} [\hat{n}_1',\mathsf{M}_1] d\sigma}{\Delta\Sigma_1} \qquad \lim_{\Delta\Sigma_1 \to 0} [\hat{n}_1'',\overline{\mathsf{M}}_1] \frac{\int_{\Delta\Sigma_1} d\sigma}{\Delta\Sigma_1} = [\hat{n}_1',\mathsf{M}_1] = -[\hat{n}_1,\mathsf{M}_1] \tag{13}$$

The evaluation includes use of the theorem of the mean for integrals, allowing $\Delta\Sigma$ to approach zero where the mean value is defined, and introducing the external normal $\hat{n}_1$ to region 1 by $\hat{n}_1 = -\hat{n}_1'$. Four integrals of the form (13) (with appropriately changed subscripts and with B written for M in two) and (12)

are substituted in (11) and multiplied through by 2δ.   The surface density of moving charge $|$ is introduced in the form

$$l_1 + l_2 = \delta i_1 + \delta i_2 \tag{14}$$

It will be recalled that the surface function $|$ is defined specifically because there may be rapid changes in the distribution of moving charges as a boundary is approached from the interior. Consequently, a section of thickness δ must be considered separately along all surfaces and boundaries where asymmetrical conditions prevail.   A similar situation does not obtain for the functions P and E.   The former is a volume function defined by interpolation throughout the interior of each region; the latter is defined at all points in space and in bodies or charged regions in terms of the several *volume* densities.   Both functions are continuous and slowly varying in the interior of each region. It follows, therefore, that the quantity $2\delta(\dot{P} + \epsilon_0\dot{E})$ defined for thin layers, each of thickness δ and one in each region along their common boundary, will not differ significantly from the same quantity defined for two similar slices, one in the interior of each region.   That is, the function $2\delta(\dot{P} + \epsilon_0\dot{E})$ describes nothing that is peculiar to the surface; its magnitude is necessarily extremely small because of the minuteness of δ and the fact that $\dot{P} + \epsilon_0\dot{E}$ assumes no abnormally large values in the thin surface section on each side of the boundaries.   Consequently (11) may be written

$$\nu_0[\hat{n}_1, B_1] + \nu_0[\hat{n}_2, B_2] = -\{l_1 + l_2 - [\hat{n}_1, M_1] - [\hat{n}_2, M_2]\} \tag{15}$$

Here all normals are *external* to the regions indicated by the subscripts.   In shorthand notation

$$\nu_0[\hat{n}, B] = -\{l - [\hat{n}, M]\} \tag{16}$$

It follows by similar reasoning that the second field equation

$$\text{curl } E = -\dot{B} \tag{17}$$

leads to the following surface equation:

$$[\hat{n}, E] = 0 \tag{18}$$

The Maxwell-Lorentz equations written for surface effects on the boundary between regions 1 and 2 have the following form:

$$\epsilon_0(\mathsf{A}_1,\mathsf{E}_1) + \epsilon_0(\mathsf{A}_2,\mathsf{E}_2) = -\{\eta_1 + \eta_2 + (\mathsf{A}_1,\mathsf{P}_1) + (\mathsf{A}_2,\mathsf{P}_2)\}$$

$$= -\{\bar{\eta}_1 + \bar{\eta}_2\} \tag{19a}$$

$$[\mathsf{A}_1,\mathsf{E}_1] + [\mathsf{A}_2,\mathsf{E}_2] = 0 \tag{19b}$$

$$\nu_0[\mathsf{A}_1,\mathsf{B}_1] + \nu_0[\mathsf{A}_2,\mathsf{B}_2] = -\{\mathsf{I}_1 + \mathsf{I}_2 + [\mathsf{A}_1, -\mathsf{M}_1] + [\mathsf{A}_2, -\mathsf{M}_2]\}$$

$$= -\{\overline{\eta_m \mathsf{v}_1} + \overline{\eta_m \mathsf{v}_2}\} \tag{19c}$$

$$(\mathsf{A}_1,\mathsf{B}_1) + (\mathsf{A}_2,\mathsf{B}_2) = 0 \tag{19d}$$

In shorthand

$$\epsilon_0(\mathsf{A},\mathsf{E}) = -\{\eta + (\mathsf{A},\mathsf{P})\} = -\bar{\eta} \tag{20a}$$

$$[\mathsf{A},\mathsf{E}] = 0 \tag{20b}$$

$$\nu_0[\mathsf{A},\mathsf{B}] = -\{\mathsf{I} + [\mathsf{A}, -\mathsf{M}]\} = -\overline{\eta_m \mathsf{v}} \tag{20c}$$

$$(\mathsf{A},\mathsf{B}) = 0 \tag{20d}$$

The interpretation of the boundary equations (19) is not difficult. Equations (19a) and (19d) apply to the normal components of the vectors E and B. Thus (19a) states that the normal component of the electric vector is discontinuous in crossing a boundary surface. The magnitude of the discontinuity is the essential surface characteristic of charge $\bar{\eta}$ divided by the electric constant ϵ_0. In the same way (19d) states that the normal component of the magnetic vector is continuous across all boundaries. The interpretation of (19c) and (19b) is slightly more involved owing to the fact that the vector product of the external normal to the surface of a region and one of the field vectors actually does not specify any particular component of this field vector. It defines an axial vector that has the *magnitude of the tangential component* of the field vector at the surface and a direction normal to the plane formed by the field vector and the external normal. It follows at once that the magnitude of the discontinuity of the axial vector so defined is equal to the magnitude of the discontinuity in the tangential component of the field vector. Therefore, (19b) requires that the tangential component of the electric vector be continuous in crossing all boundaries, whereas (19c) requires the tangential component of the magnetic vector to be discontinuous by a magnitude equal to the essential surface characteristic of moving charge divided by the magnetic constant ν_0. These boundary conditions are illustrated in Fig. 3.1. The following is a tabular form using the shorthand notation as in (20).

Vector	Component	Behavior at boundary	Magnitude of discontinuity
E	Normal	Discontinuous	$-\bar{\eta}/\epsilon_0$
E	Tangential	Continuous	0
B	Tangential	Discontinuous	$-\overline{\eta_m v}/\nu_0$
B	Normal	Continuous	0

The form of the boundary equations (19) or (20) is not changed in the stationary states. It is only necessary to require all

$$\bar{\eta} \neq 0$$

(a)

$$\overline{\eta_m v} \neq 0$$

(b)
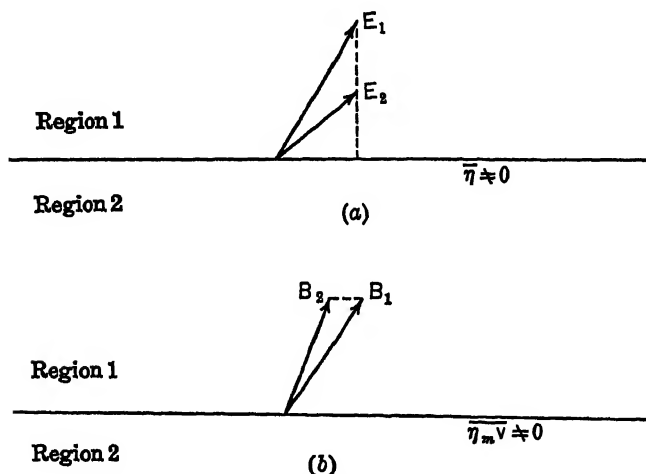
Fig. 3.1.—Electric (*a*) and magnetic (*b*) vectors on both sides of a boundary. Note that vectors with subscript 1 are just above the point on the boundary, vectors with subscript 2 are just below the point on the boundary. The surface densities are in surface layers in one or both regions.

functions to be invariant in time. The analogy between the static and steady states may be extended to include the following.

Quantity	Static state	Steady state
Operations	Vector product [ ]	Scalar product ( )

4. Field Equations as Fundamental Postulates.—The mathematical model of the electrical properties of matter is contained in the four essential characteristics that describe matter in terms of scalar and vector fields. Every point in the interior of a body

is characterized by values of the two volume functions

$$\bar{\rho} = \rho - \text{div } \mathsf{P} \tag{1}$$

$$\overline{\rho_m \mathsf{v}} = \mathsf{i} + \text{curl } \mathsf{M} + \dot{\mathsf{P}} \tag{2}$$

Every point on a boundary surface is characterized by values of the surface functions

$$\bar{\eta} = \eta + (\mathsf{n}, \mathsf{P}) \tag{3}$$

$$\overline{\eta_m \mathsf{v}} = \mathsf{l} - [\mathsf{n}, \mathsf{M}] \tag{4}$$

The mathematical model of the electrical properties of space is defined in terms of the two universal constants ϵ_0 and ν_0 and the two vector point functions E and B that are defined everywhere except in surface layers by

$$\epsilon_0 \text{ div } \mathsf{E} = \bar{\rho} \tag{5}$$

$$\text{curl } \mathsf{E} = -\dot{\mathsf{B}} \tag{6}$$

$$\nu_0 \text{ curl } \mathsf{B} = \overline{\rho_m \mathsf{v}} + \epsilon_0 \dot{\mathsf{E}} \tag{7}$$

$$\text{div } \mathsf{B} = 0 \tag{8}$$

$$\epsilon_0(\mathsf{n}, \mathsf{E}) = -\bar{\eta} \tag{9}$$

$$[\mathsf{n}, \mathsf{E}] = 0 \tag{10}$$

$$\nu_0[\mathsf{n}, \mathsf{B}] = -\overline{\eta_m \mathsf{v}} \tag{11}$$

$$(\mathsf{n}, \mathsf{B}) = 0 \tag{12}$$

The description of every point in the electromagnetic field is characterized by the nature and the distribution of the density fields that determine the essential volume and surface characteristics $\bar{\rho}$ and $\overline{\rho_m \mathsf{v}}$, $\bar{\eta}$ and $\overline{\eta_m \mathsf{v}}$. For a knowledge of these densities is sufficient to define the field vectors E and B. The third step in formulating a method for predicting and coordinating theoretical effects that are analogues of experimentally observable ones has been described. The mathematical densities identified with the electrical properties of matter have been fitted into a mathematical structure defined for all space called the electromagnetic field.

The apparently entirely arbitrary definition of two physically meaningless vectors E and B may suggest the questions: What are they good for? Where did they come from? Strictly, it is no more reasonable to ask these questions now than it would have been to ask entirely similar ones when the atomic model was introduced. But for most people conversant with physical

science, the atomic picture has become both so familiar and so plausible that it is taken for granted, whereas the Maxwell equations, though older and no less firmly established, appear strange and abstract for having been left to the last chapters of advanced treatises too long. Actually the difference is not so much one of method as of subject. In the one case a physical, in the other a mathematical, model is involved, and both have been set up here with equal boldness as a basis for further discussion of electrical effects. Both have been accepted with no justification beyond an assurance that they will serve the primary purpose of physical science: to establish and coordinate logically exact theoretical analogues of experimentally observable time sequences in nature. It is the purpose of later chapters and volumes to substantiate this assurance in terms of numerous examples and practical problems and thereby to justify and at the same time to circumscribe the usefulness of the postulated models.

It may be argued by some that a much more scientific procedure is the traditional one that begins with a so-called crucial experiment such as that of Coulomb or those of Ampère, which writes a theoretical analogue of the experimental observations and then generalizes from this. Such an argument usually overemphasizes the present significance of the so-called crucial experiments by measuring their scientific importance in terms of their great historical value. The argument further overlooks the multiple-valued nature of the inductive process that requires a continual retracing of steps in order to verify deductively that the particular generalization chosen is a convenient and useful one. It also forgets the quite illogical "boldness" of many generalizations which are ultimately always justified by the statement that they work, which means that by applying deductive methods useful theoretical analogues can be derived for special cases. Certainly the path from the observations of a special experiment to a general formula or a set of equations always traces a difficult, step-by-step, trial-and-error advance. Unquestionably it is a path that has been followed with signal success by the pioneer. But once the land has been surveyed and a map has been drawn in the form of a general mathematical model, this may be used to find a reasonably direct, deductive road to every special nook and cranny included in the generality. Surely, where the purpose is to arrive at solutions of general

or special problems and not to follow an interesting historical development, it is best to learn how to proceed from the most general equations. Thus it is interesting, but immaterial for the man interested in learning to solve electrodynamical problems, to know by what winding and devious paths, by what bold leaps, the Maxwell-Lorentz equations were finally formulated. If he nevertheless insists on seeing them "derived" from other, less general formulas such as Coulomb's law and Ampère's laws, he must content himself with the statement *that this has never been done*. The mathematical implications, the broad assumptions, the bold generalizations, in the inductive structure leading to the equations have been examined and elucidated in detail,[1] but these are not, can never be a derivation of the equations. And they may prove more confusing than enlightening to one who is interested in a coherent interpretation and a practical application of this most powerful tool for solving problems in macroscopic electrodynamics.

It is in order to obtain a unified, logical picture of the fundamental mathematical model underlying the theory and practice of electricity that the Maxwell-Lorentz equations, just as the atomic model, have been introduced as postulates rather than as conclusions. They are thus set up as fundamental principles of electrodynamics, and their proof must lie ultimately, as does that of all other mathematical and physical models, in the correctness and usefulness of deductions made from them. It is to be observed that the Maxwell equations need not depend upon the atomic picture. This was used as a convenient means of introducing the mathematical model and giving a certain physical significance to its symbols as an aid in the learning and thinking processes of all except perhaps the most mathematical minds. Other models than the simple atomic one might no doubt be devised to serve the same purpose. But the atomic one is naturally preferred because it has proved useful not only in describing electrical phenomena in electrolysis, discharges through gases, thermionics, etc., but in many other branches of physics and chemistry. A principal function of a model, physical or mathematical, is to coordinate as broad a field of observation as possible.

[1] See for example, M. Mason and W. Weaver, "The Electromagnetic Field."

Some writers have introduced densities of magnetic "charge" and magnetic "current" with the express purpose of making the field equations more symmetrical. In terms of the analogy exhibited in the stationary-state formulation as, for example, in (2.7) to (2.10), the addition of essential densities of magnetic "charge" and "current" in place of the two zeros, respectively, in (2.8) and (2.10) does not improve the symmetry. To include magnetic "current" in (2.8) and magnetic "charge" in (2.10) destroys all symmetry. In fact, in order to maintain symmetry, volume and surface densities of "free magnetic charge" and "current," volume densities of "magnetic polarization," and "magnetic magnetization" would have to be defined and coordinated with quantities already defined. Since these additional densities are not required either for the atomic model or in the solution of electromagnetic problems, they have not been included. They would be unnecessary added complications serving no useful purpose in the present formulation and in its application in electromagnetic engineering.

The field equations define the curl and the divergence of two vectors E and B in combination with the two constants ϵ_0 and ν_0. Numerical values of the latter are determined experimentally. The two vectors are the dependent variables, the space coordinates and the time are the independent variables. The space coordinates occur in the field equations not only in the definitions of the scalar and vector densities but also in the operations symbolized by the divergence and curl. The explicit solution of the equations for E and B, subject to the boundary conditions of a particular arrangement of bodies or regions, constitutes a usually necessary intermediate step in the determination of the density functions.

5. Alternative Formulation of Field Equations; Auxiliary Field Vectors and Constants.—An examination of the field and boundary equations in a purely formal way suggests a mathematically convenient rearrangement of the first and third equations. This consists in combining the terms according to their mathematical form rather than having them indicate explicitly that the divergence and curl of each of the vectors E and B are defined. Since ϵ_0 and ν_0 are constants, they may be written inside the operational symbols div and curl.

$$\text{div } (\epsilon_0 E + P) = \rho \tag{1}$$

$$\text{curl } (\nu_0 B - M) = \epsilon_0 \dot{E} + i + \dot{P} = i + \frac{\partial}{\partial t} (\epsilon_0 E + P) \tag{2}$$

$$(\hbar, \epsilon_0 E + P) = -\eta \tag{3}$$

$$[\hbar, \nu_0 B - M] = -I \tag{4}$$

For convenience in writing the following new symbols are defined:

$$D = \epsilon_0 E + P$$
$$H = \nu_0 B - M \tag{5}$$

The vectors D and H are written simply as a shorthand notation. With no other qualifications, both vary with the mode of subdivision used in defining the density functions. At a later point (Sec. 14) a definite mode of subdivision must be selected, *viz.*, one in which no bound groups are cut so that P and $-M$ are maximum and $\rho = \rho_f$, $i = i_f$, $\eta = \eta_f$, $I = I_f$. At all points in space or in bodies where the P and M fields vanish, $D = \epsilon_0 E$ and $H = \nu_0 B$. Since ϵ_0 and ν_0 are scalars, the vectors D and H point in the same direction and are proportional in magnitude, respectively, to the vectors E and B at all points where P and $-M$ vanish or are not defined. On the other hand, in bodies where P and $-M$ are nonvanishing, D and H are not in general proportional, respectively, to E and B, nor do they point in the same direction.

Much confusion exists in the literature regarding the significance of the auxiliary vectors D and H in their relation to the fundamental electromagnetic vectors E and B and in the nomenclature used for them. This is partly due to an unfortunate and historically long-standing analogy in formal nomenclature between vectors E and H which obscures the more fundamental analogy between E and B. The confusion is also in part due to a terminology originating with the ether model with its strains, stresses, and displacements. There is at present no simple way out of this confusion. Many authors[1] continue to use the old terminology and the fundamentally incorrect analogy between

[1] For example, Abraham-Becker, "The Classical Theory of Electricity and Magnetism"; J. C. Slater and N. H. Frank, "Introduction to Theoretical Physics"; S. A. Schelkunoff, "Electromagnetic Waves"; H. H. Skilling, "Fundamentals of Electric Waves"; S. Ramo and J. R. Whinnery, "Fields and Waves in Modern Radio."

E and H, D and B, instead of the analogy between E and B, D and H; others,[1] while recognizing the logical inaccuracy, have still preferred to continue with the old analogy and its nomenclature; still others have avoided the problem by virtually ignoring the auxiliary vectors D and H and developing the theory entirely in terms of E and B;[2] at least one[3] has simply interchanged the definitions of B and H. The first method is here rejected because it is directly contrary to the spirit of a clear and logically coherent description of the electrodynamical model. The second way is a good stepping stone, but in due time it is little better than the first. The third possibility is a sound one, although for some purposes the use of H and D are convenient as a shorthand and in establishing formal analogies with electric circuit theory; this is considered in greater detail at a later point. Finally, the fourth method, while neatly avoiding the difficulties of notation and nomenclature, produces the extremely perplexing situation of introducing old and familiar symbols with changed meanings. The confusion is thus shifted from the symbols and their names to their mathematical definition. This leads to great difficulties in referring from one book to another. In the present treatment, a conventional notation and the customary definitions are used but in such a way that the true analogy between the electric and magnetic quantities is indicated.[4] The difficulty regarding nomenclature is eliminated to a large extent by using as few coined names for the symbols as possible and letting these stand for themselves. Thus, reference will be made to the E vector, or the B field, or the value of H, rather than to coined names. If it is desired to distinguish especially between magnetic and electric quantities, reference will be made to the electric E vector, the electric D vector, the magnetic B and H vectors. Such names are somewhat awkward, but until a future generation has grown up to a generally accepted nomenclature, it seems to be the best course to take. Certainly to call E the electric intensity while its magnetic analogue B is called the magnetic induction and the nonanalogous quantity

[1] J. H. Van Vleck, "Electric and Magnetic Susceptibilities."

[2] M. Mason and W. Weaver, "The Electromagnetic Field."

[3] J. Frenkel, "Elektrodynamik."

[4] This agrees with J. Stratton, "Electromagnetic Theory"; G. P. Harnwell, "Principles of Electricity and Magnetism."

H is called the magnetic intensity is to be avoided. Such names as displacement, which have a pictorial meaning in the old ether model, are likewise undesirable.
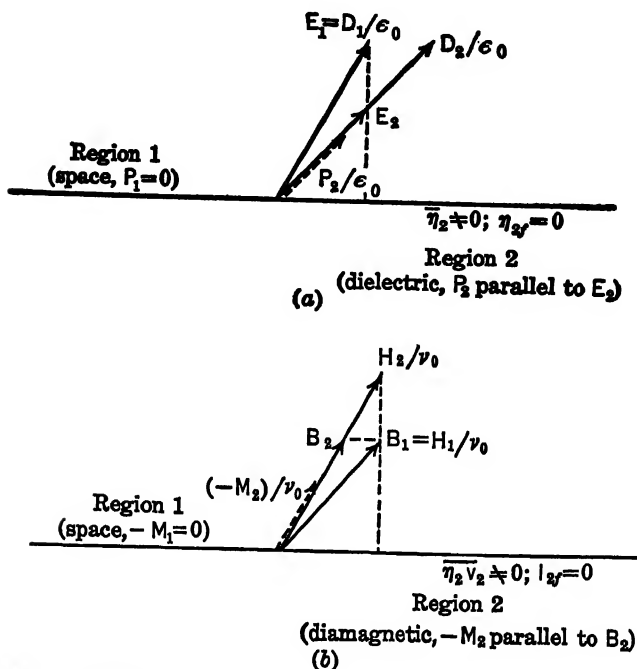
FIG. 5.1.—(a) Electric field vectors on both sides of a boundary in the special case in which region 1 is space, region 2 is dielectric with $\eta_2 = \eta_{2f} = 0$, P_2 parallel to E_2. (b) Magnetic field vectors on both sides of a boundary in the special case in which region 1 is space, region 2 is diamagnetic with $I_2 = I_{2f} = 0$; $-M_2$ parallel to B_2. Note that vectors with subscript 1 are just above the point on the boundary; the vectors with subscript 2 just below the point on the boundary.

In terms of the auxiliary vectors D and H the Maxwell equations assume a simpler appearance.

$$\text{div } D = \rho \tag{6}$$
$$\text{curl } E = -\dot{B} \tag{7}$$
$$\text{curl } H = i + \dot{D} \tag{8}$$
$$\text{div } B = 0 \tag{9}$$

The auxiliary relations are

$$D = \epsilon_0 E + P \tag{10}$$
$$H = \nu_0 B - M \tag{11}$$

The surface equations in shorthand form are

$$(\Lambda, D) = -\eta \tag{12}$$

$$[\Lambda, E] = 0 \tag{13}$$

$$[\Lambda, H] = -I \tag{14}$$

$$(\Lambda, B) = 0 \tag{15}$$

In dealing with models that have physically possible analogues it is never necessary to define a *surface* density of moving free charge. If D and H are defined using a subdivision that cuts no bound groups, η in (12) and I in (14) are, respectively, η_f and I_f. Except in the physically unrealizable but theoretically
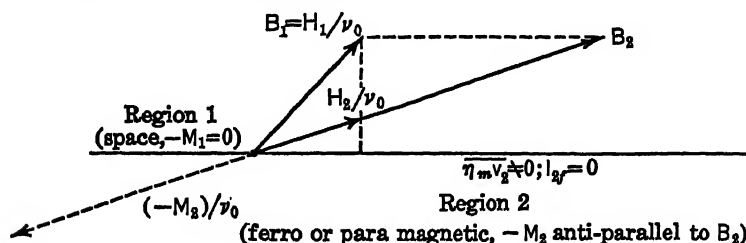
Fig. 5.2.—Magnetic field vectors on both sides of a boundary in the special case in which region 1 is space, region 2 a ferromagnetic or paramagnetic medium with $I_2 = I_{2f} = 0$. If region 2 is paramagnetic $(-M_2)$ is very small; if region 2 is ferromagnetic $(-M_2)$ is very large; in both cases $-M_2$ is directed opposite to B_2. Note that vectors with subscript 1 are just above the point on the boundary; the vectors with subscript 2 are just below the point on the boundary.

very important case of a *perfect* free-charge model, I_f is not required, so the right side of (14) is zero. The boundary conditions (12) to (15) are illustrated in Figs. 5.1, 5.2 for the special case $\eta = \eta_f = 0$, $I = I_f = 0$. Media with P parallel to and in the same direction as E are said to be *dielectric*. Media with $(-M)$ parallel to and in the same direction as B are *diamagnetic;* media with $(-M)$ parallel and directed opposite to B are *paramagnetic* if M is small and *ferromagnetic* if M is large.

Two useful auxiliary constants may be defined by combining the universal electric and magnetic constants ϵ_0 and ν_0.

Characteristic velocity of space $v_0 \equiv \sqrt{\dfrac{\nu_0}{\epsilon_0}} \equiv \dfrac{1}{\sqrt{\mu_0 \epsilon_0}}$.

$$= 2.998 \times 10^8 \doteq 3 \times 10^8 \text{ meters/second} \tag{16}$$

Characteristic impedance of space $\zeta_0 \equiv \dfrac{1}{\sqrt{\epsilon_0 \nu_0}} \equiv \sqrt{\dfrac{\mu_0}{\epsilon_0}}$

$$= 376.7 \doteq 120\pi \text{ ohms} \tag{17}$$

The dimensions of v_0 and ζ_0 are obtained directly from (2.13) and (2.14).

$$v_0 \approx \frac{L}{T} \frac{\text{meters}}{\text{seconds}} \tag{18}$$

$$\zeta_0 \approx \frac{VT}{Q} \frac{\text{volts}}{\text{amperes}} \text{ or ohms} \tag{19}$$

The volt per ampere is called the ohm. The following relations are also true:

$$\epsilon_0 = \frac{1}{\sqrt{v_0 \zeta_0}} \tag{20}$$

$$\nu_0 = \sqrt{\frac{v_0}{\zeta_0}} \tag{21}$$

INTEGRAL FORMS OF FIELD EQUATIONS

6. General Theorems.—The Maxwell-Lorentz equations that define the electromagnetic field consist of four simultaneous, partial differential equations of the first order. With the aid of general integral theorems of the calculus, it is possible to transform them into integral relations that are often convenient in the solution of problems, especially those characterized by symmetry. In many cases, the boundary conditions are sufficiently simple so that explicit expressions for E and B in terms of the density functions can be obtained in this way. The integral forms of several of the field equations have become associated with the names of men who were instrumental in establishing them as laws or theorems. The systematic derivation of the integral representations from the field equations is, of course, not their historical origin, nor will they be written in their original form.

In order to establish the integral field equations it is first necessary to describe the fundamental integral theorems that are involved. Their derivations are readily accessible in the mathematical literature, so that they will not be given.

The divergence theorem is a theorem for transforming a volume integral into an integral evaluated over the surface enclosing the volume. It is closely related to the definition of the divergence of a vector. In vector form it is expressed in terms of a continuous vector point function A which defines a vector field. If any volume V is chosen in this field, it will be contained in

a closed (mathematical) surface S. Let dV be an element of the volume, dS an element of the enclosing surface, and $\hat{n}$ an external normal to the surface.   The theorem is

$$\int_V \text{div } A \, dV = \int_S (\hat{n}, A) dS \tag{1}$$

The curl theorem is a special form of the divergence theorem. It is obtained from this by setting the vector A equal to the vector defined by the vector product $[B,C]$ where C is defined to be a constant vector.   Upon applying the divergence theorem to this vector product,

$$\int_V \text{div } [B,C] dV = \int_S (\hat{n}, [B,C]) dS \tag{2}$$

The following vector relations are readily verified using Cartesian coordinates:

$$(\hat{n}, [B,C]) = (C, [\hat{n}, B]) \tag{3}$$
$$\text{div } [B,C] = (C, \text{curl } B) - (B, \text{curl } C) \tag{4}$$

But since C is constant by definition, it follows that (4) reduces to

$$\text{div } [B,C] = (C, \text{curl } B) \tag{5}$$

Using (3) and (5) in (2) the following relation is obtained:

$$\left(C, \int_V \text{curl } B \, dV - \int_S [\hat{n}, B] dS \right) = 0 \tag{6}$$

Since the vector C is arbitrary and constant, the above scalar product of the vector C and the vector defined by the difference between the two integrals vanishes for all orientations of the vector C only if the second member of the product vanishes identically.   That is, if

$$\int_V \text{curl } B \, dV = \int_S [\hat{n}, B] dS \tag{7}$$

This is the curl theorem.   It is seen to be closely related to the definition of the curl of a vector.

Stokes's theorem is a theorem for transforming a surface integral over a cap- or cup-shaped surface into a line integral around the closed boundary of the surface.   Consider any open cap- or cup-shaped surface S which may have any form whatsoever from a flat disk enclosed by the boundary line s to a deep balloon with

only a narrow opening enclosed by the boundary line s.   Let this surface be entirely in the field of a continuous vector point function A.   The theorem is

$$\int_{S(\text{cap})} (\mathfrak{h}, \text{curl } \mathsf{A})dS = \oint_{s(\text{closed line})} (\mathsf{A}, ds) \qquad (8)$$

The line integration around the closed boundary s is to be performed in a direction such that the right-hand-screw convention is satisfied with respect to the normal $\mathfrak{h}$ to the surface S.   Stokes's theorem is seen to be closely related to the alternative definition of the curl of a vector in terms of its component normal to a surface.

Stokes's theorem and the divergence theorem may be applied to the four Maxwell-Lorentz equations to obtain integrals that are common in technical literature.   The shorthand notation using the auxiliary vectors D and H is used where convenient.

7. Gauss's Theorem for the E and D Vectors; Integral Form of the First Field Equation.—Let both sides of the field equation

$$\text{div } \mathsf{D} = \rho; \qquad \mathsf{D} = \epsilon_0 \mathsf{E} + \mathsf{P} \qquad (1)$$

be integrated over an arbitrary volume V enclosed by the surface S.   Then

$$\int_V \text{div } \mathsf{D} \, dV = \int_V \rho \, dV \qquad (2)$$

In any region in which the vector D is *continuous*, the divergence theorem may be used.   Hence, assuming continuity for the present throughout V and S, the divergence theorem applied to the left in (2) gives

$$\int_V \text{div } \mathsf{D} \, dV = \int_S (\mathfrak{h}, \mathsf{D})dS = \int_V \rho \, dV \qquad (3)$$

The second integral in (3) is the total outward normal flux of the D vector, or simply the outward normal D flux, across the closed surface S.   This name does not imply that anything physically real is flowing out of the volume.   It might be concluded in a perfectly general way from (3) that the normal D flux across any closed surface is equal to the total charge $Q = \int_V \rho \, dV$ contained within it.   But the volume integral on the right in (3) only gives the contribution to the total charge due to the volume density ρ; it takes no account of possible surface distributions

characterized by the density η which might be contained inside the enclosed envelope S. Since the D vector is discontinuous across all surfaces or boundaries where η is nonvanishing, as seen in (5.12), the divergence theorem cannot be applied directly to such a region without first excluding the surfaces of discontinuity. Consequently, the integral (3) applies only to homogeneous volumes in which there are no discontinuities in D.

In order to generalize (3) so that it may be applied to any region, suppose the volume contained within S to be composed
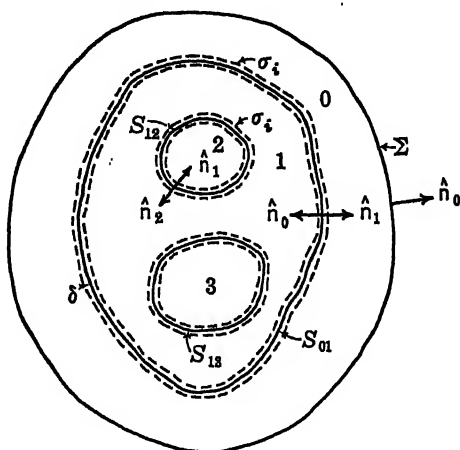
Fig. 7.1.—Regions with boundaries for obtaining Gauss's theorem for a discontinuous vector.

of several dissimilar parts as indicated in Fig. 7.1. Let a surface of integration Σ be drawn around this composite region in such a way that it is entirely within a single homogeneous medium. In each of the regions 0, 1, 2 a different set of continuous densities ρ_i, P_i with appropriate subscripts is defined. On the boundaries between these regions, surface layers of charge may be described by the function η_{ij}. In the interior of each homogeneous region the following equation is true:

$$\text{div } D_i = \rho_i \tag{4}$$

On the boundary surfaces between two dissimilar regions, such as 0 and 1, the following typical equation is true:

$$(\hat{n}_0, D_0) + (\hat{n}_1, D_1) = -(\eta_0 + \eta_1) \equiv -\eta_{01} \tag{5}$$

Here $\hat{n}_0$ is an external normal to the region $0'$; $\hat{n}_1$ is an external normal to region 1.

The boundary surfaces across which D is discontinuous are the envelopes S_{01}, S_{12}, S_{13}, etc., as shown in Fig. 7.1. Let them be excluded from the region of integration by enclosing them in surfaces σ_i constructed both inside and outside the envelopes S. The volume V is thus divided into volumes V_j. The σ_i surfaces are to be drawn at small distances δ (of atomic magnitude) from the surfaces S, so that surface layers of charge defined by the surface density η must be contained in the narrow regions each of thickness δ on each side of the boundaries S. The expression for the total outward normal flux of the D vector across all the surfaces σ_i and the outside boundary Σ now can be written down. Since the surfaces σ_i are drawn very close to the surfaces S, no serious error is made by taking the integral over each side of S instead of over the two σ surfaces enclosing it. In the same way the volume between pairs of σ surfaces is so small that it may be neglected in comparison with the rest, so that $\Sigma V_j = V$.

$$\int_{\Sigma} (\hat{n}_0, D_0) dS + \int_{S_{01}} \{(\hat{n}_0, D_0) + (\hat{n}_1, D_1)\} dS_{01}$$
$$+ \sum_i \int_{S_{1i}} \{(\hat{n}_1, D_1) + (\hat{n}_i, D_i)\} dS_{1i} = \sum_j \int_{V_j} \rho_j dV_j \quad (6)$$

All normals in (6) are external to the regions indicated by the subscripts. By now substituting the appropriate boundary conditions (5) in (6) for each pair of integrals,

$$\int_{\Sigma} (\hat{n}_0, D_0) dS = \left\{ \sum_j \int_{V_j} \rho_j dV_j + \int_{S_{01}} \eta_{01} dS_{01} + \sum_i \int_{S_{1i}} \eta_{1i} dS_{1i} \right\}$$
$$= Q \quad (7)$$

Q is defined by (7) to be the sum of all integrals of volume and surface densities of charge. Accordingly, it represents the total charge in the volume V enclosed by the surface Σ regardless of whether it is distributed throughout the volume or along surfaces or boundaries. Gauss's theorem can be formulated as follows:

Gauss's Theorem: If one or more bodies or regions of any internal structure, shape, or arrangement whatsoever are imagined enclosed in an envelope of integration Σ of entirely arbitrary

shape and size, the total outward normal flux of the D vector across the surface is equal to the total charge contained within it.

$$\int_\Sigma (\hat{n}, D)dS = Q \tag{8}$$

$$D = \epsilon_0 E + P \tag{9}$$

$$Q = \sum_j \int_{V_j} \rho_j dV_j + \sum_{ij} \int_{S_{ij}} \eta_{ij} dS_{ij} \tag{10}$$

If P is defined using a mode of subdivision that cuts through no bound groups, as is customary when the D vector is used, Q in (10) is the total *free* charge within Σ and a subscript f is written on Q, ρ, and η.   If Q and D vary in time, Gauss's theorem is true at every instant.

If the field equation (1) is written in the form

$$\epsilon_0 \operatorname{div} E = \bar{\rho}; \qquad \bar{\rho} = \rho - \operatorname{div} P \tag{11}$$

the following alternative integral theorem is obtained

$$\epsilon_0 \int_\Sigma (\hat{n}, E)dS = \bar{Q} = \sum_j \int_{V_j} \bar{\rho}_j dV_j + \sum_{ij} \int_{S_{ij}} \bar{\eta}_{ij} dS_{ij} \tag{12}$$

8. The Maxwell-Ampère Theorem of Circuitation; Integral Form of the Third Field Equation.—By integrating both sides of the field equation

$$\operatorname{curl} H = i + \frac{\partial D}{\partial t} \tag{1}$$

with

$$H = \nu_0 B - M \tag{2}$$
$$D = \epsilon_0 E + P$$

over a cap surface S the following is obtained:

$$\int_{S(\text{cap})} (\hat{N}, \operatorname{curl} H)dS = \int_{S(\text{cap})} (\hat{N}, i)dS + \frac{\partial}{\partial t} \int_{S(\text{cap})} (\hat{N}, D)dS \tag{3}$$

Assuming continuity of the vector H, Stokes's theorem applied to the integral on the left gives

$$\int_{S(\text{cap})} (\hat{N}, \operatorname{curl} H)dS = \oint_{s(\text{line})} (H, ds) = \int_{S(\text{cap})} (\hat{N}, i)dS$$
$$+ \frac{\partial}{\partial t} \int_{S(\text{cap})} (\hat{N}, D)dS \tag{4}$$

The line integral in (4) is called the circuitation of the H vector, or simply the H circuitation around the closed contour s. The positive direction around the line is determined by the right-hand-screw convention referred to an arbitrarily directed normal $\hat{\mathbf{N}}$ to the surface S.

Since Stokes's theorem requires the vector to be continuous throughout the region of integration, special consideration must be given to the case where a cap surface is crossed by two or more dissimilar regions in contact because thin sheets of current of density $\mathbf{I}$ may then flow between them. If this is
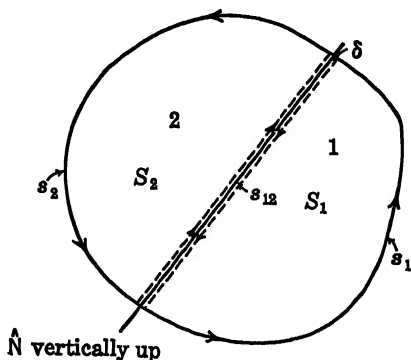
FIG. 8.1.—Region with boundaries for obtaining the Maxwell-Ampère theorem for a discontinuous vector. Small arrows indicate the directions of integration around regions 1 and 2.

the case, the H vector is discontinuous across the boundary between the two regions, as seen in the boundary condition (5.14). In order to examine this situation, consider two electrically dissimilar regions or bodies 1 and 2 in contact along a boundary surface S_{12} (Fig. 8.1). Let region 1 be characterized by H_1, D_1, and i_1; region 2 by H_2, D_2, and i_2. For any closed contour in either region, (6.8) is valid. On the other hand, suppose a contour s to be drawn in such a way that it cuts the boundary surface S_{12}. Then any cap surface bounded by s is partly in region 1, partly in region 2. For simplicity, let the cap surface S be taken to be a plane perpendicular to the direction of the boundary surface S_{12} between the two regions. Let the line traced by the boundary surface where it intersects the plane S be s_{12}. This line cuts the plane cap surface S into two parts S_1 and S_2 so that $S_{cap} = S_1 + S_2$. The closed contour s bound-

ing the plane is also cut into two parts s_1 and s_2 so that $s = s_1 + s_2$. S_1 is bounded by s_1 and one side of s_{12}; S_2 is bounded by s_2 and the other side of s_{12}. Let $\hat{N}$ be a normal perpendicular to S and hence to s_{12}. $\hat{N}$ is evidently parallel to the boundary surface S_{12} between the two regions where this is cut by the plane S.

In order to form the circuitation of the H vector completely around the contour s, it is necessary to exclude the line of discontinuity s_{12} by drawing a line parallel to it on each side at a small distance δ of atomic or molecular magnitude. The length of each of these two lines is essentially the same as that of s_{12}. A surface current of density I if it exists is necessarily confined to the strip of thickness 2δ since I is defined only in this strip. Because each region is homogeneous, it is legitimate to form a line integral around S_1 and a second line integral around S_2. The path of integration for the first of these is along s_1 and s_{12} in a counterclockwise direction when viewed from above. This follows from the fact that the normal $\hat{N}$ to the surface is directed vertically upward from the plane of the paper. Similarly, the path around S_2 will be counterclockwise along s_2 and s_{12}. The integrals may be taken along s_{12} instead of along lines parallel to s_{12} at distances δ on each side because the surface contained between these lines is negligible. One of the two integrals is

$$\oint_{s_1+s_{12}} (H_1, ds) = \int_{S_1} (\hat{N}, i) dS + \frac{\partial}{\partial t} \int_{S_1} (\hat{N}, D_1) dS \qquad (5)$$

The second integral is like (5) with subscript 1 written for 2 and 2 for 1. The sum of these two integrals is the same (neglecting the length δ) as the counterclockwise integral completely around $S_1 + S_2 = S$, plus the clockwise, or minus the counterclockwise, integral around the boundary line s_{12} at a distance δ from it. The latter integral is, of course, essentially equivalent to an integral along one side of s_{12} and back along the other side, since δ is negligible. That is,

$$\oint_{s_1+s_{12}} (H_1, ds) + \oint_{s_2+s_{12}} (H_2, ds) = \oint_{s_1+s_2} (H, ds) - \int_{s_{12}} (H_1, ds)$$
$$- \int_{s_{12}} (H_2, ds) \qquad (6)$$

The negative sign before the last two integrals indicates that

the direction of integration around the two sides of the line s_{12} has been changed from clockwise to counterclockwise. The value of H in the first integral on the right in (6) is taken to be H_1 or H_2 depending upon in what region ds is. With $s = s_1 + s_2$ and integrating along both sides of s_{12} in the same direction instead of around s_{12}, (6) may be written as follows:

$$\oint_s (H,ds) = \oint_{s_1+s_{12}} (H_1,ds) + \oint_{s_2+s_{12}} (H_2,ds) + \int_{s_{12}} (H_1 - H_2,ds) \quad (7)$$

By multiplying both sides of the boundary condition (5.14) scalarly by $\hat{N}$,

$$(\hat{N},[\hat{n}_1,H_1]) + (\hat{N},[\hat{n}_2,H_2]) = -(\hat{N},I) \quad (8)$$

Here $\hat{n}_1$ and $\hat{n}_2$ are external normals to the boundary between the regions 1 and 2. Since $\hat{n}_2 = -\hat{n}_1$, (8) can be written

$$(\hat{N},[\hat{n}_1,H_1 - H_2]) = -(\hat{N},I) \quad (9)$$

By cyclic permutation of the factors of the double product,

$$(H_1 - H_2,[\hat{N},\hat{n}_1]) = -(\hat{N},I) \quad (10)$$

But since $\hat{N}$ and $\hat{n}_1$ are not only normal to each other but also both perpendicular to s_{12}, it follows that the vector product $[\hat{N},\hat{n}_1]$ must be in the direction ds along s_{12}. Upon writing

$$[\hat{N},\hat{n}_1] = -\frac{ds}{|ds|} \quad (11)$$

and substituting in (10)

$$(H_1 - H_2,ds) = (\hat{N},I)ds \quad (12)$$

Upon combining two integrals like (5) using (7) and (12),

$$\oint_s (H,ds) = \sum_{j=1}^{2} \left\{ \int_{S_j} (\hat{N},i)dS_j + \frac{\partial}{\partial t} \int_{S_j} (\hat{N},D)dS_j \right\} + \int_{s_{12}} (\hat{N},I)ds \quad (13)$$

But

$$\sum_{j=1}^{2} \int_{S_j} (\hat{N},i)dS_j + \int_{s_{12}} (\hat{N},I)ds = I \quad (14)$$

is the total normal current crossing the cap surface S. If M is defined using a subdivision that cuts through no bound groups, as is customary when the H vector is used, I is the total moving *free* charge and the subscript f is written on I, i, and I.

$$\oint_s (\mathsf{H}, d\mathsf{s}) = I + \frac{\partial}{\partial t} \int_{S_{cap}} (\hat{\mathsf{N}}, \mathsf{D}) dS \qquad (15)$$

with

$$S_{cap} = \sum_{j=1}^{2} S_j$$

The above relations are easily generalized to include more than one surface of discontinuity. The Ampère-Maxwell circuitation theorem is expressed symbolically in (15). In words, the circuitation of the H vector around a closed contour s is equal to the total normal current plus the time rate of change of the normal D flux across any cap surface bounded by the contour.

The right side of the Ampère-Maxwell relation consists of two terms of which the first is a current in the legitimate sense of moving charge. The second term was called the "displacement current" by Maxwell, and this name continues to be used. Actually this terminology is unfortunate because the word displacement belongs to the old ether model and because the word current means specifically moving charge. Since, by definition, $\mathsf{D} = \epsilon_0 \mathsf{E} + \mathsf{P}$, this so-called displacement current actually consists of two parts. These are

$$\frac{\partial}{\partial t} \int_S (\hat{\mathsf{N}}, \mathsf{D}) dS = \epsilon_0 \int_S (\hat{\mathsf{N}}, \dot{\mathsf{E}}) dS + \int_S (\hat{\mathsf{N}}, \dot{\mathsf{P}}) dS \qquad (16)$$

It will be recalled that $\dot{\mathsf{P}}$ is the mathematical analogue of a motion of charge in the physical model due to the fluctuation of an orientation-distortion effect. It was called the volume density of polarization current (Sec. I.23). Accordingly, the integral on the right measures the total normal polarization current that traverses the cap surface S. It is legitimately called a current and may be represented by the symbol I_p. On the other hand, the first term on the right stands for the time rate of change of the E vector which has no physical analogue whatsoever. It characterizes the electrical properties of a purely mathematical space in terms of the mathematically defined

density functions.   Thus, $\epsilon_0\dot{\mathsf{E}}$ is defined in terms of moving charge or current, but it does not itself stand for moving charge in the physical model.   In the present treatment, the name current is reserved exclusively for such quantities as are direct mathematical analogues of moving charge in the physical model. Consequently, neither the time rate of change of the total normal E flux nor that of the D flux across a cap surface may be called a current.   The former is not the analogue of a current, while the latter is a mathematically convenient shorthand for two mathematically related quantities of which the one is the analogue of a current while the other is not.

An alternative formulation of the circuitation theorem proceeds from the field equation

$$\nu_0 \, \mathrm{curl} \, \mathsf{B} = \overline{\rho_m \mathsf{v}} + \epsilon_0 \frac{\partial \mathsf{E}}{\partial t} \tag{17}$$

instead of from (1).   By the same formal reasoning

$$\nu_0 \oint_s (\mathsf{B},\mathrm{d}s) = \bar{I} + \frac{\partial}{\partial t} \epsilon_0 \int_{S_{\mathrm{cap}}} (\hat{\mathsf{N}},\mathsf{E})dS \tag{18}$$

with

$$\bar{I} = \int_{S_{\mathrm{cap}}} (\hat{\mathsf{N}},\overline{\rho_m \mathsf{v}})dS + \sum_{ij} \int_{s_{ij}} (\hat{\mathsf{N}},\overline{\eta_m \mathsf{v}_{ij}})ds_{ij} \tag{19}$$

$S_{\mathrm{cap}} = \Sigma S_j$; s_{ij} is the boundary between S_i and S_j.   In this formulation the entire moving charge or current is contained in the term $\bar{I}$.

9. Faraday's Law of Circuitation; Integral Form of the Second Field Equation.—Let N be a normal to a cap surface S bounded by a closed contour s.   The components normal to S of the vectors defining each side of the second field equation (2.2) satisfy the relation

$$(\hat{\mathsf{N}}, \mathrm{curl} \, \mathsf{E}) = - \frac{\partial}{\partial t} (\hat{\mathsf{N}},\mathsf{B}) \tag{1}$$

If each side is integrated over the cap surface and Stokes's theorem is applied,

$$\int_{S(\mathrm{cap})} (\hat{\mathsf{N}}, \mathrm{curl} \, \mathsf{E})dS = \oint_s (\mathsf{E},\mathrm{d}s) = - \frac{\partial}{\partial t}\int_{S(\mathrm{cap})}(\hat{\mathsf{N}},\mathsf{B})dS \tag{2}$$

If the cap surface is cut by one or more boundaries, the integral over the cap is replaced by a sum of integrals over the several

parts of the cap. Faraday's law is symbolically given in (2). It may be stated as follows: The circuitation of the E vector in the positive direction around any closed contour in which E is continuous is equal to the negative time rate of change of the total normal B flux across any cap surface bounded by the contour. The positive direction is specified by the right-hand-screw convention referred to the arbitrarily directed normal $\hat{\mathsf{N}}$ to S.

10. Gauss's Theorem for the B Vector; Integral Form of the Fourth Field Equation.—By applying (6.1) to div $\mathsf{B} = 0$,

$$\int_V \text{div } \mathsf{B}\, dV = \int_{S(\text{closed})} (\hat{\mathsf{n}}, \mathsf{B}) dS = 0 \tag{1}$$

The equation states that the total outward normal flux of the B vector across any closed surface vanishes.

11. Summary of Integral Field Equations.—In terms of the fundamental vectors the theorems are

(Gauss) $$\epsilon_0 \int_{S(\text{closed})} (\hat{\mathsf{n}}, \mathsf{E}) dS = \bar{Q} \tag{1}$$

(Faraday) $$\oint_s (\mathsf{E}, d\mathsf{s}) = -\frac{\partial}{\partial t} \int_{S(\text{cap})} (\hat{\mathsf{N}}, \mathsf{B}) dS \tag{2}$$

(Ampère-Maxwell) $$\nu_0 \oint_s (\mathsf{B}, d\mathsf{s}) = \bar{I} + \frac{\partial}{\partial t} \epsilon_0 \int_{S(\text{cap})} (\hat{\mathsf{N}}, \mathsf{E}) dS \tag{3}$$

(Gauss) $$\int_{S(\text{closed})} (\hat{\mathsf{n}}, \mathsf{B}) dS = 0 \tag{4}$$

Here

$$\bar{Q} = \sum_j \int_{V_j} \bar{\rho}_j dV_j + \sum_{ij} \int_{S_{ij}(\text{closed})} \bar{\eta}_{ij} dS_{ij} \tag{5}$$

$$\bar{I} = \int_{S(\text{cap})} (\hat{\mathsf{N}}, \overline{\rho_m \mathsf{v}}) dS + \sum_{ij} \int_{s_{ij}} (\hat{\mathsf{N}}, \overline{\eta_m \mathsf{v}_{ij}}) ds_{ij} \tag{6}$$

S_{cap} is a cap surface with edge s; if surface currents cross it and divide it into parts S_j, S_{cap} is equivalent to $\sum_j S_j$. s_{ij} is a boundary between S_i and S_j; the sum includes the contour s if a surface current crosses this.

In terms of the auxiliary vectors

(Gauss) $$\int_{S(\text{closed})} (\mathsf{n}, \mathsf{D}) dS = Q \tag{7}$$

(Faraday) $$\oint_s (\mathsf{E}, d\mathsf{s}) = -\frac{\partial}{\partial t} \int_{S(\text{cap})} (\hat{\mathsf{N}}, \mathsf{B}) dS \tag{8}$$

(Ampère-
Maxwell)
$$\oint_s (H, ds) = I + \frac{\partial}{\partial t} \int_{S(cap)} (\hat{N}, D) dS \tag{9}$$

(Gauss)
$$\int_{S(closed)} (\hat{n}, B) dS = 0 \tag{10}$$

Here

$$Q = \sum_j \int_{V_j} \rho_j dV_j + \sum_{ij} \int_{S_{ij}} \eta_{ij} dS_{ij} \tag{11}$$

$$I = \int_{S(cap)} (\hat{N}, i) dS + \sum_{ij} \int_{s_{ij}} (\hat{N}, l_{ij}) ds_{ij} \tag{12}$$

$$\cdot D = \epsilon_0 E + P \tag{13}$$
$$H = \nu_0 B - M$$

ELECTROMAGNETIC FORCE

12. Definition of Electromagnetic Force and Torque.—One of the original postulates of the atomic model was a repulsion-attraction effect between stationary and moving charges. This in itself represented a purely qualitative assignment of a property which, up to this point, has served only as a somewhat vague cause for such separations and motions of charge as were desired in constructing the density fields. In terms of the physical model of bodies and regions composed of billions of widely separated charges in motion, the problem of attractions and repulsions between the charges of separate bodies seems hardly different from the problem of the attractions and repulsions between the charges of a single homogeneous body. However, with the physical model of each body replaced by a small number of continuous density functions, the interaction of charges in different bodies becomes specifically a problem of formulating the interaction between the density fields characterizing these bodies. One of the density fields associated with every body is the density of mass. Relative motions and conditions of equilibrium of such density fields are described by the mathematical model of mechanics in a manner that has proved itself entirely adequate in predicting experimental analogues in the case of *uncharged* bodies. It has been so successful, in fact, that the attempt is almost always made to devise mechanical models for phenomena of the most diverse kinds, often including the electromagnetic. Since a large proportion of pointer-reading analogues in so-called electrical measurements are actually experimentally observed using mechanical contrivances, a con-

nection between the mathematical model of mechanics and the mathematical model of electromagnetism must be provided. However, there seems to be no need to attempt to combine their physical models, if this involves great flights of the imagination. Mechanics has defined a convenient vector, called the force vector, for determining conditions of motion or of equilibrium between density of mass fields, or simply between masses. This vector, together with the standard of mass, is defined in terms of the following fundamental equation of motion of the center of mass of a body

$$F_M = \frac{d}{dt}(mv) \tag{1}$$

where $m = \int_\tau D \, d\tau$ and v is the velocity of the center of mass.

Since every body may be characterized by the mechanical property of mass and the electrical properties of charge and distribution of charge, it is clear that any condition of equilibrium or of motion may, in general, be explained in terms of mechanical forces and electrical repulsion-attraction effects. It is, therefore, convenient to define an electromagnetic force to represent these latter and to describe the conditions of motion or of rest of all bodies in terms of the action of both electromagnetic and mechanical forces. Thus, in order to complete the mathematical model of the electrical properties of matter in space, it is convenient to define an electromagnetic vector force which will relate the mathematical model of electrodynamics with that of mechanics. This vector must take account of the attraction-repulsion effects of bodies containing stationary and moving charges, and it must do so in terms of the density functions of all bodies in the electromagnetic field.

The resultant *electromagnetic force* F and *torque* T, acting on a body, must be so defined that the following conditions of equilibrium are satisfied for a body at rest:

$$F_M + A_e F = 0 \tag{2a}$$
$$T_M + A_e T = 0 \tag{2b}$$

Here F_M and T_M are the resultant mechanical force and torque acting on the body, and the factor A_e is a constant depending upon the relation between the electrical and the mechanical units of force. It may be defined either as a fundamental constant to be

determined experimentally as the mechanical equivalent of electricity, or it may be assigned an arbitrary dimensionless numerical value. The latter alternative is chosen in the practical system of units in which A_e is dimensionless and equal to one. If there is no mechanical force to balance the electrical force, the body is accelerated. A mechanical force of inertia may then be written in the familiar form

$$\mathsf{F}_M = -\frac{d}{dt}(m\mathsf{v}) \tag{3}$$

Here v is the velocity of the center of mass of the body of mass m. With $A_e = 1$, the force equation (2) then becomes the following equation of motion:

$$\mathsf{F} - \frac{d}{dt}(m\mathsf{v}) = 0 \tag{4}$$

The definition of the electromagnetic force F which is to be used in (2) may be looked upon as a third fundamental postulate of electrodynamics. Its justification, like that of the atomic picture or of the field equations, must be sought in the ultimate success of the mathematical model of electromagnetism in predicting theoretical analogues of experimentally observable pointer readings.

The electromagnetic force F and vector torque T acting on a volume τ characterized by the densities ρ, η, $\overline{\rho_m\mathsf{v}}$, $\overline{\eta_m\mathsf{v}}$ are defined to be

$$\mathsf{F} = \int_\tau \{\rho\mathsf{E} + [\overline{\rho_m\mathsf{v}},\mathsf{B}]\}d\tau + \int_\Sigma \{\bar{\eta}\mathsf{E} + [\overline{\eta_m\mathsf{v}},\mathsf{B}]\}d\sigma \tag{5}$$

$$\mathsf{T} = \int_\tau [\mathsf{r},d\mathsf{F}_\tau] + \int_\Sigma [\mathsf{r},d\mathsf{F}_\sigma] \tag{6}$$

In (6) r is the vector from an arbitrary origin to the element $d\tau$ or $d\sigma$; $d\mathsf{F}_\tau$ is the integrand of the volume integral, $d\mathsf{F}_\sigma$ the integrand of the surface integral in (5). In (5) and (6) E and B are calculated from the field equations in terms of the continuous volume and surface densities of all bodies and regions *exterior* to τ. If τ is at rest relative to the observer,[1] the densities $\bar{\rho}$, $\bar{\eta}$,

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[1] If τ is not at rest relative to the observer, $\overline{\rho_m\mathsf{v}}$, originally defined in terms of motion relative to τ, must be generalized as suggested in Sec. 2 to include all the charge moving relative to the observer.

$\overline{\rho_m v}$, and $\overline{\eta_m v}$ are the essential characteristics of charge and moving charge already defined as follows:

$$\tilde{\rho} \equiv \rho - \operatorname{div} \mathsf{P}; \qquad \overline{\rho_m v} \equiv \mathsf{i} + \operatorname{curl} \mathsf{M} + \dot{\mathsf{P}} \tag{7}$$
$$\tilde{\eta} \equiv \eta + (\mathsf{n,P}); \qquad \overline{\eta_m v} \equiv \mathsf{I} - [\mathsf{n,M}]$$

Since these densities are obtained by interpolation from averages defined at the center of each volume or surface cell, it is clear that the values of E and B, and hence of F calculated from them, can have a meaning only at points that are far from the density fields contributing to E and B as compared with the dimensions of the volume cells used in defining them. That is, the electromagnetic force acting on a certain volume τ_1 due to charges and currents in a second region τ_2 can be calculated from (5) only if each point of τ_1 is very much farther from every point in τ_2 than the dimension of a typical volume cell in τ_2. In defining the continuous densities in τ_2 by interpolation, the average electrical properties of each volume cell were, in effect, assumed to be concentrated at its center. The implication is that the densities correctly represent these average properties only if it may be assumed without serious error that all charges in each volume cell are at the same distance from the point where E and B are calculated as is the center of the cell in question. Evidently this restriction is not important insofar as the mutual interaction of bodies as a whole is concerned.

Although the definition of electromagnetic force and the force equation are fundamental in the formulation of electromagnetic theory and its coordination with mechanics, many problems can be solved without reference to either of these. The reason is that the part of the problem which is associated with mechanical force is often solved in advance in the ammeter, voltmeter, or other device used in obtaining pointer readings which is calibrated directly in terms of electrical quantities.

13. Dynamical Equation.—The electromagnetic force F is defined by (12.5) specifically for use in equation (12.2). Since this applies to conditions of equilibrium of bodies as a whole, the integrals in (12.5) are necessarily evaluated over the entire volume and surface of the body which is acted upon by the mechanical force F_M. There is, however, no reason why the resultant electromagnetic force F acting on a body as a whole should not be considered to be the vector sum of any number of

components.   For example, the components acting on different
sections of the body might be considered separately.   That is,
the component of electromagnetic force acting on each of a large
number of small volumes into which the body is subdivided
might be calculated.   By vector addition the same resultant
force could be obtained.   Since the densities are by definition
continuous functions in any body or region, it is quite legitimate
to calculate the average electromagnetic force acting on any
volume however small, provided only that it is a part of a larger
region in which the density functions are properly defined.
It is not legitimate to use (12.5) to calculate the electromagnetic
force on a region that is not much larger than a typical volume
cell, and which is not a part of a more extensive, homogeneous
body, because statistical conditions cannot be assumed to prevail
and no density functions can be defined.   However, if the
electromagnetic field due to all bodies surrounding a region in
which volume and surface densities are properly defined is
sufficiently slowly varying so that it may be assumed to be
sensibly constant over the entire region, a simplification is
possible.   In this case E and B may be removed from under the
signs of integration in (12.5) and there remains

$$F = qE + [q_m v, B] \tag{1}$$

with

$$q = \int_\tau \bar{\rho} \, d\tau + \int_\Sigma \bar{\eta} \, d\sigma$$
$$q_m v = \int_\tau \overline{\rho_m v} \, d\tau + \int_\Sigma \overline{\eta_m v} \, d\sigma \tag{2}$$

In order to evaluate (2) it must be expressed in component form,
using any convenient system of coordinates, the integrations
carried out for each component separately, and the results
combined.

It is evidently immaterial whether the total charge q in τ
is determined by first constructing continuous density func-
tions and integrating or by direct summation of the charges.
The same is true for $q_m v$.   It follows that (1) may be used for as
small a region as desired if q and $q_m v$ are calculated by direct
summation.   If q and q_m coincide, so that all charges are in
motion as in an electron stream, (1) reduces to the following
simple form

$$F = q\{E + [v, B]\} \tag{3}$$

This expression may be used to define the electromagnetic force acting on a small group of charges or even on a single charge moving with nonrandom velocity v, if E and B are sensibly constant over its extension, and if it is at a distance from all charged regions which is large compared with the dimensions of the volume cells used in constructing the densities entering into the definitions of E and B.

As a consequence of the restrictions implied in writing (3), this cannot be used to calculate the electromagnetic force acting on a charge in the interior of a body or region.   It may be used to calculate the component of electromagnetic force on an interior charge due to the entire charge distribution *outside* of a sphere drawn around the charge and with radius large compared with a typical volume cell.   The force due to the charge distribution within this sphere, the so-called local force, cannot be calculated using (3).   For certain symmetrical distributions, it can be shown to be zero.   In other cases, more or less well-founded assumptions must be made.

If the entire electromagnetic force F acting on a charge q moving with mean velocity v is given by (3) and no other forces are involved, the charge will be accelerated.   If its mass is M, the mechanical force resisting acceleration is

$$F_M = -\frac{d}{dt}(M v) \tag{4}$$

so that the equation of motion is

$$q\{E + [v,B]\} = \frac{d}{dt}(M v) \tag{5}$$

If the charge is an electron, $q = e = -|e|$, $M = m$, and (5) is called the dynamical equation of the electron.

SIMPLE MEDIA

14. Constitutive Parameters for Polarization, Magnetization, and Current.—In describing the physical model of matter it was assumed that the normal state for any region or body containing electric charges was that of complete statistical equilibrium with all charges moving in a random distribution.   As a consequence of the postulated principle of mutual interaction, such a condition can be achieved only by an extreme separation of a

region or body from all others. It will be recalled, however, that all the density functions were introduced precisely in order to take account of nonuniform and nonrandom distributions produced by the proximity of other charged regions. Thus, in particular, the volume density of polarization was defined as a means of describing mathematically an average distortion-orientation effect of the charges in the bound-charge model. The volume density of magnetization was similarly constructed to describe an average circulation-orientation effect in a closely bound configuration of moving charges called the spin model. The volume density of convection current was designed to represent an average drift of charge. The density functions so introduced were defined exclusively in terms of the effects called polarization, magnetization, or drift of charge, and not at all in terms of the external influences acting to produce and maintain them.   Indeed, no attempt was made to explain physically or to define mathematically how the conditions of charge on one body might produce nonrandom conditions in another.   It was merely assumed that they could. With the definition of the electromagnetic vectors E and B by the field equations, and the definition of the electromagnetic force acting on a region in terms of these vectors, means have been provided for interconnecting mathematically the density fields characterizing charged regions which are sufficiently near each other to interact significantly.

As a first consideration, it is to be noted that the average values of E and B calculated at a particular point are completely determined through the field equations by the density fields of all bodies. If one of these should be removed to an extremely remote location, new and different values of E and B would necessarily be computed at the same point.   The change would be a composite measure both of the direct contribution of the density fields of the removed body to E and B and of its effect in changing the density fields of all other bodies that contribute to E and B.   Thus, the computed values of the field vectors in the two cases differ not only because one body is removed and its density fields vanish, but also because the density fields of all bodies depend upon each other and the removal of one changes those remaining.   It is thus clear that a complicated functional interrelationship exists between the densities characterizing all bodies. Since E and B are functions of all density fields, it is

evidently possible to express the interconnection of the densities P, $-M$, and i_f of one body with those of all others, with the following functional symbolism:

$$P = P\{E,B\} \tag{1}$$
$$-M = M\{E,B\} \tag{2}$$
$$i_f = i_f\{E,B\} \tag{3}$$

Here i_f is the volume density of drifting *free* charge only, and (1), for example, means that P is a function of E and B.

In order to discover convenient and useful forms of these general functional relationships according to which the qualitatively described bound-charge, spin, and free-drift models may be characterized mathematically, a reconsideration of the simple physical models invented as a basis for the construction of the densities P, $-M$, and i_f is suggested. It will be recalled that in the case of all three functions special models endowed with particular kinds of internal constitutive forces were used. These were assumed to act against an external influence such as an electromagnetic force seeking to disturb a normal, random distribution. And the final condition of balance or equilibrium was called, respectively, in the three cases, a condition of polarization, magnetization, or steady motion of charge. Thus, polarization was pictured to be the direct result of an attraction of one kind of charge and a repulsion of the other in a closely bound configuration of an atom, molecule, or group of molecules. This attraction-repulsion tended to distort the configuration by separating the relative statistical rest positions of the positive and negative charges forming the group. It was assumed that this distorting electromagnetic force was opposed by strong internal forces of constitution tending to maintain the normal, random distribution with its coincident statistical rest positions of the positive and negative charge. Since the resulting condition of polarization is one of balance, it is evidently a static state involving no nonrandom drift or circulation of charge. The electromagnetic force (12.5) acting on the charge in each volume τ therefore reduces to the electrostatic force

$$F = \int_\tau \bar{\rho}E \, d\tau \tag{4}$$

The mean statistical force acting to separate positive and negative charges must be of the form (4) but written for the positive

(or the negative) charge only.   Without actually attempting to analyze the force on each charge in the individual atom, it is clear that its statistical effect involves only the field vector E and not the vector B.   Consequently, one may express the functional relationship between the volume density of polarization P in one body and the density fields of all bodies in the form

$$P = P\{E\} \tag{5}$$

In the case of the spin model, an analogous situation arises. The picture of magnetization in the physical model is that of a steady circulation about a definitely oriented axis superimposed upon the random motions and orbits of the charges.   It is produced by an electromagnetic force acting against strong internal forces tending to maintain the normal state of random orientation.   A force that is to act on moving charges to maintain a circulation about a definite axis must act at right angles to the direction of motion of the charges.   This is true only of the magnetic term in the electromagnetic force.   It is

$$F = \int_\tau [\overline{\rho_m v}, B] d\tau \tag{6}$$

The electrostatic term (4), on the other hand, can contribute nothing to a microscopic circulation, since E (as calculated from charge distributions outside of τ) is a slowly varying function in each region not only in magnitude but also in direction. Without investigating the nature of the interior forces acting to produce an oriented circulation insofar as the individual atom is concerned, it may be expected that its statistical effect involves only the vector B and not the vector E.   Therefore, it is reasonable to express the functional relationship between the volume density of magnetization M in one body and the density fields of all bodies in the form

$$-M = M\{B\} \tag{7}$$

A steady drift of electric charge throughout the interior of a region is most simply described in terms of a free-charge model under the action of the same repulsion-attraction influence that was pictured as producing a distortion-orientation in the bound-charge model.   By definition, a free-charge model contains a considerable proportion of free electrons or mobile ions

which are statistically free to wander independent of any constitutive forces that may act to maintain a charge distribution characteristic of a particular atomic structure. An electromagnetic force acting on mobile electrons or ions sets these in motion and continues to act to accelerate them as long as they are prevented from accumulating at a boundary. If the charges are not accelerated beyond a certain constant velocity, it must be concluded that internal forces (other than the inertial ones of charges moving in a nonrandom manner) have been brought into play. Without considering the nature of these forces, it is clear that the electromagnetic force that maintains a steady drift of charge must act in a direction parallel to the direction of motion of the charges. A force acting at right angles to the direction of motion necessarily produces a circulation and not a unidirectional drift. It follows, therefore, that a steady drift of free charges must be maintained by an electromagnetic force that involves directly only the vector E and not the vector B.[1] Consequently, it is reasonable to write a functional relationship of the form

$$i_f = i_f\{\mathsf{E}\} \tag{8}$$

The specialization of the functional relationships (1) to (3) to suit the atomic models that may be characterized, respectively, by polarization, magnetization, and steady drift of free charge has led to the simpler forms (5), (7), (8). These are still quite general and leave practically unrestricted the nature of the internal forces that act against the electromagnetic force in each case to maintain the condition of equilibrium represented by a stationary state of polarization, magnetization, or steady drift. In order to express these relations in a more useful form, let it be assumed that the body or region in question is isotropic and homogeneous. Let it be assumed, in addition, that the general functions (5), (7), (8) may be expanded in converging

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[1] It is to be noted that the vector E is functionally related to B in the unsteady state through the definition of its curl $\mathsf{E} = -\dot{\mathsf{B}}$. Every nonstationary condition involves both a value of $\dot{\mathsf{B}}$ different from zero and a nonvanishing value of curl E and hence of E. This is equivalent to stating that an unsteady electric force of the form (4) but with E a function of the time exists to maintain an unsteady convection current density if in a region containing free charges.

power series of the form

$$P = P_0 + P_1E + P_2EE + \cdots \tag{9}$$
$$-M = -M_0 + M_1B + M_2BB + \cdots \tag{10}$$
$$i_f = i_0 + i_1E + i_2EE + \cdots \tag{11}$$

(If the body is not isotropic, each component of the vectors P, $-M$, and i_f along principal axes must be expanded separately in terms of different coefficients.) In (9) to (11) the parameters with subscripts are independent of the electromagnetic vectors; in homogeneous bodies they are also independent of the coordinates.

The terms of zeroth order in (9) to (11) are independent of the field vector in each case. They represent an intrinsic, permanent polarization, magnetization, or drift of charge which is a fundamental characteristic of a particular structure. In the case of polarization and magnetization, nonvanishing values of P_0 and M_0 require a statistically permanent nonrandom orientation of bound-charge and spin groups, respectively, which are themselves individually characterized by permanent electric and magnetic moments. Although such structures have not been included in the description of matter, which is based on the assumption that random conditions prevail in the absence of external agencies, they are, nevertheless, physically reasonable and may be incorporated in the mathematical formulation by permitting P_0 and M_0 to differ from zero. Bodies that are well represented by mathematical analogues in which P_0 and M_0 do not vanish are said to be permanently or intrinsically polarized or magnetized and are called electrets and magnets, respectively. Ferromagnetic substances are analyzed in terms of M_0. Such bodies will not be considered explicitly in this formulation which is limited to bodies in which polarization and magnetization vanish when no external forces are acting. Accordingly, it is required that

$$M_0 = 0 = P_0 \tag{12}$$

In the case of freely drifting charge, a nonvanishing value of the zeroth-order term i_0 requires a permanent convection or conduction current to exist in an isolated, homogeneous body. Special models in which such a flow is possible (once it has been started by an external agency) are considered in a study of

superconductivity. They are excluded from this treatment, which is limited to models in which

$$i_0 = 0 \qquad (13)$$

With the atomic model specialized to the extent that the zeroth-order terms are excluded from the series (9) to (11), an extremely intricate mathematical formulation still exists except in cases in which the series converge so rapidly that all second- and higher order terms are negligible compared with the leading first-order term. For a homogeneous, isotropic body it is in this case possible to write the following simple expressions:

$$\mathsf{P} = P_1 \mathsf{E} = \epsilon_0 \chi \mathsf{E} \qquad (14)$$
$$-\mathsf{M} = M_1 \mathsf{B} = \nu_0 \chi_m \mathsf{B} \qquad (15)$$
$$i_f = i_1 \mathsf{E} = \sigma \mathsf{E} \qquad (16)$$

The parameters P_1, M_1, and i_1 are assumed to be independent of the field vectors and the coordinates. (In a nonisotropic body the parameters P_1, M_1, and i_1 are not scalars but linear vector functions.) Atomic models of such inner structure that the simple relations (14) to (16) are true with the three parameters constant are called *simple models* or *simple media*. In such models the parameters are assigned the symbols $\epsilon_0 \chi$, $\nu_0 \chi_m$, and σ.

The mathematical description of the characteristic structural properties of simple models is confined to the three constant parameters χ, χ_m, and σ. Like the density functions they can describe only average statistical properties of regions containing large numbers of units. The parameter χ, which characterizes the polarizability of a model, is called the *electric susceptibility*. The parameter χ_m, which represents the magnetizability, is assigned the name *magnetic B susceptibility*.[1] Finally, σ, which is related to the average mobility of charges, is named the *conductivity*. Since both P and M depend upon the mode of subdivision of a body into volume and surface cells, the values of the parameters χ and χ_m vary correspondingly. It is, therefore, necessary to specify a particular mode in order that these

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[1] The symbol χ_{mH} is assigned to the conventionally used magnetic H susceptibility. This is defined by writing $\mathsf{M} = \chi_{mH}\mathsf{H}$, instead of as in (15), choosing H instead of B as the fundamental magnetic vector and treating M instead of $-\mathsf{M}$ as the analogue of P. See also the footnote for (15.6b).

parameters may be defined uniquely. *This is accomplished by selecting a subdivision into volume cells which cuts through no closely bound groups or whirls.* Since i_f is independent of the mode of subdivision, no ambiguity can arise in defining σ.

If it is desired to consider more intricate models in which the general functional relations (9) to (11) are not adequately represented in the linear form (14) to (16), it is possible to retain additional terms in the series and define second- and higher order constitutive parameters. As a consequence, second- and higher powered terms in E and B must appear in the field equations along with second- and higher order polarization, magnetization, and conduction parameters. The mathematical complexity is in this way enhanced beyond hope of reasonably simple solution. An alternative procedure is to define continuous densities in addition to the six already in use. In the bound-charge model, for example, it is possible to introduce density functions to represent the distribution of quadrupoles and higher order multipoles in addition to the representation of the distribution of dipoles by the volume density of polarization. This method is also complicated but is used in a study of substances composed of polar molecules. A superficially simple way for side-stepping the difficulty is to write the linear relations (14) to (16) and allow the parameters χ, χ_m, and σ to be functions of the field vectors E or B instead of requiring them to be constants. Such a procedure naturally excludes the possibility of an exact solution for E and B, since the parameters appearing in their definitions are unknown functions of E and B. In some instances, however, it is in this way possible to obtain approximate or partial solutions by first determining the variation of the parameter with the field vector, as in the case of ferromagnetic substances.

If the constitutive parameters are functions neither of the coordinates nor of the field vectors but of a nonelectromagnetic variable such as the temperature, no change in (14) to (16) is necessary if the temperature is held constant and appropriate values of the parameters are used. This is true provided there is no fundamental change in the internal conditions upon which the phenomena of polarization, magnetization, or steady drift depend. Such effects might be due to a modification in structure at extremely high or low temperatures, to changes in state, etc., which are beyond the scope of the present discussion. An impor-

tant case is the dependence of the parameters upon time.   It is to be noted that the linear relations (14) to (16) actually assume an instantaneous response of the charges in a body to changes in the value of E or B if the parameters are assumed to be constants.   This is an impossible requirement in any system in which inertial or other forces not independent of the motion of the charges must be overcome, so that in writing (14) to (16) it is implicitly assumed that a time of lag in response or of relaxation is sufficiently short to play no significant part in the time intervals during which the simple constitutive relations are to be used.   These relations are certainly not valid during short intervals of unsteadiness in changes from one stationary state to another.   If an unsteady state prevails which involves periodic changes in E and B, the adequacy of the simple relations (14) to (16) depends entirely upon the relative magnitudes of the period involved and of the time of relaxation associated with the accompanying changes in the condition of polarization, of magnetization, or of steady drift.   If the time of lag or of relaxation is small compared with the period of E or B, it is possible to define sensibly constant parameters, so that one may properly speak of simple media.   If the order of magnitude of the two time intervals is the same, the constitutive parameters become complicated functions of the frequency and of the detailed molecular or atomic structure.   It follows that to define a simple model by requiring one or all of the relations (14) to (16) to be a good approximation implies an upper frequency limit for periodic. variations of the density fields contributing to E or B.

15. Field Equations in Simple Media; Dielectric Constant, Permeability.—In simply polarizing, magnetizing, and conducting models, *i.e.*, in regions in which P, $-M$, and i_f are directly proportional to E or B, the field equations are simpler in form. This follows from the fact that instead of three density fields, which are complicated functions of internal and external forces, three constant statistical parameters appear.   In order to write down the simplified equations, it is convenient to consider first the form of the auxiliary vectors D and H as defined by the relations

$$D = \epsilon_0 E + P \tag{1}$$
$$H = \nu_0 B - M \tag{2}$$

Upon substituting the value $P = \epsilon_0\chi E$ and $-M = \nu_0\dot{\chi}_m B$,

$$D = (1 + \chi)\epsilon_0 E \tag{3}$$
$$H = (1 + \chi_m)\nu_0 B \tag{4}$$

Here the terms

$$(1 + \chi) \text{ and } (1 + \chi_m) \tag{5}$$

are dimensionless numerical factors depending only upon the characteristic electric or magnetic structure of the models for which χ and χ_m are defined. Each of these parenthetical factors is conveniently represented by a symbol that stands for the properties of simple polarizability or simple magnetizability. They are

$$\epsilon_r \equiv (1 + \chi) \quad \text{or} \quad \chi = (\epsilon_r - 1) \tag{6a}$$
$$\nu_r \equiv (1 + \chi_m) \equiv \frac{1}{\mu_r} \quad \text{or} \quad \chi_m = (\nu_r - 1) = \left(\frac{1}{\mu_r} - 1\right)^{*} \tag{6b}$$

The quantity ϵ_r is called the *relative dielectric constant* or *relative permittivity* of a simply polarizable medium; the quantity ν_r is called the *relative diamagnetic constant* or *relative reluctivity* of a simply magnetizable medium. The reciprocal of ν_r is μ_r; it is called the *relative permeability* of a simply polarizable medium. Because the products $\epsilon_0\epsilon_r$ and $\nu_0\nu_r$ occur very frequently, it is convenient to introduce special symbols for them. Let

$$\epsilon \equiv \epsilon_0\epsilon_r; \qquad \nu \equiv \nu_0\nu_r = \frac{1}{\mu} = \frac{1}{\mu_r\mu_0} \tag{7}$$

The names *absolute dielectric constant* or *permittivity* for ϵ, *absolute diamagnetic constant* or *reluctivity* for ν, and *absolute permeability* for μ will be used. In terms of these symbols

$$D = \epsilon_r\epsilon_0 E = \epsilon E \tag{8}$$
$$H = \nu_r\nu_0 B = \nu B \tag{9}$$
$$P = (\epsilon_r - 1)\epsilon_0 E = (\epsilon - \epsilon_0)E \tag{10}$$
$$-M = (\nu_r - 1)\nu_0 B = (\nu - \nu_0)B \tag{11}$$

Since χ and χ_m, and hence ϵ_r and ν_r, are *defined in terms of* P *and* $-M$, *using a subdivision into volume cells that cuts no closely bound groups*, it follows that there can be no contribu-

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* The conventionally defined magnetic H susceptibility is given by

$$\chi_{mH} = -\mu_r\chi_m = (\mu_r - 1) \tag{6c}$$

tions from cut parts of such groups either to the volume density of charge ρ or to the volume density of current i. Therefore, these are limited to contributions due to free charge so that $\rho = \rho_f;\ i = i_f$. Using (8) and (9) and the constitutive relation

$$i_f = \sigma E \qquad (12)$$

the field equations reduce to the following form in which ρ_f is the only density function appearing explicitly:

$$\epsilon\,\text{div}\ E = \rho_f \qquad (13)$$
$$\text{curl}\ E = -\dot{B} \qquad (14)$$
$$\nu\,\text{curl}\ B = \sigma E + \epsilon\dot{E} \qquad (15)$$
$$\text{div}\ B = 0 \qquad (16)$$

These relations are valid in simply polarizing, magnetizing, and conducting media. It is to be noted that if values of P and $-M$ are determined using (10) and (11) and values of ϵ_r and ν_r are obtained from these equations in any particular case, the values so obtained *must* be interpreted in terms of a subdivision that cuts no closely bound polarized or magnetized groups.

Since P and M are defined entirely in terms of volume cells, there can be no contributions to surface densities of charge or moving charge from cut-off surface slices of magnetized or polarized units. Any surface density must be due to free charge only. The surface equations, or boundary conditions, are

$$\epsilon_1(\hat{n}_1,E_1) + \epsilon_2(\hat{n}_2,E_2) = -\eta_{1f} - \eta_{2f} \quad \text{or} \quad \epsilon(\hat{n},E) = -\eta_f \quad (17)$$
$$[\hat{n}_1,E_1] + [\hat{n}_2,E_2] = 0 \quad \text{or} \quad [\hat{n},E] = 0 \quad (18)$$
$$\nu_1[\hat{n}_1,B_1] + \nu_2[\hat{n}_2,B_2] = -l_{1f} - l_{2f} \quad \text{or} \quad \nu[\hat{n},B] = -l_f \quad (19)$$
$$(\hat{n}_1,B_1) + (\hat{n}_2,B_2) = 0 \quad \text{or} \quad (\hat{n},B) = 0 \quad (20)$$

The simpler forms on the right are a shorthand for the explicit expressions on the left. It may seem surprising that surface densities appear in these equations when a subdivision into volume cells only was used in defining P and M. This is due to the fact that in the above representation a subdivision into surface and volume cells is retained for the free charges and a separate subdivision into volume cells only is used for the bound charges and current whirls.

The analogy between the static and the steady states may be extended further to include the following constitutive parameters in the special case of simple media.

Quantity	Static state	Steady state
Constitutive parameters in simple models	Electric susceptibility χ Relative dielectric constant ϵ_r Absolute dielectric constant or permittivity ϵ	Magnetic susceptibility χ_m Relative diamagnetic constant or reluctivity ν_r Absolute diamagnetic constant or reluctivity ν

16. Conductors and Nonconductors in the Stationary States.
The physical basis for the mathematical description of matter
is an atomic model constructed of free charges and bound
groups. Two extreme cases may be distinguished in terms of a
relative abundance and a complete absence of free charges
(electrons or ions). Models in which free charges are relatively
plentiful are called conductors; models in which they are completely
or practically absent are called nonconductors or dielectrics.
Intermediate models between these extremes may be
grouped together under the names semiconductors or imperfect
dielectrics. The gradation between nonconductors and conductors
is a gradual one, and it is possible to construct atomic
models of physically available substances that have almost any
desired proportion of free electrons from none to a maximum
in models (such as those of silver and copper) in which the two
outer electron shells are both only partly filled. For this
reason, it might seem unprofitable to construct special mathematical
models for the two extreme cases. Actually, it is
precisely for conductors and nonconductors that theoretical
functions can be constructed which are most successful in their
agreement with experimental analogues. The experimental
behavior of a large class of elements, the metals, is well portrayed
by the mathematical model of a conductor. The properties of
another large group of substances including glass, ceramic
materials, rubber, and polystyrene are correctly predicted in
terms of the model of a nonconductor. It is indeed fortunate
that this is the case, for the general mathematical model of the
semiconductor is complicated, whereas considerable simplification
is possible by introducing conditions appropriate for the two
extremes.

The fundamental definition of a conductor from the electrostatic
point of view is an atomic model that contains sufficient

free charges so that the mean statistical electromagnetic force acting on each interior charge (whether free or a part of a bound configuration) is at all times zero. Since the electrostatic force acting on a charge depends only upon E, it must be concluded that the interior of a conductor is characterized by

$$E = 0 \tag{1}$$

Since polarization in simple models is directly proportional to E, it follows that

$$P = \epsilon_0 \chi E = 0 \tag{2}$$

Furthermore, it follows from the field equation

$$\epsilon_0 \text{ div } E = \rho - \text{ div } P \tag{3}$$

that (1) and (2) lead to the condition

$$\rho = 0 \tag{4}$$

In order to maintain the validity of (1), (2), and (4), every change in the electromagnetic force due to external density fields must be accompanied by a redistribution of free charges along the surface of the conductor. Moving charge, however, is by definition excluded from the static state. Therefore it is necessary to require the redistribution of charge which ends when $E = 0$ to occur so rapidly that the short interval of lag between the instant of rearrangement of the distant external charges contributing to E and the moment when E is again zero in the interior is insignificant. During this interval, an *unsteady state necessarily prevails* in which certainly E and $\dot{E}$ and, if polarizable units are present, P and $\dot{P}$ all differ from zero. These considerations suggest that a suitable criterion for defining a conductor is the time required for E to become sensibly zero in the interior after a change has been made from one static state to another. In particular, the interval of unsteadiness may be required to be so brief in a conductor that it may be assumed to disappear in a time that is short compared with experimentally measurable intervals. In a semiconductor, on the other hand, the electric intensity in the interior eventually reduces to zero, but the period of unsteadiness may be of long duration. In a nonconductor, a nonvanishing electrostatic intensity can be maintained indefinitely.

A steady drift of free charge can occur only in a conductor and only if an agency is provided on its boundaries which continuously maintains a uniform density of positive charge over one part and a similar density of negative charge over another. Such an agency is called a generator or a region of negative conductivity. It is considered in a later section. The separated layers of positive and negative charges maintained in this way lead to a nonvanishing value of E and of the electrostatic force in the interior of the conductor. As a result, a continuous redistribution of charge in a direction tending to reduce E to zero takes place in the conductor. Since charges moving with statistically constant velocities are statistically in equilibrium, it follows that a steady drift can continue in a region only if the electrostatic force resulting from the charge distribution on the generator surfaces is exactly balanced throughout the interior by a force of a "frictional" type opposing a nonrandom acceleration of free charges and tending to maintain a constant drift velocity. Such a force must increase with the drift velocity in order that a greater electrostatic accelerating force may be balanced when the charges move with a correspondingly greater velocity. In terms of the atomic model, the internal force limiting the velocity of a nonrandom stream is attributed to an interaction between the charges in the steady stream and the statistically stationary bound groups of charges constituting the atoms or molecules between and around which the drift takes place. This interaction leads to an acceleration of the random motion of all the charges and, hence, of each atom and molecule as a whole. Depending upon the magnitude of the interaction, characteristic of a particular structure, a larger or smaller electrostatic force is required *not* to maintain a given steady drift of free charges, but rather to support the continuous random acceleration of all charges which inevitably accompanies every nonrandom flow. The statistical reaction of atoms and molecules to random acceleration will not be described in detail. It is sufficient to ascribe their statistical resistance to an increase in velocity to microscopic mechanical forces of inertia—so-called "thermal" forces. With this terminology, the steady state is a condition of equilibrium between electrostatic forces that act to produce a *nonrandom* acceleration of *free charges* and internal inertial forces seeking to prevent a random acceleration of all

charges and bound groups. The two forces are linked by a characteristic coupling interaction between a nonrandom drift of free charges and the random motion of all charges which prevents the existence of the former without the simultaneous occurrence of the latter. The net effect of the interaction on the drift may be specified by calling it a "frictional" force acting to balance exactly the electrostatic accelerating force. Instead of indefinitely accelerating the free charges in the direction of the drift, this latter acts through the coupling interaction to accelerate simultaneously the random motion of all charges.

The resulting condition of equilibrium between "frictional" and electromagnetic forces acting on the drifting free charges is expressed (for simple conductors) by the steady-state constitutive relation

$$i_f = \sigma E \tag{5}$$

Here E is the average electric field in the interior of a conductor due to all density fields including those of the conductor itself; i_f is the volume density of drifting charge; σ is the conductivity, a material constant. Since E is directly proportional to i_f in a simple conductor, it is clear that the average internal "frictional" forces must be directly proportional and opposite in direction to the drift velocity of the stream.

According to (5) a change in i_f must be accompanied by a proportional change in E which occurs *instantaneously*. As before, it is necessary to conclude that a redistribution of charge from one condition of equilibrium (whether static or steady state) to another necessarily involves a time of redistribution during which an unsteady state prevails in which E, P, and i, as well as their time rates of change, are nonvanishing functions of the time. Consequently, a conductor may be defined in the steady state in terms of the time required for the constitutive relation $i_f = \sigma E$ to become valid. This time is the same as that required for i_f to become constant. If the constant is zero, the static state prevails. The material constant σ is a convenient parameter for characterizing the abundance and the mobility of free charges, as well as the magnitude of the interaction between charges engaged in a steady drift and the random motion of all others. It is evident from (5) that

$$\sigma = 0 \tag{6}$$

defines a nonconductor, since no free drift of charge is possible, regardless of the magnitude of E and of the electrostatic force. Accordingly, a conductor is characterized by

$$\sigma > 0 \tag{7}$$

Since the value of i_f increases with σ for a given E, it is clear that the larger σ the more free charges must be available, and the smaller must be the restraining "frictional" forces. Therefore, the larger σ, the better is the conductor.

The qualitative definitions of conductors and nonconductors as interpreted from the stationary states may be summarized as follows:

A good *conductor* is an atomic model characterized by a conductivity σ which is sufficiently large that a transient, nonrandom drifting of charge existing in a change from one stationary state to another is reduced to a negligibly small value in an experimentally insignificant length of time. The constitutive relation $i_f = \sigma E$ is then valid during all experimentally significant time intervals.

A *semiconductor* is an atomic model that is characterized by a conductivity σ sufficiently small that a transient, nonrandom motion of electric charge existing while passing from one stationary state to another may have significant values during a time that is comparable with experimentally significant and measurable intervals. The relation $i_f = \sigma E$ is not valid during this time, but it is ultimately obeyed.

A good *nonconductor* or dielectric is an atomic model characterized by a conductivity that is so near zero that no significant nonrandom drift of charge is observed. The relation $i_f = \sigma E$ has no application.

Since all these definitions involve in a fundamental way an interval of unsteadiness, a more specific formulation is possible only in terms of the nonstationary state.

17. General Definition of a Conductor.—In order to define mathematically a homogeneous, simple conductor, it is necessary to establish a convenient criterion for the rapidity with which ρ_f vanishes or i_f becomes constant in the interior of the conducting medium after a change in the distribution of charge on the surface or in neighboring bodies. In the absence of detailed formulas for the constitutive parameters ϵ_r, ν_r, and σ in terms of atomic

and molecular structure, such a definition is possible only if it is
assumed that time lags in polarization, in magnetization, and in
current responses to changes in E and B are smaller than an
experimentally observable interval t_{ex}.   That is, the definition
of a conductor includes the requirement that the constitutive
relations (15.10), (15.11), (15.12), *viz.*,

$$P = (\epsilon_r - 1)\epsilon_0 E; \qquad -M = (\nu_r - 1)\nu_0 B; \qquad i_f = \sigma E \quad (1)$$

are true at all times within experimental observation.   Subject
to (1) the first and third field equations assume the forms (15.13),
(15.15).   These are

$$\epsilon \operatorname{div} E = \rho_f \qquad\qquad (2a)$$
$$\nu \operatorname{curl} B = \sigma E + \epsilon \dot{E} \qquad\qquad (2b)$$

Upon taking the divergence of (2b) and recalling that the opera-
tion div curl acting on any vector yields zero,

$$\operatorname{div} (\sigma E + \epsilon \dot{E}) = 0 \qquad\qquad (3)$$

Using (2a) to eliminate E the following linear differential equation
in ρ_f results:

$$\dot{\rho}_f + \frac{\sigma}{\epsilon} \rho_f = 0 \qquad\qquad (4)$$

The solution of this equation is[1]

$$\rho_f = \rho_0 \, e^{-(\sigma/\epsilon)t} \qquad\qquad (5)$$

Here ρ_0 is the value of ρ_f at $t = 0$.   From (5) it is clear that
ρ_f reduces to a negligibly small value in a time that depends
upon the ratio (ϵ/σ).   For convenience let this quantity be
called after Maxwell the *time of relaxation* and denoted by

$$T_R \equiv \frac{\epsilon}{\sigma} \qquad\qquad (6)$$

The time of relaxation is the time required for ρ_f to reduce to
$1/e$ of its initial value ρ_0.   It is noteworthy that it is independent
of $\nu = 1/\mu$.   It follows from (6) that when

$$\sigma = 0; \qquad \rho_f = \rho_0 = \text{const.} \qquad\qquad (7)$$
$$\sigma \doteq \infty; \qquad \rho_f \doteq 0 \qquad\qquad (8)$$

[1] I. S. Sokolnikoff and E. S. Sokolnikoff, "Higher Mathematics for
Engineers and Physicists," 2d ed., Sec. 85, p. 284.

In a perfect nonconductor as defined by (7), the volume density of charge is everywhere constant. Since closely bound groups such as atoms and molecules are electrically neutral as a whole, it is in most cases correct to set the constant ρ_0 in (7) equal to zero. In a *perfect* conductor as defined by (8), the volume density of charge vanishes at all times.

If it is assumed that for all practical purposes it is adequate for ρ_f to reduce to 1 per cent of its initial value in a time that is short compared with an experimentally observable interval, then a conductor can be defined by the following conditions:

$$e^{-t/T_R} << 1 \quad \text{or} \quad e^{t/T_R} \geqslant 100; \quad t << t_{ex} \quad (9)$$

Upon taking the logarithm of (9) and noting that $e^{4.6} = 100$,

$$T_R = \frac{\epsilon}{\sigma} \leqslant \frac{t}{4.6} \quad \text{or} \quad \frac{\sigma}{\epsilon} \geqslant \frac{4.6}{t}; \quad t << t_{ex} \quad (10)$$

By assigning a definite time t that is short compared with an experimentally significant interval t_{ex} in any particular case, a conductor is defined by (10) to be a model in which the ratio of the material parameters σ/ϵ is greater than a specified magnitude. Let this definition be expressed by the following shorthand symbolism which implies that a time limit has been established and that (1) and (10) are both true. A conductor is defined by

$$\sigma >> 0 \quad (11a)$$

If (11a) is true, $\rho_f = 0$ within the required degree of accuracy during all significant intervals of time. In periodic phenomena, the time of relaxation T_R may be required to be negligible compared with $T_p/2\pi$ where T_p is the period. With

$$T_p = \frac{1}{f} = \frac{2\pi}{\omega},$$

(11a) becomes more specifically

$$T_R << \frac{T_p}{2\pi}; \quad \frac{\omega \epsilon}{\sigma} << 1 \quad (11b)$$

This is discussed in greater detail in conjunction with (24) below.

In terms of the field equation (2b), the definition of a conductor (10) is equivalent to requiring

$$\sigma E >> \epsilon \dot{E} \quad (12)$$

Hence, in a conductor, (2b) reduces to

$$\nu_0 \text{ curl } B \doteq \sigma E \tag{13}$$

With (1); (12) is equivalent to

$$i_f >> \dot{P} + \epsilon_0 \dot{E} \tag{14}$$

With (1) and (8), (2a) gives

$$\text{div } E = 0 \tag{15a}$$

so that with $\epsilon E = P + \epsilon_0 E$

$$\text{div } P = 0 \tag{15b}$$

Since P is thus seen to play no significant part in the definitions of E and B in a conductor, it may be concluded that although no proof has been given that P necessarily vanishes in a conductor, its effect is certainly negligibly small so that it is as well to write

$$P \doteq 0 \tag{16}$$

If an unsteady state prevails in which there is a periodically varying distribution of charge so that

$$E_{\text{inst}} = E \cos (\omega t + \theta_E); \quad B_{\text{inst}} = B \cos (\omega t + \theta_B) \tag{17}$$

a time lag in the instantaneous values of polarization, magnetization, and current density may be expressed by writing (1) in the form

$$P_{\text{inst}} = \chi \epsilon_0 E \cos (\omega t + \theta_E - \theta_P) \tag{18a}$$

$$-M_{\text{inst}} = \chi_m \nu_0 B \cos (\omega t + \theta_B - \theta_M) \tag{18b}$$

$$i_{f\text{inst}} = \bar{\sigma} E \cos (\omega t + \theta_E - \theta_i) \tag{18c}$$

Here the parameters χ, χ_m, $\bar{\sigma}$ as well as the phase angles θ_P, θ_M, θ_i are functions of ω. The definition of a conductor in the periodic unsteady state includes the following conditions:

$$\theta_P << 1; \quad \theta_M << 1; \quad \theta_i << 1$$

$$\bar{\chi} \doteq \chi; \quad \bar{\chi}_m \doteq \chi_m; \quad \bar{\sigma} \doteq \sigma \tag{19}$$

Upon substituting (17) in (2b) and in (13) [to which (2b) reduces by definition in a conductor], the following two equations are obtained:

curl $B \cos (\omega t + \theta_B)$

$$= \frac{\sigma}{\nu} E[\cos (\omega t + \theta_E) - \frac{\omega \epsilon}{\sigma} \sin (\omega t + \theta_E)] \tag{20}$$

$$\text{curl } B \cos (\omega t + \theta_B) \doteq \frac{\sigma}{\nu} E \cos (\omega t + \theta_{\boldsymbol{E}}) \qquad (21)$$

Using the trigonometric formula

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \cos \left(x - \arctan \frac{a}{b} \right) \qquad (22)$$

(20) becomes

$$\text{curl } B \cos (\omega t + \theta_B)$$
$$= \frac{\sigma}{\nu} E \sqrt{1 + \left(\frac{\omega \epsilon}{\sigma} \right)^2} \cos \left(\omega t + \theta_{\boldsymbol{E}} + \arctan \frac{\omega \epsilon}{\sigma} \right) \qquad (23)$$

Subject to (10), (23) must reduce to (21). A more convenient, though equivalent definition of a conductor for the harmonic unsteady state than is (10) may be formulated by requiring (11b) to be satisfied

$$\frac{\omega \epsilon}{\sigma} = \omega T_R << 1 \qquad (24)$$

Subject to (24),

$$\sqrt{1 + \left(\frac{\omega \epsilon}{\sigma} \right)^2} \doteq 1; \qquad \arctan \frac{\omega \epsilon}{\sigma} \doteq \frac{\omega \epsilon}{\sigma} << 1 \qquad (25)$$

subject to (25), (23) is equivalent to (21) to the desired degree of approximation.

The definitions of conductors, semiconductors, and non-conductors may be summarized as follows:

Conductor: $\sigma/\epsilon \geqq 4.6/t$ (shorthand, $\sigma >> 0$) where t is a time that is short compared with experimentally measurable intervals. For harmonic time dependence, $\sigma/\epsilon >> \omega$, where $\omega = 2\pi f$ and f is the highest frequency. In a conductor the constitutive relations

$$P = (\epsilon_r - 1)\epsilon_0 E, \qquad -M = (\nu_r - 1)\nu_0 B, \qquad i_f = \sigma E$$

are true at all times within experimental observation; P is always negligibly small; ρ_f vanishes.

Semiconductor: $\sigma > 0$; $i_f \to \sigma E$; $\rho_f \to 0$; the constitutive relations are obeyed after an observable time interval or with an observable time lag.

Nonconductor: $\sigma = 0$; $i_f = 0$ for all values of E; $\rho_f = $ const. $(= 0)$.

In simple conductors and nonconductors (in which it may be assumed that $\rho_f = 0$) the field equations and the corresponding boundary equations for media 1 and 2 are

$$\text{div } \mathsf{E} = 0 \tag{26a}$$
$$\text{curl } \mathsf{E} = -\dot{\mathsf{B}} \tag{26b}$$
$$\nu \text{ curl } \mathsf{B} = \sigma\mathsf{E} + \epsilon\dot{\mathsf{E}} \tag{26c}$$
$$\text{div } \mathsf{B} = 0 \tag{26d}$$
$$\epsilon_1(\mathsf{n}_1,\mathsf{E}_1) + \epsilon_2(\mathsf{n}_2,\mathsf{E}_2) = -\eta_{1f} - \eta_{2f} \tag{27a}$$
$$[\mathsf{n}_1,\mathsf{E}_1] + [\mathsf{n}_2,\mathsf{E}_2] = 0 \tag{27b}$$
$$\nu_1[\mathsf{n}_1,\mathsf{B}_1] + \nu_2[\mathsf{n}_2,\mathsf{B}_2] = -\mathsf{l}_{1f} - \mathsf{l}_{2f} \tag{27c}$$
$$(\mathsf{n}_1,\mathsf{B}_1) + (\mathsf{n}_2,\mathsf{B}_2) = 0 \tag{27d}$$

Because ρ and P vanish in their interior, the electrical properties of conductors may be represented mathematically in terms of the surface density η_f alone instead of in terms of $\bar{\rho}$ and $\bar{\eta}$. In the solution of most boundary-value problems involving practically available conductors and frequencies, no surface layers of moving free charge which are so thin that they cannot be described adequately in terms of i_f are encountered. Therefore, l_f is not required and need not be defined. It follows that a representation entirely in terms of η_f, i_f, and M is possible. This is not true of problems involving the idealized case of *perfect conductors* with $\sigma \to \infty$. For these a surface function l_f rather than a representation in terms of an equivalent $[\mathsf{n},\mathsf{M}]$ is often desirable because values of M actually associated physically with spin atoms may be encountered also. The function l_f need be used only in problems involving infinite conductivity. For ordinary conductors and boundaries between two such conductors numbered 1 and 2

$$\text{div } \mathsf{E} = 0 \tag{28a}$$
$$\text{curl } \mathsf{E} = -\dot{\mathsf{B}} \tag{28b}$$
$$\nu \text{ curl } \mathsf{B} = \sigma\mathsf{E} \tag{28c}$$
$$\text{div } \mathsf{B} = 0 \tag{28d}$$
$$\epsilon_0(\mathsf{n}_1,\mathsf{E}_1) + \epsilon_0(\mathsf{n}_2,\mathsf{E}_2) = -\eta_{1f} - \eta_{2f} \tag{29a}$$
$$[\mathsf{n}_1,\mathsf{E}_1] + [\mathsf{n}_2,\mathsf{E}_2] = 0 \tag{29b}$$
$$\nu_1[\mathsf{n}_1,\mathsf{B}_1] + \nu_2[\mathsf{n}_2,\mathsf{B}_2] = 0 \tag{29c}$$
$$(\mathsf{n}_1,\mathsf{B}_1) + (\mathsf{n}_2,\mathsf{B}_2) = 0 \tag{29d}$$

In nonconductors free charges are present in negligible amount or not at all so that the entire representation may be expressed

in terms of P and M alone.   The following equations are valid for nonconductors and boundaries between nonconductors:

$$\text{div } E = 0 \tag{30a}$$

$$\text{curl } E = -\dot{B} \tag{30b}$$

$$\nu \text{ curl } B = \epsilon\dot{E} \tag{30c}$$

$$\text{div } B = 0 \tag{30d}$$

$$\epsilon_1(\mathfrak{n}_1,E_1) + \epsilon_2(\mathfrak{n}_2,E_2) = 0 \tag{31a}$$

$$[\mathfrak{n}_1,E_1] + [\mathfrak{n}_2,E_2] = 0 \tag{31b}$$

$$\nu_1[\mathfrak{n}_1,B_1] + \nu_2[\mathfrak{n}_2,B_2] = 0 \tag{31c}$$

$$(\mathfrak{n}_1,B_1) + (\mathfrak{n}_2,B_2) = 0 \tag{31d}$$

Since the right-hand members of the boundary conditions (31) are all zero, the following relations for the magnitudes of E and B on the two sides of the boundary may be written.   All directions are referred to the external normal to region 1.   The notation (n,E) in the argument of a trigonometric function means the angle between $\mathfrak{n}$ and E.

$$\epsilon_1 E_1 \cos (n_1,E_1) = \epsilon_2 E_2 \cos (n_1,E_2) \qquad \text{or} \qquad \epsilon_1 E_{1n} = \epsilon_2 E_{2n} \tag{32a}$$

$$E_1 \sin (n_1,E_1) = E_2 \sin (n_1,E_2) \qquad \text{or} \qquad E_{1t} = E_{2t} \tag{32b}$$

$$\nu_1 B_1 \sin (n_1,B_1) = \nu_2 B_2 \sin (n_1,B_2) \qquad \text{or} \qquad \nu_1 B_{1t} = \nu_2 B_{2t} \tag{33c}$$

$$B_1 \cos (n_1,B_1) = B_2 \cos (n_1,B_2) \qquad \text{or} \qquad B_{1n} = B_{2n} \tag{33d}$$

From these the following general ratios are obtained:

$$\frac{\tan (n_1,E_1)}{\tan (n_1,E_2)} = \frac{\epsilon_1}{\epsilon_2} \tag{34a}$$

$$\frac{\cot (n_1,B_1)}{\cot (n_1,B_2)} = \frac{\nu_1}{\nu_2} \tag{34b}$$

$$\frac{E_2}{E_1} = \sqrt{\frac{\epsilon_1^2}{\epsilon_2^2} + \left(1 - \frac{\epsilon_1^2}{\epsilon_2^2}\right) \sin^2 (n_1,E_1)} \tag{35a}$$

$$\frac{B_2}{B_1} = \sqrt{\frac{\nu_1^2}{\nu_2^2} + \left(1 - \frac{\nu_1^2}{\nu_2^2}\right) \cos^2 (n_1,B_1)} \tag{35b}$$

It is sometimes convenient to represent a distribution of moving free charges entirely in terms of P and $-M$.   This is possible if the free-charge model is first replaced by a *mathematically equivalent* bound-charge model which is characterized by the same essential densities.   These are then given by

$$\overline{\rho_m\mathsf{v}} = \text{curl } M + \dot{P} \qquad \bar{\rho} = -\text{div } P \tag{36}$$

$$\overline{\eta_m\mathsf{v}} = -[\mathfrak{n},M] \qquad \bar{\eta} = (\mathfrak{n},P)$$

Once this change in model and representation has been made, a relative dielectric constant (permittivity) and a relative diamagnetic constant (reluctivity) may be defined in the usual way so that (30) and (31) apply.   It is well to note that the definition of dielectric and diamagnetic constants *implies a linear relation between* P *and* E and *between* $-$M *and* B and, therefore, a bound-charge model of *special* structure.   The fact that a bound-charge model can always be constructed to be equivalent to a *particular* distribution of moving free charge does not mean that the equivalence is necessarily maintained under the action of diverse external forces.   This certainly would not be the case in general, especially if linear relations between P and E and $-$M and.B are assumed to be maintained.   A conducting medium does not behave like a dielectric, even though it is possible to represent a particular distribution of free charge in terms of an equivalent polarization.

18. Impressed and Intrinsic Electric Fields; Negative Conductivity.—A steady flow of electric charge can be maintained around a closed path constructed of a single homogeneous conductor that is not continuously exposed to the action of electromagnetic forces only if its conductivity is infinite.   This is the special case of superconductivity which is excluded from the present discussion.   In all circuits in which σ is not infinite and in which a steady or unsteady convection or conduction current exists, electromagnetic forces must be active continuously. These are maintained by so-called intrinsic or impressed electric fields.

An intrinsic electric field in a circuit of conductors may be ascribed to a so-called region of negative conductivity.   Since a region of positive conductivity is one in which a positive electric force is required to maintain a steady drift of charge, a region of negative conductivity is evidently one in which a negative electric force is required to support such a drift.   An alternative, but equivalent, way of expressing such a condition is to state that a positive electric force is required to *prevent* such a drift.   This obviously implies that an intrinsic force of some kind must be continuously active to maintain it.   That is, a region of negative conductivity is endowed with the property of separating positive and negative charges and accumulating these in separate layers with densities that increase until the

electrostatic force drawing the unlike charges together is sufficient to balance the intrinsic charge-separating force that is driving them apart.   Such a charge-separating force may be attributed to electrochemical, electromechanical, thermoelectric, photoelectric, or other effects beyond the scope of this treatise.   It is assumed without further discussion that atomic models may be constructed which are endowed with the characteristic property of separating positive and negative charge in opposition to electrostatic forces.   They are called regions of negative conductivity or generators.

Between the positively and negatively charged boundary surfaces or terminals of a charge-separating region, a nonvanishing value of the electric vector may be calculated, so that if the terminals are joined by a conductor a drift of electric charge through it results.   The magnitude of the drift depends upon the particular value of the positive conductivity σ of the simple conductor and upon the effective negative conductivity $\sigma_{\overrightarrow{\text{eff}}}$ of the charge-separating region.   The net effect is a steady stream completely around the circuit formed by the conductor and the negative conductor.   In the former, the charges in the drift are acted upon by electrostatic forces due to charged layers; in the latter, they are acted upon by oppositely directed and larger intrinsic separating forces as well.   The steady flow represents a condition of equilibrium between the accelerating forces (intrinsic minus electrostatic in the charge-separating region, electrostatic in the conductor) and the "frictional" forces due to the interaction between charges moving in the steady drift and all charges and groups of charges moving at random.   Such "frictional" forces are not limited to the region of positive conductivity but also exist in the region of negative conductivity.   Consequently, the effective negative conductivity $\sigma_{\overrightarrow{\text{eff}}}$ may be regarded as the sum of a conductivity $-\sigma^e$ and a smaller, positive conductivity σ.   A simple negative conductor may then be defined as one in which the relation

$$i_f = \sigma_{\overrightarrow{\text{eff}}} \mathsf{E} \tag{1}$$

obtains and where

$$\sigma_{\overrightarrow{\text{eff}}} = \sigma - \sigma^e; \qquad \sigma^e > \sigma \tag{2}$$

By combining (2) with (1), and factoring out the positive conductivity σ

$$i_f = \sigma \left(E - \frac{\sigma^\bullet}{\sigma} E \right) \tag{3}$$

Let the vector

$$E^\bullet \equiv -\frac{\sigma^\bullet}{\sigma} E \tag{4}$$

be defined. With (4), (3) becomes

$$i_f = \sigma(E + E^\bullet) \tag{5}$$

From (5) it is clear that a region in which the vector $E^\bullet$ vanishes is an ordinary simple conductor with positive conductivity σ. A nonvanishing vector $E^\bullet$ may be called an impressed or, more specifically, an intrinsic electric field. It is a fundamental characteristic of a charge-separating region. It is equal and opposite to that electric field which must be maintained in the region by accumulations of charge in order to provide an electric force that exactly balances the intrinsic separating forces and so prevents a further separation of charge. Therefore, $E^\bullet$ is a direct measure of the charge-separating properties of the region per unit volume, assuming that these are uniformly distributed. The generalized constitutive relation (5) may be written instead of the simpler one

$$i_f = \sigma E \tag{6}$$

whenever it is desired to indicate expressly the nonvanishing of an intrinsic or impressed electric field. In general, only (6) need be written with the understanding that E may include an impressed intensity $E^\bullet$ wherever this differs from zero. In the analysis of electric circuits, it is often convenient to assume a circuit to be driven by one (or more) charge-separating region maintained between adjacent surfaces across the conductor at A and B. Such a region is called a *slice generator* or a *point generator* if the problem is one-dimensional.

TRANSFORMATIONS OF FIELD AND FORCE EQUATIONS

The four Maxwell-Lorentz equations defining the electro-magnetic field and the force equation relating this to mechanics constitute the general mathematical model of electromagnetism. It interrelates the electrical properties of all bodies in terms of the essential density fields that characterize them individually. Bodies or regions which are so constructed that this interrelation assumes the simple form defined by three constitutive relations are called simple media. For them, the field equations assume a form that involves no density fields, but three constitutive parameters are involved.

In order actually to make use of the field and force equations to predict experimental pointer readings in any particular case, it is necessary to establish two sets of analogues between theoretical quantities and experimental ones. The first of these is the specification of initial conditions and boundary conditions in the form of mathematical hypotheses that are the analytical analogues of the actual measurable physical conditions in question. The second is the formulation of a result in terms of theoretical analogues of experimentally observable pointer readings. No general rules can be given for the discovery of useful analogues among the innumerable symbolic forms of mathematics and the pointer readings of a particular contrivance. Whether it be intuition, genius, or hard work of the trial-and-error kind called experience, which leads to such discoveries, is left to the preference of the individual. Whatever it may be, it certainly does not lend itself to general formulation and can best be considered a part of the study of each particular problem. A useful and more generally convenient method is one that reverses the procedure. Instead of attempting to discover theoretical analogues to serve as hypotheses and conclusions to fit a given physical phenomenon, it is possible simply to assume a wide variety of mathematical hypotheses and, using these, calculate

an equally wide variety of conclusions.   The attempt can then be made to find theoretical analogues among them to fit a set of observed pointer readings.   Such a procedure, of course, can be successful only because of the vast amount of work that has already been done and from which the nature of the most useful mathematical functions and hypotheses can be anticipated. The Maxwell-Lorentz equations define precisely such a set of useful functions.

In order to suit the particular circumstances of a variety of problem types and facilitate their solution, it is found convenient to transform the Maxwell-Lorentz definitions and the force equation in very general ways.   These transformations are obtained by standard mathematical manipulation.   They do not represent the introduction of anything fundamentally new into electromagnetic theory, but merely provide alternative forms in which it may be expressed.   The most important of these are the potential formulation of the field equations and the energy formulation of the force and field equations.

POTENTIAL FUNCTIONS AND THE GENERAL WAVE EQUATION

1. Gradient of a Scalar Field.—A continuous mathematical function that assigns a scalar to every point in a region is called a scalar point function.   Such a function may characterize a certain property of a physical model such as volume density of charge, volume density of mass, temperature, humidity, pressure, elevation above sea level, but it is not at all necessary that a scalar point function have a physical significance.   It may equally well associate purely mathematical magnitudes devoid of all physical meaning with points in space.

An important property of a scalar point function is its space rate of change.   Consider, for example, a scalar point function ϕ that is continuous with its derivatives.   The simplest form for this function is that of a constant.   If ϕ is everywhere constant, the rate of change of ϕ at every point and in every direction vanishes.

$$\phi = C; \qquad \frac{d\phi}{ds} = 0 \tag{1}$$

If the function varies so that a different number is assigned to each point, $d\phi/ds$ does not vanish.   At every point it measures the rate of increase of the function ϕ in the direction in which ds

is taken. It may vary not only from one location to another in the region, but also in different directions relative to a single point. Thus, the space rate of change of a scalar point function itself defined a continuous function that assigns not one magnitude to every point, but a *magnitude to every direction* at that point. That is, the property associated with the space rate of change of a scalar point function is characterized not only by magnitude, but also by direction. Therefore, the magnitude $d\phi/ds$ at a particular point and in a particular direction (as specified by the orientation of the element ds) is the component in the direction of ds of a vector. The component G_s of a vector G in a given direction specified by the unit vector $\$$ is the scalar product

$$G_s = (\$, G) \tag{2}$$

Hence, if $d\phi/ds$ is the component of a vector G in the direction of ds specified by the unit vector $\$$

$$\frac{d\phi}{ds} = G_s = (\$, G) \tag{3}$$

This component vanishes in a direction in which ϕ remains constant. In such a direction

$$\frac{d\phi}{ds} = 0 = (\$, G) \tag{4}$$

Since $\$$ cannot vanish and G can be zero if $d\phi/ds$ vanishes in all directions, which is assumed not to be the case, there is no way to satisfy the right side of (4) except by requiring G to be perpendicular to $\$$ when this points in a direction in which ϕ is constant. Therefore the vector G, characterizing the space rate of change of ϕ, is always in a direction normal to surfaces along which ϕ is constant. The maximum value of $d\phi/ds$ is in a direction that makes the cosine of the angle between $\$$ and G equal to unity. That is,

$$\left(\frac{d\phi}{ds}\right)_{max} = |G| = G \tag{5}$$

Hence, the vector G defines both the magnitude and the direction of the maximum space rate of increase of ϕ. It is directed perpendicular to the surfaces $\phi = $ constant. The vector G which defines the direction and magnitude of the maximum

space rate of increase of a scalar point function is given the name *gradient of the scalar point function.* It defines a vector point function. The operation of taking the derivative of a scalar function in the direction in which this (the derivative) is a maximum is assigned the special symbol grad. That is,

$$\left(\frac{d}{ds}\right) \text{ in direction of maximum} \equiv \text{grad} \tag{6}$$

Accordingly

$$\left(\frac{d\phi}{ds}\right)_{\text{max}} = |\text{grad } \phi| = G \tag{7}$$

$$G = \text{grad } \phi \tag{8}$$

In other words, the symbol grad is a differential operator that specifies differentiation with respect to the space coordinates in the direction in which the derivative has its maximum value. More precisely, the operator grad operating on a scalar point function ϕ produces a vector point function grad ϕ, the direction of which at every point is that of the greatest space rate of increase of ϕ and the magnitude of which measures the maximum rate of increase $(d\phi/ds)_{\text{max}}$ of that function.

Fig. 1.1.—Parallelepiped in scalar field.

Alternatively, grad ϕ is a vector which has for a component in any direction the rate of change $d\phi/ds$ in that direction.

The gradient of a scalar field may be defined in a way that is analogous to the definition of the divergence and the curl of a vector field. Thus,

$$\text{grad } \phi = \lim_{\Delta\tau\to0} \frac{\int_{\Sigma} \hat{n}\phi \, d\tau}{\Delta\tau} \tag{9}$$

Here $\hat{n}$ is an external normal to the surface Σ enclosing the volume $\Delta\tau$. It is easily shown that this definition is equivalent to that given above. Let the volume $\Delta\tau$ be chosen so that two sides always lie along surfaces on which ϕ is constant, while the other four sides are perpendicular to these surfaces as in Fig. 1.1. Everywhere on the top surface ΔS_1, $\phi = \phi_1$; everywhere on the bottom surface ΔS_2, $\phi = \phi_2$ with $\phi_1 > \phi_2$; on the four sides

ΔS_i, ϕ varies continuously from ϕ_1 to ϕ_2. Let the mean value of ϕ on each side be $\bar{\phi}$. With this notation and assuming $\Delta\tau$ to be sufficiently small, (9) becomes

$$\text{grad } \phi = \lim_{\Delta\tau \to 0} \left\{ \frac{\hbar_1\phi_1\Delta S_1 + \hbar_2\phi_2\Delta S_2 + \sum_{i=3}^{6} \bar{\phi}_i\hbar_i\Delta S_i}{\Delta\tau} \right\} \quad (10)$$

As the volume is made smaller and smaller, $\Delta S_1 \to \Delta S_2$ and the opposite pairs of surfaces ΔS_i become equal. The average values $\bar{\phi}$ on opposite side surfaces ΔS_i also become equal, so that $\hbar_i\bar{\phi}_i\Delta S_i$ for opposite sides are alike in magnitude and opposite in sign. Hence the sum in (10) vanishes in the limit leaving

$$\text{grad } \phi = \lim_{\Delta\tau \to 0} \frac{\hbar_1\Delta S_1(\phi_1 - \phi_2)}{\Delta\tau} = \lim_{\Delta\tau \to 0} \frac{\hbar_1\Delta S_1\Delta\phi}{\Delta S_1\Delta s} \quad (11)$$

In (11) Δs is the distance between the parallel upper and lower surfaces. Finally

$$\text{grad } \phi = \hbar_1 \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \hbar_1 \frac{d\phi}{ds} \quad (12)$$

Here $\hbar_1$ points in a direction perpendicular to a surface of constant ϕ at the point where $d\phi/ds$ is defined and in the direction of increasing ϕ.

The form (9) is convenient for expressing the gradient in various systems of orthogonal coordinates.[1]

As an example of the gradient, let $\phi = h$ measure the height above sea level at every point on a mountain range. Let $s = \sqrt{x^2 + y^2}$ measure distance in the horizontal plane. The

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[1] In rectangular coordinates the volume $\Delta\tau$ is chosen to be a cube $\Delta x\,\Delta y\,\Delta z$ with center at $x = 0$, $y = 0$, $z = 0$. In this case

$$\text{grad } \phi = \lim_{\Delta x\,\Delta y\,\Delta z \to 0}$$
$$\left[\frac{\hat{x}\{\phi_{x+\frac{1}{2}\Delta x} - \phi_{x-\frac{1}{2}\Delta x}\}\Delta y\,\Delta z + \hat{y}\{\phi_{y+\frac{1}{2}\Delta y} - \phi_{y-\frac{1}{2}\Delta y}\}\Delta z\,\Delta x + \hat{z}\{\phi_{z+\frac{1}{2}\Delta z} - \phi_{z-\frac{1}{2}\Delta z}\}\Delta x\,\Delta y}{\Delta x\,\Delta y\,\Delta z} \right]$$

Using Taylor's series

$$\phi_{x+\frac{1}{2}\Delta x} - \phi_{x-\frac{1}{2}\Delta x} = \phi + \frac{1}{2}\Delta x\left(\frac{\partial\phi}{\partial x}\right) + \cdots - \left\{\phi - \frac{1}{2}\Delta x\left(\frac{\partial\phi}{\partial x}\right) + \cdots\right\}$$
$$= \left(\frac{\partial\phi}{\partial x}\right)\Delta x + \cdots$$

The functions and derivatives on the right are evaluated at the center of the

lines $\phi = h = $ const. or $d\phi/ds = 0$ are level contours that neither rise nor descend.   The direction specified by grad ϕ at any point gives the direction of steepest ascent from that point, and the magnitude of grad ϕ is the numerical value of the steepest upward slope, or the tangent of the largest angle of upward inclination. On the other hand, $-$grad ϕ is directed opposite to grad ϕ at every point so that it gives the direction and magnitude of steepest downward slope.   If grad $\phi = 0$ at a point, there is no direction of steepest rise, the slope in all directions is zero.   This is true on a level plateau or on the uppermost part of a rounded mountain peak.   If water is poured out at any point, it starts to flow in the direction of $-$grad ϕ.

2. Operational Notation; Directional Derivative; Green's Theorem.—The operation of taking the gradient of a scalar or the divergence or curl of a vector involves the vector operator

$$\nabla \equiv \lim_{\Delta\tau \to 0} \frac{\int_\Sigma \hbar \, d\sigma}{\Delta\tau} \tag{1}$$

Thus,

$$\text{grad } \phi \equiv \nabla\phi \equiv \lim_{\Delta\tau \to 0} \frac{\int_\Sigma \hbar\phi \, d\sigma}{\Delta\tau} \tag{2}$$

$$\text{div } \mathsf{A} \equiv (\nabla,\mathsf{A}) \equiv \lim_{\Delta\tau \to 0} \frac{\int_\Sigma (\hbar,\mathsf{A}) d\sigma}{\Delta\tau} \tag{3}$$

$$\text{curl } \mathsf{A} \equiv [\nabla,\mathsf{A}] \equiv \lim_{\Delta\tau \to 0} \frac{\int_\Sigma [\hbar,\mathsf{A}] d\sigma}{\Delta\tau} \tag{4}$$

The concise operational notation using the operator ∇ (nabla) is in common use.   It is not so suggestive for the beginner as

cube.   Accordingly, and writing first-order terms only

$$\text{grad } \phi = \lim_{\Delta x\,\Delta y\,\Delta z \to 0}$$
$$\left[\frac{ \mathfrak{x}\left(\frac{\partial\phi}{\partial x}\right)\Delta x\,\Delta y\,\Delta z + \mathfrak{y}\left(\frac{\partial\phi}{\partial y}\right)\Delta x\,\Delta y\,\Delta z + \mathfrak{z}\left(\frac{\partial\phi}{\partial z}\right)\Delta x\,\Delta y\,\Delta z}{\Delta x\,\Delta y\,\Delta z} \right]$$

Allowing $\Delta x\,\Delta y\,\Delta z$ to approach zero at xyz

$$\text{grad } \phi = \mathfrak{x}\,\frac{\partial\phi}{\partial x} + \mathfrak{y}\,\frac{\partial\phi}{\partial y} + \mathfrak{z}\,\frac{\partial\phi}{\partial z} \tag{13}$$

the symbols grad, div, and curl, so that it will not be used extensively in this introductory text.

The rate of change of a scalar point function in a direction specified by the unit vector $\hat{a}$ is the directional derivative of the scalar function. It is

$$\frac{d\phi}{ds} \text{ in direction } \hat{a} = (\hat{a}, \text{grad } \phi) \equiv (\hat{a}, \text{grad})\phi \equiv (\hat{a}, \nabla)\phi \quad (5)$$

The form $(\hat{a}, \text{grad})\phi$ or $(\hat{a}, \nabla)\phi$ is often used to mean the scalar product $(\hat{a}, \text{grad } \phi)$. The combination $(\hat{a}, \text{grad})$ is then interpreted to be an operator which, when operating on a scalar point function ϕ, specifies the rate of change $d\phi/ds$ in the *direction of* $\hat{a}$, just as grad is an operator which, when operating on a scalar point function ϕ, specifies the rate of change $d\phi/ds$ in the *direction of the maximum* rate of change $(d\phi/ds)_{\text{max}}$.

The operational symbol $(\hat{a}, \text{grad})$ for directional derivative is also defined for use with a vector point function. That is, the rate of change of a vector point function B in a direction specified by the unit vector $\hat{a}$ or the directional derivative of the vector B in the direction $\hat{a}$ is denoted by[1]

$$\frac{d\mathsf{B}}{ds} \text{ in direction } \hat{a} = (\hat{a}, \text{grad})\mathsf{B} \quad (6)$$

Green's theorem is an important integral theorem involving the gradient. It may be obtained directly from the divergence theorem (Sec. II.6.1) by setting

$$\mathsf{A} = u \text{ grad } v \qquad \qquad (9)$$

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[1] In rectangular coordinates this operation is

$$(\hat{a}, \text{grad})\mathsf{B} = \hat{x}(\hat{a}, \text{grad } B_x) + \hat{y}(\hat{a}, \text{grad } B_y) + \hat{z}(\hat{a}, \text{grad } B_z) \quad (7a)$$

$$(\hat{a}, \text{grad})\mathsf{B} = \hat{x}\frac{dB_x}{ds} + \hat{y}\frac{dB_y}{ds} + \hat{z}\frac{dB_z}{ds} \text{ with } ds \text{ in the direction } \hat{a} \quad (7b)$$

The following notation is also used:

$$(\mathsf{A}, \text{grad})\mathsf{B} = \hat{x}(\mathsf{A}, \text{grad } B_x) + \hat{y}(\mathsf{A}, \text{grad } B_y) + \hat{z}(\mathsf{A}, \text{grad } B_z) \quad (8a)$$

or

$$(\mathsf{A}, \text{grad } \mathsf{B}) = \hat{x}\left\{A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right\}$$
$$+ \hat{y}\left\{A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right\}$$
$$+ \hat{z}\left\{A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right\} \quad (8b)$$

with u and v arbitrary scalar point functions.   Thus

$$\int_V \text{div } (u \text{ grad } v)dV = \int_S (\mathbf{n}, u \text{ grad } v)dS \qquad (10)$$

Using the vector relation[1]

$$\text{div } \phi\mathbf{A} = \phi \text{ div } \mathbf{A} + (\mathbf{A}, \text{ grad } \phi) \qquad (11)$$

the integral on the left may be transformed into two integrals. The integral on the right may be changed in form by noting that

$$(\mathbf{n}, \text{ grad } v) = \text{grad}_n v = \frac{\partial v}{\partial n} \qquad (12)$$

With (11) and (12), (10) becomes

$$\int_V u \text{ div grad } v \, dV + \int_V (\text{grad } u, \text{ grad } v)dV = \int_S u \frac{\partial v}{\partial n} dS \quad (13)$$

The symbol ∇^2 (nabla squared) is in common use for the important operation of taking the divergence of the gradient of a scalar.

$$\nabla^2 v \equiv \text{div grad } v \qquad (14)$$

Hence,

$$\int_V u\nabla^2 v \, dV = \int_S u \frac{\partial v}{\partial n} dS - \int_V (\text{grad } u, \text{ grad } v)dV \quad (15)$$

This is one form of Green's theorem. A second form, called Green's symmetrical theorem, may be obtained from (15) by interchanging u and v and subtracting the expression so obtained from (15).   The result is

$$\int_V (u\nabla^2 v - v\nabla^2 u)dV = \int_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS \qquad (16)$$

3. Potential Functions. *Definition.*—A *scalar potential* is a scalar point function the *negative* gradient of which is a vector point function.   In other words, any scalar point function ϕ is a scalar potential if $-\text{grad } \phi$ defines a vector point function $\mathbf{K}$. A surface on which ϕ is constant is called an *equipotential surface*.

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[1] In Cartesian coordinates

$$\text{div } \phi\mathbf{A} = \text{div } \phi(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z) = \frac{\partial}{\partial x}(\phi A_x) + \frac{\partial}{\partial y}(\phi A_y) + \frac{\partial}{\partial z}(\phi A_z)$$
$$= \phi \text{ div } \mathbf{A} + (\mathbf{A}, \text{ grad } \phi)$$

A vector point function is not necessarily the gradient of a scalar potential. The class of vector functions that can be derived as the negative gradients of scalar potentials is limited by the following theorem:

THEOREM:[1] A necessary and sufficient condition that a vector field K possess a *scalar* potential ϕ is that

$$\operatorname{curl} K = 0 \tag{1}$$

If this is true, the vector K is called a *potential vector* and its field is said to be *irrotational*. It may be derived from its associated scalar potential by setting

$$K = -\operatorname{grad} \phi \tag{2}$$

Definition.—A *vector potential* is a vector point function the curl of which is also a vector point function. In other words, a vector point function A is a vector potential if curl A defines a second vector point function C. The class of vector functions that can be derived from vector potentials is limited by the following theorem:

THEOREM:[1] A necessary and sufficient condition that a vector field C possess a vector potential A is that

$$\operatorname{div} C = 0 \tag{3}$$

If (3) is true, C is called a *solenoidal* vector; its field is *rotational*. It may be derived from a vector potential by setting

$$C = \operatorname{curl} A \tag{4}$$

Any single-valued vector point function S that together with its derivatives is finite and continuous and vanishes at infinity can be derived from a scalar potential ϕ and a vector potential A in a form known as the Helmholtz theorem.

$$S = -\operatorname{grad} \phi + \operatorname{curl} A \tag{5}$$

4. Equations of d'Alembert, Poisson, and Laplace.—The direct determination of E and B by solving the Maxwell-Lorentz equations is in most cases very difficult. The solution is usually facilitated by defining potential functions in terms of the field vectors, because it is possible to transform the four first-order

[1] I. S. SOKOLNIKOFF and E. S. SOKOLNIKOFF, "Higher Mathematics for Engineers and Physicists," 2d ed., pp. 422, 423.

field equations into two second-order equations in the potential functions that are formally integrable in a general way. From the integrals so obtained, and the definitions of the potential functions, it is possible to calculate E and B and, from these, the distributions of current and charge using the boundary conditions. Frequently, current and charge are more conveniently determined directly from the potential functions than from the field vectors.

An examination of the four field equations discloses that the magnetic vector B is solenoidal because its divergence vanishes. Hence B can be derived from a vector potential A defined by

$$\text{curl } A = B \tag{1}$$

The vector point function A defined (incompletely) in (1) is of fundamental importance especially in the theory of antennas. It is called the *magnetic vector potential*.

In order to define a scalar potential it is necessary to find a vector with vanishing curl. From the symmetry of electric and magnetic quantities the second field equation should be used for this purpose, since this is the electric analogue of the magnetic fourth equation. The second equation is

$$\text{curl } E = -\dot{B} \tag{2}$$

This is easily transformed with (1) to read

$$\text{curl } \{E + \dot{A}\} = 0 \tag{3}$$

The vector $(E + \dot{A})$ is a potential vector because its curl vanishes. It can be derived from a scalar potential ϕ defined by

$$-\text{grad } \phi = E + \dot{A} \tag{4}$$

This also is a fundamentally important relation. The scalar potential defined by (4) is called the *electric scalar potential*. It is evident that if the scalar and vector potentials ϕ and A are known, the electromagnetic vectors E and B may be calculated directly from

$$E = -\text{grad } \phi - \dot{A} \tag{5}$$
$$B = \text{curl } A \tag{6}$$

With the scalar and vector potentials defined, the next step is to eliminate E and B from the field equations. Before pro-

ceeding to do this, it is well to note that the statement that ϕ and A have been defined is not strictly true insofar as A is concerned. Whereas the scalar ϕ is defined completely except for an additive constant by (4), the vector A is not defined completely by (1). In order to define a vector, both its curl and its divergence must be specified. Up to the present, nothing has been said about the divergence of A, and to this extent at least the vector A is still arbitrary. Advantage will be taken of this fact below.

Since the potential functions ϕ and A are defined in terms of the second and fourth field equations, they must still be made to satisfy the first and third.

$$\epsilon_0 \text{ div } \mathsf{E} = \bar{\rho} \tag{7}$$

$$\nu_0 \text{ curl } \mathsf{B} = \overline{\rho_m \mathsf{v}} + \epsilon_0 \dot{\mathsf{E}} \tag{8}$$

Direct substitution of (5) and (6) in (7) and (8) gives

$$\text{div grad } \phi + \text{div } \dot{\mathsf{A}} = -\frac{\bar{\rho}}{\epsilon_0} \tag{9}$$

$$\text{curl curl } \mathsf{A} = \frac{\{\overline{\rho_m \mathsf{v}} - \epsilon_0 \text{ grad } \phi - \epsilon_0 \ddot{\mathsf{A}}\}}{\nu_0} \tag{10}$$

It is customary to introduce the symbol ∇^2 or Δ (del) to stand for two quite different operations, depending upon whether it operates on a scalar ϕ or a vector A. The definition of the Laplacian operator in the two cases is

$$\Delta\phi \equiv \nabla^2\phi \equiv \text{div grad } \phi \tag{11}$$

$$\Delta\mathsf{A} \equiv \nabla^2\mathsf{A} \equiv \text{grad div } \mathsf{A} - \text{curl curl } \mathsf{A} \tag{12}$$

With this symbolism (9) and (10) become

$$\nabla^2\phi + \text{div } \dot{\mathsf{A}} = -\frac{\bar{\rho}}{\epsilon_0} \tag{13}$$

$$\nabla^2\mathsf{A} - \text{grad div } \mathsf{A} = \frac{\{-\overline{\rho_m \mathsf{v}} + \epsilon_0 \text{ grad } \phi + \epsilon_0\ddot{\mathsf{A}}\}}{\nu_0} \tag{14}$$

Since the vector potential A is still not completely defined, it is possible to assign any convenient value to div A. The value that is chosen is that which separates the variables A and ϕ in (13) and (14). Let the definition of A be completed (except for an additive constant vector) by writing

$$\text{curl } \mathsf{A} = \mathsf{B} \tag{15}$$

$$\text{div } \mathsf{A} = -\frac{\epsilon_0}{\nu_0}\frac{\partial \phi}{\partial t} = -\frac{1}{v_0^2}\frac{\partial \phi}{\partial t} \tag{16}$$

The last step in (16) makes use of (II.5.16) defining the characteristic velocity v_0. The second relation may be written in the form

$$\text{div } \mathsf{A} + \frac{\epsilon_0}{\nu_0}\frac{\partial \phi}{\partial t} = 0 \tag{17}$$

It now closely resembles the generalized equation of continuity of electric charge

$$\text{div } \overline{\rho_m \mathsf{v}} + \frac{\partial \bar{\rho}}{\partial t} = 0 \tag{18}$$

It will be seen at once [in (19) and (20)] that, subject to (17), A is defined in terms of $\overline{\rho_m \mathsf{v}}$ and ϕ in terms of $\bar{\rho}$. The resemblance between (17) and (18) is evidently more than formal. (17) is called the equation of continuity for potentials. If it is satisfied, (13) and (14) reduce to

$$\nabla^2 \phi - \frac{\epsilon_0}{\nu_0}\ddot{\phi} = -\frac{\bar{\rho}}{\epsilon_0} \tag{19}$$

$$\nabla^2 \mathsf{A} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{A}} = -\frac{\overline{\rho_m \mathsf{v}}}{\nu_0} \tag{20}$$

If desired $1/v_0^2$ may be written instead of (ϵ_0/ν_0). These are the fundamental d'Alembert equations governing the potential fields ϕ and A.

In the stationary states (19) and (20) reduce to a form known as Poisson's equation.

$$\nabla^2 \phi = -\frac{\bar{\rho}}{\epsilon_0} \tag{21}$$

$$\nabla^2 \mathsf{A} = -\frac{\mathsf{I}}{\nu_0} \tag{22}$$

The definitions of the potential functions in the stationary states are

$$-\text{grad } \phi = \mathsf{E} \tag{23}$$

$$\left.\begin{array}{l} \text{curl } \mathsf{A} = \mathsf{B} \\ \text{div } \mathsf{A} = 0 \end{array}\right\} \tag{24}$$

Since (21) involves only the essential volume density of charge $\bar{\rho} = \rho - \text{div } \mathsf{P}$ and the electric constant ϵ_0, it is a static-state

equation. It is analogous to the steady-state equation (22)
that involves the essential volume density of moving charge
$\imath = \mathsf{i} + \text{curl } \mathsf{M}$ and the magnetic constant ν_0. The formal
analogy between the static and steady states may be extended as
follows:

Quantity	Static state	Steady state
Potential function...	Electric scalar potential ϕ	Magnetic vector potential A
Operation...........	∇^2 (operating on a scalar)	∇^2 (operating on a vector)

A special form of Poisson's equation called Laplace's equation
is

$$\nabla^2\phi = 0 \tag{25}$$
$$\nabla^2\mathsf{A} = 0 \tag{26}$$

5. Boundary Conditions for Potential Functions.—The general
boundary conditions for the electromagnetic vectors are given
in (II.3.19). They may be written as follows at the boundary
between two media 1 and 2.

$$(\mathsf{n}_1,\mathsf{E}_1) + (\mathsf{n}_2,\mathsf{E}_2) = -\frac{\bar{\eta}}{\epsilon_0} \tag{1a}$$

$$[\mathsf{n}_1,\mathsf{E}] + [\mathsf{n}_2,\mathsf{E}] = 0 \tag{1b}$$

$$[\mathsf{n}_1,\mathsf{B}] + [\mathsf{n}_2,\mathsf{B}] = -\frac{\overline{\eta_m\mathsf{v}}}{\nu_0} \tag{1c}$$

$$(\mathsf{n}_1,\mathsf{B}) + (\mathsf{n}_2,\mathsf{B}) = 0 \tag{1d}$$

The shorthand notation $\bar{\eta} \equiv \bar{\eta}_1 + \bar{\eta}_2$, $\overline{\eta_m\mathsf{v}} \equiv \overline{\eta_m\mathsf{v}_1} + \overline{\eta_m\mathsf{v}_2}$ is used.
Normals are external to the region indicated by the subscript.
The electromagnetic vectors in (1) can be replaced by the poten-
tial functions using (4.5,6), *viz.*,

$$\mathsf{E} = -\text{grad } \phi - \dot{\mathsf{A}} \tag{2}$$
$$\mathsf{B} = \text{curl } \mathsf{A} \tag{3}$$

Upon substituting (2) in (1a) using

$$(\mathsf{n}, \text{grad } \phi) = \frac{\partial\phi}{\partial n} \tag{4}$$

this becomes

$$\frac{\partial\phi_1}{\partial n_1} + \frac{\partial\phi_2}{\partial n_2} + (\mathsf{n}_1,\dot{\mathsf{A}}_1) + (\mathsf{n}_2,\dot{\mathsf{A}}_2) = \frac{\bar{\eta}}{\epsilon_0} \tag{5}$$

Before substituting (2) in (1b) let this be multiplied vectorially by a unit vector $\mathfrak{s}$ tangent to the surface of discontinuity.   Thus,

$$[\mathfrak{s},[\mathfrak{n}_1,E_1]] + [\mathfrak{s},[\mathfrak{n}_2,E_2]] = 0 \tag{6}$$

Using the vector formula

$$[A,[B,C]] = B(A,C) - C(A,B) \tag{7a}$$

and noting that since $\mathfrak{s}$ and $\mathfrak{n}$ are mutually perpendicular $(\mathfrak{s},\mathfrak{n}) = 0$

$$[\mathfrak{s},[\mathfrak{n},E]] = \mathfrak{n}(\mathfrak{s},E) \tag{7b}$$

Hence (1b) is equivalent to

$$\mathfrak{n}_1(\mathfrak{s},E_1) + \mathfrak{n}_2(\mathfrak{s},E_2) = 0 \tag{8}$$

or, since $\mathfrak{n}_1 = -\mathfrak{n}_2$,

$$(\mathfrak{s},E_1) - (\mathfrak{s},E_2) = 0 \tag{9}$$

Upon substituting (2) in (9) using (4) with $\mathfrak{s}$ written for $\mathfrak{n}$,

$$\frac{\partial \phi_1}{\partial s} - \frac{\partial \phi_2}{\partial s} + (\mathfrak{s},\dot{A}_1) - (\mathfrak{s},\dot{A}_2) = 0 \tag{10}$$

Since the vector and scalar potentials are to be used exclusively to compute either the electromagnetic vectors or the density functions in terms of which they are defined, it is sufficient to impose such boundary conditions on ϕ and A as are consistent with (1); it is not necessary that they be as general as possible. Thus (5) and (10) will certainly be satisfied if at the boundary

$$A_1 - A_2 = 0 \tag{11}$$

$$\left(\frac{\partial \phi}{\partial n_1}\right)_1 + \left(\frac{\partial \phi}{\partial n_2}\right)_2 = \frac{\bar{\eta}}{\epsilon_0} \tag{12}$$

$$\left(\frac{\partial \phi}{\partial s}\right)_1 - \left(\frac{\partial \phi}{\partial s}\right)_2 = 0 \tag{13}$$

No condition has been imposed on ϕ itself.   For all boundaries characterized only by $\bar{\eta}$ and $\overline{\eta_m v}$, it is possible to write

$$\phi_1 - \phi_2 = 0 \tag{14}$$

[This condition is not correct if a surface density of polarization k (I.10) is involved.   In this case, $\phi_1 - \phi_2 = k$.   The function k is not required in the analysis of problems in electromagnetic

engineering. In a similar way, (11) would not be correct if a surface density of magnetization (I.20) were defined.]

In order to determine the boundary conditions for the normal and tangential derivatives of the vector potential by substituting (3) in (1c,d), it is necessary to express curl A in terms of its components parallel and perpendicular to the boundary surface. This may be done in terms of a set of rectangular axes defined by the unit vectors $\hat{n}$, $\hat{s}$, $\hat{p}$ arranged to form a right-handed system with $\hat{n} = \hat{n}_2 = -\hat{n}_1$ perpendicular to the boundary, with $\hat{s}$ parallel to the surface *and in the direction of* $\overline{\eta_m v}$, with $\hat{p}$ parallel to the

Fig. 5.1.—Rectangular coordinates at a boundary.

surface and perpendicular to $\hat{s}$. This is illustrated in Fig. 5.1. The curl is conveniently expressed in the form

$$\text{curl } A = \begin{vmatrix} \hat{n} & \hat{s} & \hat{p} \\ \dfrac{\partial}{\partial n} & \dfrac{\partial}{\partial s} & \dfrac{\partial}{\partial p} \\ A_n & A_s & A_p \end{vmatrix} \tag{15}$$

Upon substituting (15) for B in (1c) three terms are obtained on the left which are multiplied, respectively, by the vectors $[\hat{n},\hat{n}]$, $[\hat{n},\hat{s}]$, $[\hat{n},\hat{p}]$. Of these the first term vanishes because $[\hat{n},\hat{n}] = 0$; the second term must also vanish because it is perpendicular to $\hat{s}$ which is by definition the direction of $\overline{\eta_m v}$; the third term, multiplied by the vector $[\hat{n},\hat{p}] = -\hat{s}$, is parallel to $\overline{\eta_m v}$. Consequently, (15) substituted in (1c) yields

$$\hat{s}\left(\frac{\partial A_s}{\partial n} - \frac{\partial A_n}{\partial s}\right)_1 - \hat{s}\left(\frac{\partial A_s}{\partial n} - \frac{\partial A_n}{\partial s}\right)_2 = -\frac{\overline{\eta_m v}}{\nu_0} \tag{16}$$

It is seen in (4.20) that A is always parallel to the moving charge in terms of which it is calculated. Since $\overline{\eta_m v}$ in (16) is by definition in the direction of $\hat{s}$ it can contribute nothing to A_n or A_p.

so that

$$\left(\frac{\partial A_n}{\partial s}\right)_1 - \left(\frac{\partial A_n}{\partial s}\right)_2 = 0 \tag{17}$$

and

$$\hat{s}\left(\frac{\partial A_s}{\partial n}\right)_1 - \hat{s}\left(\frac{\partial A_s}{\partial n}\right)_2 = -\frac{\overline{\eta_m v}}{v_0} \tag{18}$$

or, in terms of $\hat{n}_1 = -\hat{n} = -\hat{n}_2$,

$$\left(\frac{\partial A_s}{\partial n_1}\right)_1 + \left(\frac{\partial A_s}{\partial n_2}\right)_2 = \frac{\overline{\eta_m v}}{v_0} \tag{19}$$

Here $\dot{A}_s$ is parallel to $\overline{\eta_m v}$.   From the term in $[\hat{n}, \hat{s}]$,

$$\left(\frac{\partial A_n}{\partial p}\right)_1 - \left(\frac{\partial A_n}{\partial p}\right)_2 = \left(\frac{\partial A_p}{\partial n_1}\right)_1 + \left(\frac{\partial A_p}{\partial n_2}\right)_2 = 0 \tag{20}$$

with A_p perpendicular to $\overline{\eta_m v}$ but tangent to the boundary.

Upon substituting (15) in (1d), terms multiplied by $(\hat{n}, \hat{n})$, $(\hat{n}, \hat{s})$, $(\hat{n}, \hat{p})$ are obtained.   Since $(\hat{n}, \hat{s})$ and $(\hat{n}, \hat{p})$ are zero, (1d) yields merely

$$\left(\frac{\partial A_p}{\partial s} - \frac{\partial A_s}{\partial p}\right)_1 - \left(\frac{\partial A_p}{\partial s} - \frac{\partial A_s}{\partial p}\right)_2 = 0 \tag{21}$$

Since A_s and A_p are independent, (21) may be separated to give

$$\left(\frac{\partial A_p}{\partial s}\right)_1 - \left(\frac{\partial A_p}{\partial s}\right)_2 = 0 \tag{22}$$

$$\left(\frac{\partial A_s}{\partial p}\right)_1 - \left(\frac{\partial A_s}{\partial p}\right)_2 = 0 \tag{23}$$

Summarizing, the following boundary conditions obtain for the scalar potential:

$$\phi_1 - \phi_2 = 0 \tag{24}$$

$$\epsilon_0 \left(\frac{\partial \phi}{\partial n_1}\right)_1 + \epsilon_0 \left(\frac{\partial \phi}{\partial n_2}\right)_2 = \bar{\eta} \equiv \bar{\eta}_1 + \bar{\eta}_2 \tag{25}$$

$$\left(\frac{\partial \phi}{\partial s}\right)_1 - \left(\frac{\partial \phi}{\partial s}\right)_2 = 0 \tag{26}$$

That is, ϕ is continuous across a boundary as is its tangential derivative; its normal derivative is discontinuous by $\bar{\eta}/\epsilon_0$.

The boundary conditions for the vector potential are

$$A_1 - A_2 = 0 \tag{27}$$

$$\nu_0 \left(\frac{\partial A_s}{\partial n_1}\right)_1 + \nu_0 \left(\frac{\partial A_s}{\partial n_2}\right)_2 = \overline{\eta_m v} \equiv \overline{\eta_m v_1} + \overline{\eta_m v_2} \tag{28}$$

$$\left(\frac{\partial A_{n,p}}{\partial n_1}\right)_1 + \left(\frac{\partial A_{n,p}}{\partial n_2}\right)_2 = 0 \tag{29}$$

$$\left(\frac{\partial A_{n,s,p}}{\partial p}\right)_1 - \left(\frac{\partial A_{n,s,p}}{\partial p}\right)_2 = 0 \tag{30}$$

$$\left(\frac{\partial A_{n,s,p}}{\partial s}\right)_1 - \left(\frac{\partial A_{n,s,p}}{\partial s}\right)_2 = 0 \tag{31}$$

The vector potential is continuous across a boundary as is the tangential derivative of its tangential component; the normal derivative of its tangential component in the direction of $\overline{\eta_m v}$ is discontinuous by $\overline{\eta_m v}/\nu_0$; the normal derivative of its tangential component at right angles to $\overline{\eta_m v}$ is continuous. More briefly, only the normal derivative of the component of the vector potential *parallel* to the surface current is discontinuous.

6. Equations of d'Alembert, Poisson, and Laplace in Simple Media.—The potential equations that were obtained in the preceding section involve the essential densities $\bar{\rho} = \rho - \text{div } P$, and $\overline{\rho_m v} = i + \text{curl } M + \dot{P}$. The solution of these equations for ϕ and A thus assumes a knowledge of all the volume densities ρ, P, i, and M. In media in which the constitutive parameters ϵ_r, ν_r, and σ may be used, considerable simplification is achieved by reducing the number of density functions that must be known. In writing equations for simple media it is advantageous to consider two cases separately. The first of these involves simply polarizing and magnetizing media, so that the parameters ϵ_r and ν_r appear and the densities P and $-M$ do not. However, no specific assumptions are made regarding the volume density of convection current i_f so that it appears explicitly. The second formulation is like the first in assuming simply polarizing and magnetizing media, but it assumes simply conducting media as well, so that all three parameters ϵ_r and ν_r and σ appear and P, $-M$, and i_f do not.

In simply polarizing and magnetizing media, the first and third field equations have the following form:

$$\epsilon \text{ div } E = \rho_f \tag{1}$$

$$\nu \text{ curl } B = i_f + \epsilon \dot{E} \tag{2}$$

If these equations are compared with (4.7) and (4.8), they are found to be formally exactly like the more general ones.   It is only necessary to write

$$\left. \begin{array}{l} \epsilon \text{ for } \epsilon_0 \\ \nu \text{ for } \nu_0 \\ \rho_f \text{ for } \bar{\rho} \\ i_f \text{ for } \overline{\rho_m \mathbf{v}} \end{array} \right\} \tag{3}$$

By defining scalar and vector potentials according to

$$-\text{grad } \phi = \mathbf{E} + \dot{\mathbf{A}} \tag{4}$$

$$\text{curl } \mathbf{A} = \mathbf{B} \tag{5a}$$

$$\text{div } \mathbf{A} = -\frac{\epsilon}{\nu} \frac{\partial \phi}{\partial t} \tag{5b}$$

equations corresponding to (4.19) and (4.20) are obtained.   They are

$$\nabla^2 \phi - \frac{\epsilon}{\nu} \ddot{\phi} = -\frac{\rho_f}{\epsilon} \tag{6}$$

$$\nabla^2 \mathbf{A} - \frac{\epsilon}{\nu} \ddot{\mathbf{A}} = -\frac{i_f}{\nu} \tag{7}$$

The corresponding steady-state equations are

$$\nabla^2 \phi = -\frac{\rho_f}{\epsilon} \tag{8}$$

$$\nabla^2 \mathbf{A} = -\frac{i_f}{\nu} \tag{9}$$

The second form of the field equations in simple media is derived from the field equations in the form

$$\epsilon \text{ div } \mathbf{E} = \rho_f \tag{10}$$

$$\nu \text{ curl } \mathbf{B} = \sigma \mathbf{E} + \epsilon \dot{\mathbf{E}} \tag{11}$$

By defining scalar and vector potentials in terms of the relations

$$-\text{grad } \phi = \mathbf{E} + \dot{\mathbf{A}} \tag{12}$$

$$\left\{ \begin{array}{l} \text{curl } \mathbf{A} = \mathbf{B} \\[2mm] \text{div } \mathbf{A} = -\dfrac{\{\sigma\phi + \epsilon\dot{\phi}\}}{\nu} \end{array} \right. \tag{13a} \tag{13b}$$

the following relations are derived:

$$\nabla^2 \phi - \frac{\sigma}{\nu} \phi - \frac{\epsilon}{\nu} \ddot{\phi} = -\frac{\rho_f}{\epsilon} \tag{14}$$

$$\nabla^2 A - \frac{\sigma}{\nu} \dot{A} - \frac{\epsilon}{\nu} \ddot{A} = 0 \tag{15}$$

These equations differ from the previous ones in that terms in ϕ and $\dot{A}$ appear together with the conductivity σ. Since it has been shown that in conductors ρ_f vanishes and that in non-conductors ρ_f is a constant which may in most cases be set equal to zero, the following symmetrical equations are usually valid:

$$\nabla^2 \phi - \frac{\sigma}{\nu} \dot{\phi} - \frac{\epsilon}{\nu} \ddot{\phi} = 0 \tag{16}$$

$$\nabla^2 A - \frac{\sigma}{\nu} \dot{A} - \frac{\epsilon}{\nu} \ddot{A} = 0 \tag{17}$$

The potential equation of continuity that must also be satisfied is

$$\operatorname{div} A + \frac{\sigma}{\nu} \phi + \frac{\epsilon}{\nu} \dot{\phi} = 0 \tag{18}$$

Depending upon the particular problem to be investigated, one of the three forms of the potential equations may be selected. It is to be noted that the last form involves no volume densities and only the three constitutive parameters.

The boundary conditions for the potential functions at a boundary between two simple media are obtained as in Sec. 5 but proceeding from (II.15.17) to (II.15.20). The results are

$$\phi_1 - \phi_2 = 0 \tag{19}$$

$$\epsilon_1 \left(\frac{\partial \phi}{\partial n_1}\right)_1 + \epsilon_2 \left(\frac{\partial \phi}{\partial n_2}\right)_2 = \eta_{1f} + \eta_{2f} \tag{20}$$

$$\left(\frac{\partial \phi}{\partial s}\right)_1 - \left(\frac{\partial \phi}{\partial s}\right)_2 = 0 \tag{21}$$

$$A_1 - A_2 = 0 \tag{22}$$

$$\nu_1 \left(\frac{\partial A_s}{\partial n_1}\right)_1 + \nu_2 \left(\frac{\partial A_s}{\partial n_2}\right)_2 = l_{1f} + l_{2f} \tag{23}$$

$$\left(\frac{\partial A_{n,p}}{\partial n_1}\right)_1 + \left(\frac{\partial A_{n,p}}{\partial n_2}\right)_2 = 0 \tag{24}$$

$$\left(\frac{\partial A_{n,s,p}}{\partial p}\right)_1 - \left(\frac{\partial A_{n,s,p}}{\partial p}\right)_2 = 0 \tag{25}$$

$$\left(\frac{\partial A_{n,s,p}}{\partial s}\right)_1 - \left(\frac{\partial A_{n,s,p}}{\partial s}\right)_2 = 0 \tag{26}$$

The four first-order field equations defining the vectors E and B have been replaced by two mutually independent potential equations. These equations have been written both in the general form and in the special forms suited to simple media. An alternative mathematical model of space is in this way provided. Instead of describing the electrical properties of space in terms of the two vector point functions E and B which define the electromagnetic field, they may be described in terms of one scalar and one vector point function which together define the potential field. The advantage of the potential formulation is a mathematical one; in general, the potential equations may be solved more readily than the field equations.

7. Polarization and Magnetization Potentials.—Instead of making use of the scalar and vector potentials ϕ and A in the solution of electromagnetic problems, it is sometimes convenient to introduce two vector functions which are called polarization and magnetization potentials. Alternative names are, respectively, the Hertzian vector and the Fitzgerald vector. (Some writers call both functions Hertzian vectors.) The new functions can be defined only when the essential volume densities of charge and moving charge $\bar{\rho}$ and $\overline{\rho_m \mathsf{v}}$ are expressed entirely in terms of the volume densities of polarization and magnetization. However, by a suitable choice of an equivalent although perhaps physically meaningless model and an appropriate mode of subdivision, the functions $\bar{\rho}$ and $\overline{\rho_m \mathsf{v}}$ can always be written exclusively in terms of a *mathematically equivalent* combination of M and P. Thus let

$$\bar{\rho} = -\operatorname{div} \mathsf{P} \tag{1}$$

$$\overline{\rho_m \mathsf{v}} = \operatorname{curl} \mathsf{M} + \dot{\mathsf{P}} \tag{2}$$

The polarization potential Z and the magnetization potential Y are defined as completely as required by writing

$$\phi = -\operatorname{div} \mathsf{Z} \tag{3}$$

$$\mathsf{A} = \frac{\epsilon_0}{\nu_0} \dot{\mathsf{Z}} + \operatorname{curl} \mathsf{Y} \tag{4}$$

It is to be noted that Z and Y independently satisfy the potential equation of continuity (4.17). By substituting (3) and (4) in the potential equations (4.19) and (4.20) and making use of the identities (4.11) and (4.12) with appropriate changes in variable,

$$\operatorname{div}\left\{\operatorname{grad} \operatorname{div} \mathsf{Z} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Z}} + \frac{\mathsf{P}}{\epsilon_0}\right\} = 0 \qquad (5)$$

$$\frac{\epsilon_0}{\nu_0}\frac{\partial}{\partial t}\left\{\nabla^2\mathsf{Z} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Z}} + \frac{\mathsf{P}}{\epsilon_0}\right\} + \nabla^2 \operatorname{curl} \mathsf{Y} - \frac{\epsilon_0}{\nu_0}\operatorname{curl} \ddot{\mathsf{Y}}$$
$$+ \frac{1}{\nu_0}\operatorname{curl} \mathsf{M} = 0 \quad (6)$$

The quantity $-\operatorname{curl} \operatorname{curl} \mathsf{Z}$ may be added inside the braces in (5) because div curl operating on any vector gives identically zero. With the vector identity (4.12) written in terms of Z, (5) reduces to

$$\operatorname{div}\left\{\nabla^2\mathsf{Z} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Z}} + \frac{\mathsf{P}}{\epsilon_0}\right\} = 0 \qquad (7)$$

By making use of the identity (4.12) written in terms of Y and noting that div curl operating on a vector and curl grad[1] operating on a scalar give identically zero, it is easily proved that

$$\nabla^2 \operatorname{curl} \mathsf{Y} = \operatorname{curl} \nabla^2\mathsf{Y} \qquad (8)$$

Accordingly (6) may be written as follows:

$$\frac{\epsilon_0}{\nu_0}\frac{\partial}{\partial t}\left\{\nabla^2\mathsf{Z} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Z}} + \frac{\mathsf{P}}{\epsilon_0}\right\} + \operatorname{curl}\left\{\nabla^2\mathsf{Y} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Y}} + \frac{\mathsf{M}}{\nu_0}\right\} = 0 \quad (9)$$

Although less restrictive definitions are certainly possible, both (7) and (9) are evidently true if Z and Y are required to satisfy the following relations:

$$\nabla^2\mathsf{Z} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Z}} = -\frac{\mathsf{P}}{\epsilon_0} \qquad (10)$$

$$\nabla^2\mathsf{Y} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Y}} = -\frac{\mathsf{M}}{\nu_0} \qquad (11)$$

Thus (10) and (11) are sufficient but not necessary conditions to satisfy (9). These equations are exactly the same in form as (4.20). It is seen that Z depends entirely on P so that it is

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[1] This is readily proved in Cartesian coordinates. Thus

$$\operatorname{curl} \operatorname{grad} \phi = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix} = 0$$

appropriately called the polarization potential; similarly, Y depends entirely on M so that it is given the name magnetization potential.   Because P and M are mutually independent, Z and Y are also independent.   Just as $-\mathsf{M}$ is the magnetic analogue of P, $-\mathsf{Y}$ is the magnetic analogue of Z.

If (3) and (4) are substituted in (4.5) and (4.6), expressions for computing the field vectors E and B directly from Z and Y are obtained.   They are

$$\mathsf{E} = -\mathrm{grad}\ \phi - \dot{\mathsf{A}} = \mathrm{grad\ div}\ \mathsf{Z} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Z}} - \mathrm{curl}\ \dot{\mathsf{Y}} \quad (12)$$

$$\mathsf{B} = \mathrm{curl}\ \mathsf{A} = \frac{\epsilon_0}{\nu_0}\mathrm{curl}\ \dot{\mathsf{Z}} + \mathrm{curl\ curl}\ \mathsf{Y} \quad (13)$$

Since Z and Y are mutually independent, the electromagnetic field can always be separated into one part, called a *field of electric type*, that depends entirely on Z and a second part, called a *field of magnetic type*, that depends only on Y.   The electric type field is

$$\mathsf{E} = \mathrm{grad\ div}\ \mathsf{Z} - \frac{\epsilon_0}{\nu_0}\ddot{\mathsf{Z}} \quad (14)$$

$$\mathsf{B} = \frac{\epsilon_0}{\nu_0}\mathrm{curl}\ \dot{\mathsf{Z}} \quad (15)$$

The magnetic type field is

$$\mathsf{E} = -\mathrm{curl}\ \dot{\mathsf{Y}} \quad (16)$$
$$\mathsf{B} = \mathrm{curl\ curl}\ \mathsf{Y} \quad (17)$$

An important field of the electric type is that computed from the distribution of current and charge in a straight, cylindrical, highly conducting antenna characterized entirely by the densities i_f and η_f.   Such an antenna can always be replaced by a *mathematically equivalent* cylinder of dielectric characterized entirely in terms of P so that only the polarization potential Z is required in solving a particular boundary-value problem.   Use is made of Z in solving the boundary-value problem of a straight antenna over an imperfectly conducting plane (Volume II).

The field computed from a circular ring of wire characterized entirely in terms of the density i is of the magnetic type.   The loop of wire can be replaced by a mathematically equivalent disk characterized by M alone—a so-called magnetic shell.   In this

case the magnetization potential Y is sufficient in solving a boundary-value problem. Use is made of Y in solving the boundary-value problem of a loop antenna over an imperfectly conducting plane (Volume II).

8. Telegraphist's Equations.—An alternative form of the field equations for simply magnetizing, polarizing, and conducting media may be obtained in the form of two second-order equations in which the variables E and B are separated. Proceeding from the field equations

$$\epsilon \operatorname{div} \mathsf{E} = \rho_f \tag{1}$$

$$\operatorname{curl} \mathsf{E} = -\dot{\mathsf{B}} \tag{2}$$

$$\nu \operatorname{curl} \mathsf{B} = \sigma \mathsf{E} + \epsilon \dot{\mathsf{E}} \tag{3}$$

$$\operatorname{div} \mathsf{B} = 0 \tag{4}$$

the first step is to take the curl of the second equation and differentiate the third with respect to time.

$$\operatorname{curl} \operatorname{curl} \mathsf{E} = -\operatorname{curl} \dot{\mathsf{B}} \tag{5}$$

$$\nu \operatorname{curl} \dot{\mathsf{B}} = \sigma \dot{\mathsf{E}} + \epsilon \ddot{\mathsf{E}} \tag{6}$$

Eliminating B,

$$\operatorname{curl} \operatorname{curl} \mathsf{E} + \frac{\sigma}{\nu} \dot{\mathsf{E}} + \frac{\epsilon}{\nu} \ddot{\mathsf{E}} = 0 \tag{7}$$

By reversing the operations, taking the time derivative of the second field equation and the curl of the third

$$\nu \operatorname{curl} \operatorname{curl} \mathsf{B} = \sigma \operatorname{curl} \mathsf{E} + \epsilon \operatorname{curl} \dot{\mathsf{E}} \tag{8}$$

$$\operatorname{curl} \dot{\mathsf{E}} = -\ddot{\mathsf{B}} \tag{9}$$

Upon substituting to eliminate E

$$\operatorname{curl} \operatorname{curl} \mathsf{B} + \frac{\sigma}{\nu} \dot{\mathsf{B}} + \frac{\epsilon}{\nu} \ddot{\mathsf{B}} = 0 \tag{10}$$

The vector symbol ∇^2 operating on a vector has already been defined in (4.12). Using this definition

$$\operatorname{curl} \operatorname{curl} \mathsf{B} = -\nabla^2 \mathsf{B} + \operatorname{grad} \operatorname{div} \mathsf{B} \tag{11a}$$

$$\operatorname{curl} \operatorname{curl} \mathsf{E} = -\nabla^2 \mathsf{E} + \operatorname{grad} \operatorname{div} \mathsf{E} \tag{11b}$$

From (1) and (4) it follows that

$$\operatorname{grad} \operatorname{div} \mathsf{B} = 0, \quad \text{because} \quad \operatorname{div} \mathsf{B} = 0 \tag{12a}$$

grad div E = grad $\dfrac{\rho_f}{\epsilon}$ (= 0, if ρ_f is constant or zero as in
conductors and nonconductors
that are simple media)   (12b)

Hence, with (11) and (12), and assuming grad $\rho_f = 0$, (7) and (10) become

$$\nabla^2 E - \frac{\sigma}{\nu}\dot{E} - \frac{\epsilon}{\nu}\ddot{E} = 0 \tag{13}$$

$$\nabla^2 B - \frac{\sigma}{\nu}\dot{B} - \frac{\epsilon}{\nu}\ddot{B} = 0 \tag{14}$$

These equations are formally like the potential equation (6.17). They are frequently called the telegraphist's equations. They are readily solved in one-dimensional form, as shown in Chapter V. It is to be noted that (13) and (14) are derived from (2) and (3), so that (1) and (4) must be satisfied by solutions of (13) and (14).

ENERGY FUNCTIONS

9. Concept of Energy.—In preceding sections the mathematical description of space in terms of the electric and magnetic vectors as defined by the field equations was transformed by defining scalar and vector potential functions. In a similar way the force equation, which interconnects the mathematical models of mechanics and electromagnetism, may be expressed in a different and frequently useful form by defining two new functions called energy functions.

In order to understand the significance of electromagnetic energy functions in a general scheme of technical science, it is well to recall that the quest of physical science is essentially a search for something permanent and hence useful in making predictions in a world in which observation and thought alike seem destined to perpetual change. Mathematical physics has found such permanence, such immunity from change, in the differential equations that it has introduced as parts of mathematical models of the physical world. Although new and better models continue to be invented, any particular one that has proved successful in predicting theoretical analogues for recorded pointer readings in a certain class of phenomena is always found to make predictions of the same phenomena with

the same degree of accuracy. But scientists and particularly engineers are frequently not satisfied with an invariance in time of suitably defined mathematical equations or functions. Wherever possible, they seek to find or devise quantities in mechanical models that are analogues of the mathematical functions appearing in the equations. This procedure is called giving physical significance to mathematical concepts. As a next and usually not justifiable step, they try to *identify* such mechanical models with the world of pointer readings and assume that something corresponding to the original mathematical function or equation *actually exists in nature*, and perhaps remains constant through the ages. Such a belief is expressed in the law of conservation of energy, if by energy is meant a physically real "substance" which is assumed to exist in nature.

The origin of the concept of energy is found in the mathematical model of macroscopic mechanics. It follows from the definition of mechanical force

$$\mathbf{F}_M = m\frac{d\mathbf{v}}{dt} \tag{1}$$

A study of the line integral of force, called work, together with the relation

$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{ds}\frac{ds}{dt} = \frac{d\mathbf{v}}{ds}v \tag{2}$$

leads to the concept of energy.

$$\int_{s_0}^{s} (\mathbf{F}_M, d\mathbf{s}) = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 \tag{3}$$

The left side in (3) defines the work done by the force $\mathbf{F}_M$ in the motion of the mass m from s_0 to s; the right side is the change in a function called the kinetic energy. The left side of (3) can be integrated directly when $(\mathbf{F}_M, d\mathbf{s})$ is a perfect differential. This is true when $\mathbf{F}_M$ is a potential vector so that it can be derived as the negative gradient of a scalar potential V. Thus, provided that curl $\mathbf{F}_M = 0$,

$$(\mathbf{F}_M, d\mathbf{s}) = -dV \tag{4}$$

with V a scalar function. By carrying out the integration,

$$V + \tfrac{1}{2}mv^2 = V_0 + \tfrac{1}{2}mv_0^2 \tag{5}$$

The right side, and hence also the left side of (5), is a constant

for any motion of m.   Here, then, is a mathematical quantity $V + \frac{1}{2}mv^2$ that is fundamentally involved in any motion of the mass m in a so-called conservative mechanical system and that has the significant characteristic of being invariant in time. The constant is denoted by W and is called the *total energy of the system*.   The scalar potential function V is given the name potential energy; the function $\frac{1}{2}mv^2$ has already been called the kinetic energy.

The energy terminology may be generalized to include non-conservative mechanical systems by suitably defining thermal, electrical, chemical, and many other energy functions in such a way that an increase in one such function defined for a given region is always accompanied by a corresponding decrease in another energy function defined for the same or perhaps a different, quite distant region.   The statement that energy in a given region or in the universe as a whole is conserved expresses the belief that it is always possible to invent enough energy functions for the region in question, or for the entire universe, so that the time rate of change of their sum is invariant.   In many instances, it is possible to define energy-transfer functions on the boundary of a closed region which, when integrated over the boundary, measure the time rate of decrease of the energy functions defined in the region or the equal time rate of increase of the energy functions defined outside.

The generalized concept of energy and the definition of a large variety of energy functions in accordance with the postulate of invariance has proved of great value both in solving problems and in interrelating the mathematical models of the several branches of physical science.   Unfortunately, it has often been assumed that the mathematically defined energy functions must *necessarily* be analogues of a physically real "energy substance" that exists in the universe in a myriad of forms but in a forever unchanging amount.   Although such an assumption makes no difference analytically, much confusion in thought and many a physically unacceptable interpretation can be avoided if energy is looked upon as a mathematical invariant rather than as a universally constant, fluidlike substance that mysteriously changes its form while flowing from here to there.

10. Definition of the Electromagnetic Energy Function and Energy-transfer Function.—A function U to which the name

electromagnetic energy may be applied should satisfy an equation of the form

$$\left(\frac{dW}{dt} + A_q\frac{dQ}{dt} + A_e\frac{dU}{dt}\right) + (A_qT_q + A_eT) = 0 \qquad (1)$$

in a closed region involving a representation in terms of the mathematical models of macroscopic mechanics, thermodynamics, and electromagnetism. W is the mechanical energy function (or the mechanical equivalent of any other energy function); Q is the thermal energy function; T_q is the thermal transfer function of the enclosing surface; A_q is the mechanical equivalent of thermal energy or simply the mechanical equivalent of heat. It is presumed that a function U to be called the electromagnetic energy function and a function T to be called the electromagnetic energy-transfer function of the enclosing boundary can be found to satisfy this equation in such a way that A_e is a numerical and dimensional constant. This constant might be called the mechanical equivalent of electromagnetic energy, or simply the mechanical equivalent of electricity. In the practical system of units, the energy functions are so defined that A_e is dimensionless and equal to one and it will be assigned this value from here on. It is assumed in writing (1) that each energy-transfer function and the time rate of change of each energy function carries its own sign according as the forces contributing to it do active (positive) work or resist its performance (do negative work). For example, if the forces contributing to W in a closed but not isolated region τ do positive work and so supply energy to the region, while all other forces within and outside the region do negative work, i.e., receive energy, then (1) may be written as follows for the region τ with all signs explicitly written and all derivatives and transfer functions positive.

$$\frac{dW}{dt} - A_q\frac{dQ}{dt} - \frac{dU}{dt} - A_qT_q - T = 0 \qquad (2)$$

It has already been stated that electrical phenomena are inextricably entangled with mechanical ones because a large number of pointer-reading methods devised to record electrical changes in bodies use mechanical-electrical devices. Examples of these are ammeters, voltmeters, and dynamometers. It follows that

macroscopic electrical theory must take account of mechanics to the extent that the mathematical models of these two separately developed branches of physics and engineering must share at least one equation.   The interconnection between electromagnetism and mechanics might be established directly and originally in terms of the power equation (2) by suitably defining the electromagnetic energy function U.   Actually, a connection has already been established by constructing the electromagnetic force vector F to satisfy the vector-force equation

$$\mathsf{F}_M + \mathsf{F} = 0 \tag{3}$$

Here F_M is the resultant mechanical force, and F was defined in Sec. II.12.   The definition of electromagnetic energy and energy-transfer functions must be consistent with this equation and with the definition of electromagnetic force upon which it depends.   But these conditions do not suffice to limit the choice of such functions to a single pair.   In fact, a *wide variety* of quite different sets of functions may be constructed which satisfy these requirements and which all have the dimensions of energy. One pair, in particular, depends upon transformations of the force equation involving the electromagnetic vectors, while another pair uses different transformations leading to expressions in terms of the potential functions ϕ and A.   The properties of the two sets of functions are not the same, and either pair might be chosen.   Actually, engineering practice has established itself in favor of the first pair of functions involving the field vectors, and these will be defined first.

The equilibrium between electrical and mechanical forces acting on the charges in an element of volume $d\tau$ may be expressed in

$$d\mathsf{F}_M + d\mathsf{F} = 0 \tag{4}$$

Let (4) be multiplied scalarly by the mean nonrandom velocity u of the *free charges* in $d\tau$ that are actually engaged in nonrandom motion and integrated over the volume τ.

$$\int_\tau (d\mathsf{F}_M, \mathsf{u}) + \int_\tau (d\mathsf{F}, \mathsf{u}) = 0 \tag{5}$$

The first integral on the left may be interpreted to be the sum of two terms by resolving $d\mathsf{F}_M$ into an active, charge-separating force $d\mathsf{F}'_M$ and an oppositely directed "frictional" or resisting force

dF''_M, as discussed in Sec. II.18. The active force dF'_M is non-vanishing only in charge-separating regions of which at least one is assumed to be present within τ. The frictional force dF''_M resists the acceleration of the nonrandom motion and leads to an increase in random or heat motion. Accordingly, all positive work is done on the charges by dF'_M in the charge-separating region against the "thermal" forces dF''_M and the electromagnetic force dF which acts wherever there are charges. Expressed in terms of positive rates of change of energy, *i.e.*, with algebraic signs written

$$\int_\tau (dF'_M, u) = \frac{dW}{dt} \tag{6}$$

$$\int_\tau (dF''_M, u) = -A_q \frac{dQ}{dt} \tag{7}$$

W is the total mechanical, chemical, or other energy function associated with the charge-separating region. It has a positive time rate of change because it is associated with forces that do work on the moving free charge in τ. Accordingly, W decreases in time. Q is the thermal-energy function associated with the moving free charge in the region τ, which is assumed to be thermally isolated for simplicity, so that $T_q = 0$ and Q increases in time. The time rate of change of Q is negative because it is associated with forces against which work is done. With $T_q = 0$, the sum of (6) and (7) is equal to the first integral in (5) and may be substituted for it.

Using the notation of (6) and (7), (5) becomes

$$\frac{dW}{dt} - A_q \frac{dQ}{dt} + \int_\tau (dF, u) = 0 \tag{8}$$

(If the region τ is not thermally isolated, a thermal energy transfer function T_q multiplied by A_q must be subtracted on the left to account for the rate of decrease of Q due to the transfer of heat across the boundary of τ.) It is desired to define an energy function U within τ and, if necessary, an energy transfer function T on its boundary surface Σ_τ such that

$$\int_\tau (dF, u) = -\frac{dU}{dt} - T \tag{9}$$

with dU/dt and T positive.   If this can be done, (8) will have the form (2).

If surfaces of discontinuity that are characterized by surface densities of current are excluded by enclosing them in surfaces S_j so that the volume τ is divided into several volumes τ_i such that $\tau = \sum_i \tau_i$, the electromagnetic force acting on free charges engaged in nonrandom motion in an element $d\tau_i$ is

$$dF_i = \{\rho_f' E + [i_f, B]\} d\tau_i \tag{10}$$

The force acting on free charges engaged in nonrandom motion in an element $d\sigma_j$ of a boundary surface S_j within Σ_τ is

$$dF_j = \{\eta_f' E + [l_f, B]\} d\sigma_j \tag{11}$$

In accordance with the notation of Secs. I.13 and I.14

$$i_f = \rho_f' u; \qquad l_f = \eta_f' u \tag{12}$$

with ρ_f' and η_f', respectively, volume and surface densities of free charge moving with mean nonrandom velocity u.

Substitution of (10) and (11) in the integral in (9) gives

$$\int_\tau (dF, u) = \sum_i \int_{\tau_i} \{\rho_f'(E, u) + ([i_f, B], u)\} d\tau_i$$
$$+ \sum_j \int_{S_j} \{\eta_f'(E, u) + ([l_f, B], u)\} d\sigma_j \tag{13}$$

With (12),

$$([i_f, B], u) = \rho_f'([u, B], u) = \rho_f'(B, [u, u]) = 0 \tag{14}$$

because $[u, u]$ vanishes.   A similar product involving l_f vanishes in the same way.   Accordingly, with (12)

$$\int_\tau (dF, u) = \sum_i \int_{\tau_i} (i_f, E) d\tau_i + \sum_j \int_{S_j} (l_f, E) d\sigma_j \tag{15}$$

The integrals on the right in (15) may be transformed by expressing i_f and l_f in terms of the electromagnetic vectors using their fundamental definitions.   In general, these also involve the densities P and M so that i_f and l_f must be expressed either in terms of E, B, P, and M or in terms of E, B, D, and H.   The latter selection is customary.   Furthermore, since i_f is involved explicitly, a mode of subdivision in interpreting P and M

or D and H is implied that cuts through no bound groups. The following equations are used:

$$\text{curl } H = i_f + \dot{D} \tag{16}$$

$$[\hat{n}, H] = -I_f \tag{17}$$

Solving for i_f and I_f and substituting in the factors on the right in (15),

$$(i_f, E) = (\text{curl } H, E) - (\dot{D}, E) \tag{18}$$

$$(I_f, E) = -([\hat{n}, H], E) = -([H, E], \hat{n}) = ([E, H], \hat{n}) \tag{19}$$

Substitution in (15) gives

$$\int_\tau (dF, u) = \sum_i \int_{\tau_i} \{(\text{curl } H, E) - (\dot{D}, E)\} d\tau_i$$
$$+ \sum_j \int_{S_j} (\hat{n}, [E, H]) d\sigma_j \tag{20}$$

The right sides of (15) and (20) are mathematically equivalent, but they differ in a fundamental way. Whereas (15) involves integration only over those *parts of the volume τ* which contain moving free charges, *i.e.*, conductors, (20) involves integration over the *entire volume τ*. This is a consequence of the fact that the electromagnetic vectors were deliberately defined throughout space, and so long as only mathematical significance is attached to energy and to the electromagnetic field, no further comment is required. On the other hand, if energy is to be regarded as a physically meaningful "substance," the change from (15) to (20) is of fundamental significance because the "energy substance" apparently confined to conductors within Σ_τ in (15) may be interpreted as distributed throughout space in (20). *It is in passing from* (15) *to* (20) *that the idea of localizing and "storing" energy in space is made abruptly possible.*

The first term on the right in (20) may be transformed with the aid of the vector identity

$$\text{div } [E, H] = (H, \text{curl } E) - (E, \text{curl } H) \tag{21a}$$

This gives

$$(E, \text{curl } H) = -\text{div } [E, H] + (H, \text{curl } E) \tag{21b}$$

Upon introducing the field equation

$$\text{curl } E = -\dot{B} \tag{22}$$

the volume integral in (20) becomes

$$- \int_{\tau_i} \{\text{div } [E,H] + (H,\dot{B}) + (\dot{D},E)\}d\tau_i \qquad (23)$$

Let it be assumed that the boundary Σ_r enclosing the region τ is not a surface of discontinuity. The divergence theorem can then be applied to the volume τ or the volumes τ_i within Σ_r after properly enclosing boundaries where discontinuities exist within surfaces S_j

$$\sum_i \int_{\tau_i} \text{div } [E,H]d\tau_i = \sum_j \int_{S_j + \Sigma_r} (\mathbb{A},[E,H])d\sigma \qquad (24)$$

On forming (20) it is noted that all integrals over surfaces of discontinuity S_j cancel to leave

$$\int_\tau (dF,\mathsf{u}) = - \sum_i \int_{\tau_i} \{(H,\dot{B}) + (\dot{D},E)\}d\tau_i - \int_{\Sigma_r} (\mathbb{A},[E,H])d\sigma \quad (25)$$

If the expression on the right in (25) is compared with the right side of (9), it is clear that energy functions U and T may be defined to satisfy the following relations. The sum of integrals over the volumes τ_i is written as a single integral over $\tau = \sum_i \tau_i$ for simplicity and because surface currents are not required for practically available conductors so that no discontinuities can occur. The analysis was carried out including possible surface currents because the idealized case of perfect conductors where all currents are surface currents is often assumed in engineering problems. In such cases, the sum of integrals must be written for the single integral, but no change in interpretation is required.

$$\frac{dU}{dt} \equiv \int_\tau \{(H,\dot{B}) + (\dot{D},E)\}d\tau \qquad (26)$$

$$T \equiv \int_{\Sigma_r} (\mathbb{A},[E,H])d\sigma \qquad (27)$$

These are the conventional definitions for the time rate of change of an electromagnetic energy function U and for an electromagnetic energy transfer function T. Because (26) includes a term involving only the vectors E and D associated with electric phenomena and another involving only the vectors B and H associated with magnetic phenomena, it is customary to define separate time rates of change of electric and magnetic energies

U_E and U_M

$$\frac{dU_E}{dt} = \int_\tau (\dot{D},E)d\tau; \qquad \frac{dU_M}{dt} = \int_\tau (H,\dot{B})d\tau \qquad (28)$$

It is also customary to define a vector S, called the *Poynting vector*, by

$$S \equiv [E,H] \qquad (29)$$

so that T is the total outward normal flux of the Poynting vector.

$$T = \int_{\Sigma_\tau} (\hat{n},S)d\sigma \qquad (30)$$

From the point of view of general electromagnetic theory, the electromagnetic energy functions defined in (26) and (27) are not attractive because it is not possible to express them entirely in terms of the fundamental field vectors E and B. The appearance of D and H in the energy functions, as well as the original assumption that only free charges are involved in the force equation, not only implies a definite mode of subdivision but presupposes that time variations in polarization and magnetization lead to no increase in thermal energy. If the forces that oppose periodic variations in polarization and magnetization lead to an increase in the random motions of charges and groups of charges so that a rise in the thermal energy associated with τ occurs, *this is not included in dQ/dt in* (8). It is actually contained in *dU/dt*, so the electromagnetic function U as defined includes what is recognized as thermal energy unless there is no increase in heat associated with changes in time of polarization and magnetization. It is only in the special case in which thermal energy is increased exclusively as a result of free charges moving in imperfect conductors that U is strictly an *electromagnetic* energy function.

If the auxiliary vectors D and H are expanded in accordance with their definitions

$$D \equiv \epsilon_0 E + P \qquad H \equiv \nu_0 B - M \qquad (31)$$

(26) and (29) have a more complicated but also a more fundamental form.

$$\frac{dU}{dt} = \int_\tau \{\nu_0(B,\dot{B}) + \epsilon_0(E,\dot{E}) - (M,\dot{B}) + (\dot{P},E)\}d\tau \qquad (32)$$

$$S = \nu_0[E,B] - [E,M] \qquad (33)$$

The dimensions of the energy functions determined from (26) and (27) are as follows:

$$U \approx \frac{Q}{LT} \cdot \frac{VT}{L^2} \cdot L^3 \approx QV \text{ coulomb-volts or joules} \quad (34)$$

$$T \approx \frac{V}{L} \cdot \frac{Q}{LT} \cdot L^2 \approx \frac{QV}{T} \text{ volt-amperes or watts} \quad (35)$$

The coulomb-volt has the name joule; the joule per second is the watt. U is measured in joules, T in watts. Since the factor A_e in (1) is by definition dimensionless and equal to unity, the mechanical joule and watt and the electrical joule and watt are the same. Accordingly, the auxiliary dimension V in volts can be expressed directly in terms of Q, L, M, T. Since mechanical energy W has the dimensions

$$W \approx \frac{ML^2}{T^2} \text{ joules} \quad (36)$$

it follows that

$$V \approx \frac{ML^2}{T^2 Q} \frac{\text{joules}}{\text{coulombs}} \text{ or volts} \quad (37)$$

In electrical problems it is usually more convenient to retain the auxiliary dimension V than to introduce its equivalent in terms of Q, L, M, T.

11. Energy Functions in Space and in Simple Media.—If the region enclosed by the boundary Σ_r consists exclusively of space and simple media, the energy functions have a more attractive form. In space, where all densities vanish, $\mathsf{D} = \epsilon_0 \mathsf{E}$, $\mathsf{H} = \nu_0 \mathsf{B}$,

$$\frac{dU}{dt} = \int_r \{\nu_0(\mathsf{B},\dot{\mathsf{B}}) + \epsilon_0(\mathsf{E},\dot{\mathsf{E}})\} d\tau = \frac{d}{dt} \int_r \frac{1}{2} \{\nu_0 B^2 + \epsilon_0 E^2\} d\tau \quad (1)$$

$$T = \int_{\Sigma_r} (\mathsf{n},\mathsf{S}) d\sigma; \qquad \mathsf{S} = \nu_0[\mathsf{E},\mathsf{B}] \quad (2)$$

Explicit definition of the electromagnetic energy functions is suggested by (1)

$$U = U_M + U_B = \int_r \frac{1}{2} \nu_0 B^2 d\tau + \int_r \frac{1}{2} \epsilon_0 E^2 d\tau \quad (3)$$

In simply polarizing and magnetizing media the constitutive relations

$$\mathsf{P} = (\epsilon_r - 1)\epsilon_0 \mathsf{E}; \qquad -\mathsf{M} = (\nu_r - 1)\nu_0 \mathsf{B} \quad (4)$$

are assumed to obtain. This implies an *instantaneous* response in polarization and magnetization to changes in E and B and *no* associated increase in random or heat motion. The relations (4) give

$$D = \epsilon_r\epsilon_0 E = \epsilon E; \qquad H = \nu_r\nu_0 B = \nu B \tag{5}$$

The energy functions in simple media are

$$U = U_M + U_E = \int_\tau \frac{1}{2}\nu B^2 d\tau + \int_\tau \frac{1}{2}\epsilon E^2 d\tau \tag{6}$$

$$T = \int_{\Sigma_\tau} (\mathsf{n},S) d\sigma; \qquad S = \nu[E,B] \tag{7}$$

In a simply conducting region, by definition

$$i_f = \sigma E \tag{8}$$

or, if a charge-separating region characterized by an impressed or intrinsic electric field E^e (which by definition is equal in magnitude and opposite in direction to the electric field required to prevent the separation) is involved,

$$i_f = \sigma(E + E^e) \tag{9}$$

Since surface densities of moving free charge are not needed to describe simple conductors of *finite* conductivity,

$$i_f = \dot{0} \tag{10}$$

If (9) and (10) are used in (10.15), it follows that:

$$\int_\tau (dF,u) = \int_\tau \sigma\{(E + E^e,E)\} d\tau \tag{11}$$

This may be expanded into

$$\int_\tau (dF,u) = \int_\tau \sigma(E + E^e,E + E^e) d\tau - \int_\tau \sigma(E + E^e,E^e) d\tau \tag{12}$$

With (9) this gives

$$\int_\tau (dF,u) = \int_\tau \frac{i_f^2}{\sigma} d\tau - \int_\tau (i_f,E^e) d\tau \tag{13}$$

Substitution in (10.8) leads to

$$\int_\tau (dF_M,u) = \frac{dW}{dt} - A_q \frac{dQ}{dt} = \int_\tau (i_f,E^e) d\tau - \int_\tau \frac{i_f^2}{\sigma} d\tau \tag{14}$$

The first term on the right can now be identified with the power supplied to the region τ by nonelectrical forces in a charge-separating region; the second term is the power dissipated as heat. The first integral on the right in (14) vanishes except in a charge-separating region where E^o differs from zero. With (14), (6), and (7), the general power equation for simple media may be written in the alternative form

$$\int_\tau (i_f, E^o)d\tau - \int_\tau \frac{i_f^2}{\sigma} d\tau - \frac{\partial}{\partial t} \int_\tau \frac{1}{2} \{\nu B^2 + \epsilon E^2\} d\tau$$
$$- \int_{\Sigma_\tau} \nu(\hbar, [E, B])d\sigma = 0 \quad (15)$$

The power equation (15) and the associated energy functions (6) and (7) as written for simple media are formally attractive and practically useful in the solution of problems of many types, such as radiation from antennas and dissipation in wave guides. Interpreting (15), the first integral measures the time rate of decrease of the mechanical (or other nonelectrical) energy function W associated with τ. The second integral is the time rate of increase of the thermal energy function Q associated with moving *free* charges in τ. The third integral defines the time rate of increase of the electromagnetic energy function U associated with τ. The surface integral measures the time rate of increase of energy functions associated with all regions outside of τ due to the action of electromagnetic forces on charges beyond Σ_τ. If the first integral is zero in a particular case, it follows as a necessary conclusion that a nonvanishing and nontransient current i_f can exist in τ only if the surface integral is negative or the external normal $\hbar$ is reversed. This means that electromagnetic forces due to moving charges *outside* of τ are doing work on the charges *in* τ to maintain the current.

If energy is assumed to be a physical "substance" endowed with properties that admit of its localization and distribution in charge-filled or empty regions and of its flow from one region to another, the above interpretation of the power equation is usually not only modified in its terminology, but greatly specialized in its meaning in ways neither suggested nor justified by the mathematical analysis. Thus, it is argued that if the integral (6) defines the electromagnetic energy "stored" in τ the

integrand when written for simple media or for space

$$\tfrac{1}{2}\nu B^2 + \tfrac{1}{2}\epsilon E^2, \qquad \tfrac{1}{2}\nu_0 B^2 + \tfrac{1}{2}\epsilon_0 E^2 \tag{16}$$

"must" specify its spatially distributed density. Further, since nothing can be "stored" in complete emptiness, the conclusion is advanced that space itself must be filled with a medium, called the luminiferous ether, which is assigned such properties as may be necessary to permit the storage of energy. The physical consequences of the simple mathematical step from (10.15) to (10.20) are in this way made gigantic in their scope.

With the energy "substance" assumed distributed throughout space with the density specified by (16), the energy transfer function T is interpreted to measure the flow of energy across the enclosing boundary Σ_r. If the flow is outward as indicated by an outwardly directed normal, the energy density outside r is increasing, that inside decreasing. If the flow is inward as indicated by an inwardly directed normal, the reverse is true. Furthermore, since T measures the *total* flow of energy across Σ_r, it is concluded that the integrand in (7) *must* measure the flow of energy across *each element* $d\sigma$ of the surface. Accordingly, the Poynting vector S is interpreted to define the direction and magnitude of the actual flow of energy across a unit area at every point. This conclusion is not mathematically justified. The surface integral in (7) was originally obtained by the application of the divergence theorem to the volume integral, thus

$$\int_r \operatorname{div} S \, d\tau = \int_{\Sigma_r} (\mathsf{n}, S) d\sigma \tag{17}$$

This theorem has a meaning only if the integration is extended over *a completely closed surface*. It does not admit of an integration over only a *part* of a closed surface. It follows that from the mathematical point of view no meaning can be attached to an integral like that on the right in (17) if the integration is carried out over a surface that is not closed, or to the integrand itself. This is made especially clear by the fact that any integration over a specified part of a closed surface can be made to have *any desired value* by merely adding to S a suitably defined solenoidal vector C. By definition, a solenoidal vector satisfies the condition div $C = 0$ so that

$$\int_r \operatorname{div} \{S + C\} d\tau = \int_{\Sigma_r} (\mathsf{n}, S) d\sigma \tag{18}$$

There is no change in the surface integral over the closed surface Σ_r, and there is no reason, either mathematical or physical, that makes it necessary or even reasonable to prefer the vector S to the vector $S + C$ because both lead to the same power equation. The "amount of energy" that is assumed to "flow" across a surface Σ, which is only a part of the closed surface Σ_r, is arbitrary and not at all uniquely specified by $\int_\Sigma (\hbar, S) d\sigma$. All values of $\int_\Sigma (\hbar, S + C) d\sigma$ are equally reasonable with C *any solenoidal vector whatsoever*.

Consequences of assuming arbitrarily that the Poynting vector S correctly specifies the direction and magnitude of the flow of spatially distributed energy are sometimes plausible, more often amusing or absurd. Conclusions regarding the "flow of energy," which must be accepted if such an assumption is made, are outlined below for two simple examples. Other illustrations are encountered and discussed in the study of antennas and transmission circuits.

Example 1. In a circuit consisting of a battery with its terminals joined by a long wire containing a resistance, the hypothesis that the Poynting vector defines the "flow of energy" requires the energy supplied to the circuit from the battery to "flow" from the battery outward into space while an equal amount of energy "flows" radially inward from space into the resistance wire to appear there as heat. Although at the outset in defining energy functions it was assumed that work was done by components of force acting on charges to maintain a current in a wire, the Poynting-vector hypothesis requires the contrary, *viz.*, that these forces do no work directly on the charges but that they do work in "straining the ether" throughout the universe by acting to increase the "energy density" and, hence, the "store of energy." This "strain" then reacts to cause a flow of energy from space into the conductor and to move the charges and maintain the current. It is significant to note that, if a closed loop of wire without battery is placed directly beside the one connected to the battery, no "energy flows" into it and no current exists in it even though it is exposed to exactly the same "strain" in the ether. By some strange mechanism, the energy that is transferred to the infinite reaches of space by a battery

always returns without loss precisely to that wire which is connected to the battery.

If the power equation is accepted as a relation between mathematical functions without the arbitrary assumptions about localized energy and its flow, the only conclusion is that the energy functions associated with the battery decrease and those associated with the wire increase at the same rate.   This is all that is required.   There is no indication from the mathematical formulation that a physical transfer of anything takes place.

Example 2.   If a bar magnet is positively charged by electrostatic means, the Poynting vector hypothesis requires that a spatially distributed "energy" existing throughout space rotate around the magnet so long as this is charged, in spite of the fact that all terms in the power equation reduce identically to zero.   Thus a static condition of charge is strangely required to produce a whirling of a "universal energy."

Whenever the power equation is used in subsequent analysis, it is treated as a mathematical relation between mathematical functions.   All unjustified assumptions regarding the nature, distribution, and mode of flow of energy are rejected as unnecessary and undesirable.

12. A Power Equation in Terms of Scalar and Vector Potentials.—A simple and useful power equation quite different in its form from (11.15) and not susceptible to an interpretation in terms of a "spatially distributed energy" may be formulated using the scalar and vector potentials instead of the electric and magnetic vectors.   By combining the relations

$$\mathsf{E} = -\operatorname{grad} \phi - \dot{\mathsf{A}} \tag{1}$$

and

$$i_f = \sigma(\mathsf{E} + \mathsf{E}^e) \tag{2}$$

the following is obtained:

$$\mathsf{E}^e = \frac{i_f}{\sigma} + \operatorname{grad} \phi + \dot{\mathsf{A}} \tag{3}$$

Let (3) be multiplied scalarly by i_f and integrated over an arbitrary volume τ.   The result is

$$\int_\tau (i_f, \mathsf{E}^e)d\tau = \int_\tau \frac{i_f^2}{\sigma} d\tau + \int_\tau (i_f, \operatorname{grad} \phi)d\tau + \int_\tau (i_f, \dot{\mathsf{A}})d\tau \tag{4a}$$

This is equivalent to

$$\int_\tau (i_f, E^e) d\tau = \int_\tau \frac{i_f^2}{\sigma} d\tau - \int_\tau (i_f, E) d\tau \qquad (4b)$$

Using the vector identity

$$\operatorname{div} \phi i_f = \phi \operatorname{div} i_f + (i_f, \operatorname{grad} \phi) \qquad (5)$$

and the divergence theorem

$$\int_\tau \operatorname{div} \phi i_f d\tau = \int_\Sigma \phi(\mathfrak{n}, i_f) d\sigma \qquad (6)$$

(4) can be transformed into

$$\int_\tau (i_f, E^e) d\tau = \int_\tau \frac{i_f^2}{\sigma} d\tau - \int_\tau \phi \operatorname{div} i_f d\tau + \int_\tau (i_f, \dot{A}) d\tau$$
$$+ \int_\Sigma \phi(\mathfrak{n}, i_f) d\sigma \qquad (7)$$

[Care must be taken in (7) not to confuse the element of surface $d\sigma$ with the conductivity σ.] In (7) Σ includes Σ_τ, a surface completely enclosing τ, and all additional surfaces within τ across which i_f (or ϕ) is discontinuous. Equation (7) can be modified further using the equations of continuity for free charge.

$$\operatorname{div} i_f + \dot{\rho}_f = 0 \qquad (8)$$

With (8), (7) becomes

$$\int_\tau (i_f, E^e) d\tau - \int_\tau \frac{i_f^2}{\sigma} d\tau - \int_\tau \dot{\rho} \phi \, d\tau - \int_\tau (i_f, \dot{A}) d\tau$$
$$- \int_\Sigma \phi(\mathfrak{n}, i_f) d\sigma = 0 \qquad (9)$$

This equation very much resembles (11.15). The first two terms are exactly the same, the next two are also one electric and one magnetic volume integral, the last term is a surface integral including the enclosing boundary. However, a fundamental difference exists between (9) and (11.15). Whereas (11.15) involves integration over all space where E and B are nonvanishing, (9) involves integration *only over that part of space where electric currents and charges differ from zero.* In free space, *all* terms in (9) vanish. This fact makes an interpretation of (9) in terms of "energy" distributed in completely evacuated space

impossible. In particular, the surface integral certainly does not suggest or permit an interpretation in terms of a "flow of energy *in space*" across Σ_r. Actually, the surface integral vanishes everywhere except in conductors.

A further transformation of (9) is possible with the surface equation of continuity.

$$\text{div } l_f + \dot{\eta}_f - (\hbar, i_f) = 0 \tag{10}$$

Excluding perfect conductors l_f need not be defined since i_f is adequate. Hence (10) reduces to

$$(\hbar, i_f) = \dot{\eta}_f \tag{11}$$

and (9) becomes

$$\int_\tau (i_f, E^\bullet) d\tau - \int_\tau \frac{i_f^2}{\sigma} d\tau - \int_\tau \dot{\rho}_f \phi \, d\tau - \int_\tau (i_f, \dot{A}) d\tau$$
$$- \int_\Sigma \dot{\eta}_f \phi \, d\sigma = 0 \quad (12)$$

This is the general equation.

As discussed in Sec. II.17, ρ_f vanishes in conductors and nonconductors *if the time dependence is periodic*. In this important case, (12) reduces to

$$\int_\tau (i_f, E^\bullet) d\tau - \int_\tau \frac{i_f^2}{\sigma} d\tau - \int_\tau (i_f, \dot{A}) d\tau - \int_\Sigma \dot{\eta}_f \phi \, d\tau = 0 \quad (13)$$

This is a power equation expressed in terms of scalar and vector potentials and written in its most useful form for periodic phenomena. The general analysis of electric circuits (Chapter VI) and of antennas (Volume II) might be based upon (13). Actually, the more fundamental relation (3) is chosen as a starting point both in Chapter VI and Volume II.

PERIODIC TIME DEPENDENCE

13. Complex Field Equations and Potential Equations.—The Maxwell-Lorentz equations are a set of linear, partial differential equations in which the space coordinates and the time are the independent variables. In order to separate these variables, it is possible to set

$$\bar{\varrho}_{\text{inst}} = \bar{\varrho} e^{j\omega t} \tag{1}$$

$$\overline{\varrho_m \mathsf{v}_{\text{inst}}} = \overline{\varrho_m \mathsf{v}} e^{j\omega t} \tag{2}$$

$$\mathbf{E}_{\text{inst}} = \mathbf{E}e^{j\omega t} \tag{3}$$

$$\mathbf{B}_{\text{inst}} = \mathbf{B}e^{j\omega t} \tag{4}$$

if simply periodic variations in time are involved.   If the time dependence is more complicated, a Fourier series or Fourier integral form may be used, or the Heaviside operator p may be written for $j\omega$ and operational methods with the unit or step functions employed.   Complex quantities are in boldface type; if they are scalars, it is boldface italic or boldface Greek; if vectors, boldface Gothic.   The name vector and the Gothic symbols are reserved for true space vectors only.   Complex scalars are not vectors.   In (1), $\mathfrak{p}_{\text{inst}}$ is a complex instantaneous *scalar*.   Its complex amplitude is $\mathfrak{p}$, its real amplitude is $\bar{p}$.

$$\mathfrak{p} = \bar{p}_r + j\bar{p}_i = \bar{p}e^{j\theta_p} = \bar{p}(\cos\,\theta_p + j\,\sin\,\theta_p) \tag{5}$$

$$\bar{p} = \sqrt{\bar{p}_r^2 + \bar{p}_i^2}; \qquad \tan\,\theta_p = \frac{\bar{p}_i}{\bar{p}_r} \tag{6}$$

$$\mathfrak{p}_{\text{inst}} = \bar{p}e^{j(\omega t + \theta_p)} = \bar{p}\{\cos\,(\omega t + \theta_p) + j\,\sin\,(\omega t + \theta_p)\} \tag{7}$$

Similar relations may be written for any scalar with the time dependence (1).   In (3), $\mathbf{E}_{\text{inst}}$ is the complex instantaneous value of a vector.   Its complex amplitude is the complex vector $\mathbf{E}$; its real amplitude is the real vector E; the scalar magnitude of the real vector E is E.

$$\mathbf{E} = \mathbf{E}_r + j\mathbf{E}_i = \mathbf{E}e^{j\theta_R} = \mathbf{E}(\cos\,\theta_R + j\,\sin\,\theta_R) \tag{8}$$

$$E = \sqrt{E_r^2 + E_i^2}; \qquad \tan\,\theta_R = \frac{E_i}{E_r} \tag{9}$$

$$\mathbf{E}_{\text{inst}} = \mathbf{E}e^{j(\omega t + \theta_R)} = \mathbf{E}\{\cos\,(\omega t + \theta_R) + j\,\sin\,(\omega t + \theta_R)\} \tag{10}$$

Corresponding formulas apply to any vector with the time dependence (3).   From these relations it is clear that the complex time dependence (7) includes:

$$\bar{p}_{\text{inst}} = \bar{p}\,\cos\,(\omega t + \theta_p) \tag{11}$$

if only the real part is retained, and

$$\bar{p}_{\text{inst}} = \bar{p}\,\sin\,(\omega t + \theta_p) \tag{12}$$

if only the imaginary part is used.

By selecting the real part *or* the imaginary part of (7), a problem involving the real time dependence (11) *or* (12) may be represented by (7).   It is, in fact, possible to solve two problems with simple harmonic time dependence of the form given by (11) and

(12) *simultaneously* by writing (7) instead of (11) or (12). If interest is in one of these only, as is usually the case, either the real part of the final complex solution is selected to correspond to the time dependence (11), *or* the imaginary part of the complex solution is chosen to correspond to the time dependence (12). *The part not selected may be discarded completely.* By using the complex exponential time dependence instead of the trigonometric, the time and space variables in the field equations are separated and two problems are solved simultaneously very much more simply than either one alone. With (1) to (4) the field equations as well as any transformations of these involve *complex* quantities up to the final solution. The complex field equations and potential equations and related quantities are listed below. For simplicity in writing, the following shorthand is used:

$$\beta_0^2 \equiv \frac{\omega^2 \epsilon_0}{\nu_0} = \omega^2 \epsilon_0 \mu_0 = \frac{\omega^2}{v_0^2} \tag{13}$$

Tabulation of the General Case

Electromagnetic Vector

Equations for interior

$$\epsilon_0 \, \text{div} \, \mathbf{E} = \bar{\varrho} \tag{14a}$$

$$\text{curl} \, \mathbf{E} = -j\omega \mathbf{B} \tag{14b}$$

$$\nu_0 \, \text{curl} \, \mathbf{B} = \overline{\varrho_m \mathbf{v}} + j\omega\epsilon_0 \mathbf{E} \tag{14c}$$

$$\text{div} \, \mathbf{B} = 0 \tag{14d}$$

Boundary conditions

$$\epsilon_0(\hat{\mathfrak{n}}_1, \mathbf{E}_1) + \epsilon_0(\hat{\mathfrak{n}}_2, \mathbf{E}_2) = -\bar{\mathfrak{n}}_1 - \bar{\mathfrak{n}}_2 \tag{15a}$$

$$[\hat{\mathfrak{n}}_1, \mathbf{E}_1] + [\hat{\mathfrak{n}}_2, \mathbf{E}_2] = 0 \tag{15b}$$

$$\nu_0[\hat{\mathfrak{n}}_1, \mathbf{B}_1] + \nu_0[\hat{\mathfrak{n}}_2, \mathbf{B}_2] = -\overline{\mathfrak{n}_m \mathbf{v}_1} - \overline{\mathfrak{n}_m \mathbf{v}_2} \tag{15c}$$

$$(\hat{\mathfrak{n}}_1, \mathbf{B}_1) + (\hat{\mathfrak{n}}_2, \mathbf{B}_2) = 0 \tag{15d}$$

Density functions

$$\bar{\varrho} \equiv \varrho - \text{div} \, \mathbf{P} \tag{16a}$$

$$\overline{\varrho_m \mathbf{v}} \equiv \mathbf{i} + \text{curl} \, \mathbf{M} + j\omega \mathbf{P} \tag{16b}$$

$$\bar{\mathfrak{n}} \equiv \mathfrak{n} + (\hat{\mathfrak{n}}, \mathbf{P}) \tag{17a}$$

$$\overline{\mathfrak{n}_m \mathbf{v}} \equiv \mathbf{I} - [\hat{\mathfrak{n}}, \mathbf{M}] \tag{17b}$$

Equations of continuity

$$\text{div } \overline{\varrho_m \mathbf{v}} + j\omega\overline{\varrho} = 0 \tag{18a}$$

$$\text{div } \{\overline{\mathbf{n}_m \mathbf{v}_1} + \overline{\mathbf{n}_m \mathbf{v}_2}\} + j\omega\{\overline{\mathbf{n}_1} + \overline{\mathbf{n}_2}\} - (\mathbf{\hbar}_1, \overline{\varrho_m \mathbf{v}_1}) - (\mathbf{\hbar}_2, \overline{\varrho_m \mathbf{v}_2})$$
$$= 0 \tag{18b}$$

Potential Functions

Equations

$$\nabla^2\phi + \beta_0^2\phi = -\frac{\overline{\varrho}}{\epsilon_0} \tag{19a}$$

$$\nabla^2\mathbf{A} + \beta_0^2\mathbf{A} = -\frac{\overline{\varrho_m \mathbf{v}}}{\nu_0} \tag{19b}$$

Equation of continuity

$$\text{div } \mathbf{A} + j\frac{\beta_0^2}{\omega}\phi = 0 \tag{19c}$$

Relation for the electromagnetic vectors

$$\mathbf{E} = -\text{grad }\phi - j\omega\mathbf{A} = -j\frac{\omega}{\beta_0^2}\{\text{grad div }\mathbf{A} + \beta_0^2\mathbf{A}\} \tag{20a}$$

$$\mathbf{B} = \text{curl }\mathbf{A} \tag{20b}$$

Polarization and Magnetization Potentials

Assumed conditions

$$\overline{\varrho} = -\text{div }\mathbf{P} \tag{21a}$$

$$\overline{\varrho_m \mathbf{v}} = \text{curl }\mathbf{M} + j\omega\mathbf{P} \tag{21b}$$

$$\overline{\mathbf{n}} = (\mathbf{\hbar}, \mathbf{P}) \tag{22a}$$

$$\overline{\mathbf{n}_m \mathbf{v}} = -[\mathbf{\hbar}, \mathbf{M}] \tag{22b}$$

Equations

$$\nabla^2\mathbf{Z} + \beta_0^2\mathbf{Z} = -\frac{\mathbf{P}}{\epsilon_0} \tag{23a}$$

$$\nabla^2\mathbf{Y} + \beta_0^2\mathbf{Y} = -\frac{\mathbf{M}}{\nu_0} \tag{23b}$$

Relations for the potential functions

$$\phi = -\text{div }\mathbf{Z} \tag{24a}$$

$$\mathbf{A} = j\frac{\beta_0^2}{\omega}\mathbf{Z} + \text{curl }\mathbf{Y} \tag{24b}$$

Relations for the electromagnetic vectors

$$\mathbf{E} = \text{grad div }\mathbf{Z} + \beta_0^2\mathbf{Z} - j\omega\text{ curl }\mathbf{Y} \tag{25a}$$

$$\mathbf{B} = j\frac{\beta_0^2}{\omega}\text{curl }\mathbf{Z} + \text{curl curl }\mathbf{Y} \tag{25b}$$

It is important to note that according to (20a) the electric vector can be computed directly from the vector potential, so that the scalar potential is not required.

Since Z and Y are independent, the electromagnetic field defined in (25a,b) can always be considered in two parts, the so-called *electric type field* derived from Z and the *magnetic type field* derived from Y.

14. Complex Field Equations and Potential Equations in Simple Media.—Simple media have been defined to be bodies or regions in which the following constitutive relations obtain:

$$\mathsf{P} = \epsilon_0 \chi \mathsf{E} = (\epsilon_r - 1)\epsilon_0 \mathsf{E} \tag{1}$$

$$-\mathsf{M} = \nu_0 \chi_m \mathsf{B} = (\nu_r - 1)\nu_0 \mathsf{B} \tag{2}$$

$$\mathsf{i}_f = \sigma \mathsf{E} \tag{3}$$

It has already been pointed out that no atomic models can satisfy these relations *exactly* except in the stationary states. If a harmonic unsteady state prevails with E and B varying periodically according to

$$\mathsf{E}_{\text{inst}} = \mathsf{E} \cos (\omega t + \theta_E) \tag{4}$$

$$\mathsf{B}_{\text{inst}} = \mathsf{B} \cos (\omega t + \theta_B) \tag{5}$$

the relations (1) to (3) are usually not obeyed instantaneously. Depending upon the frequency and the particular structure of the body under consideration, there is a larger or smaller time lag of P behind E, of $-\mathsf{M}$ behind B, and of i_f behind E. Instead of (1) to (3), relations of the form (II.17.18) are necessary. In complex notation the relations for the complex instantaneous values of P, $-\mathsf{M}$, and i_f are

$$\mathsf{P}_{\text{inst}} = \epsilon_0 \chi \mathsf{E} e^{j(\omega t - \theta_P)} = (\epsilon_r - 1)\epsilon_0 \mathsf{E} e^{j(\omega t - \theta_P)} \tag{6}$$

$$-\mathsf{M}_{\text{inst}} = \nu_0 \chi_m \mathsf{B} e^{j(\omega t - \theta_M)} = (\nu_r - 1)\nu_0 \mathsf{B} e^{j(\omega t - \theta_M)} \tag{7}$$

$$\mathsf{i}_{f\text{inst}} = \sigma \mathsf{E} e^{j(\omega t - \theta_i)} \tag{8}$$

Here the parameters χ, χ_m, σ are real functions of the frequency, as are the phase angles θ_P, θ_M, θ_i. It is now possible to define *complex* constitutive parameters χ, χ_m, σ, which are functions of the frequency as well as of the structure of the body or region,

$$\chi = \chi' - j\chi'' = \chi e^{-j\theta_P} \tag{9}$$

$$\varepsilon_r = \epsilon_r' - j\epsilon_r'' = (1 + \chi) = (1 + \chi') - j\chi'' \tag{10}$$

$$\chi_m = \chi_m' - j\chi_m'' = \chi_m e^{-j\theta_M} \tag{11}$$

$$\nu_r = \nu_r' - j\nu_r'' = (1 + \chi_m) = (1 + \chi_m') - j\chi_m'' \tag{12}$$

$$\sigma = \sigma' - j\sigma'' = \sigma e^{-j\theta_i} \tag{13}$$

At sufficiently low frequencies the times of reorientation or of relaxation are small compared with the period, and the imaginary parts ϵ_r'', ν_r'', σ'' are negligible compared with the real parts ϵ_r', ν_r', σ', and these latter are sensibly equal to the stationary-state quantities.

With (10), (12), and (13) the following generalized complex constitutive relations between harmonic amplitudes may be written to define simple media in the case of simple harmonic time dependence:

$$\mathbf{P} = (\epsilon_r - 1)\epsilon_0\mathbf{E} = (\epsilon - \epsilon_0)\mathbf{E} \tag{14}$$

$$-\mathbf{M} = (\nu_r - 1)\nu_0\mathbf{B} = (\nu - \nu_0)\mathbf{B} \tag{15}$$

$$\mathbf{i}_f = \delta\mathbf{E} \tag{16}$$

Also,

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon\mathbf{E} \tag{17a}$$

$$\mathbf{H} = \nu_0\mathbf{B} - \mathbf{M} = \nu\mathbf{B} \tag{17b}$$

The shorthand

$$\epsilon = \epsilon_0\epsilon_r = \epsilon' - j\epsilon'' \tag{18a}$$

$$\nu = \nu_0\nu_r = \nu' - j\nu'' \tag{18b}$$

is used. ϵ is called the absolute complex dielectric constant or the absolute complex permittivity, ν the absolute complex diamagnetic constant or the absolute complex reluctivity.

Since ϵ and ν are complex, $\mathbf{D}$ and $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{B}$ are not in phase. Upon substituting (14) to (16) in the complex field equations (13.14) and (13.15), these become

$$\epsilon \operatorname{div} \mathbf{E} = \varrho_f \tag{19a}$$

$$\operatorname{curl} \mathbf{E} = -j\omega\mathbf{B} \tag{19b}$$

$$\nu \operatorname{curl} \mathbf{B} = (\delta + j\omega\epsilon)\mathbf{E} \tag{19c}$$

$$\operatorname{div} \mathbf{B} = 0 \tag{19d}$$

$$\epsilon_1(\mathsf{n}_1, \mathbf{E}_1) + \epsilon_2(\mathsf{n}_2, \mathbf{E}_2) = -\mathsf{n}_{1f} - \mathsf{n}_{2f} \tag{20a}$$

$$[\mathsf{n}_1, \mathbf{E}_1] + [\mathsf{n}_2, \mathbf{E}_2] = 0 \tag{20b}$$

$$\nu_1[\mathsf{n}_1, \mathbf{B}_1] + \nu_2[\mathsf{n}_2, \mathbf{B}_2] = -\mathsf{l}_{1f} - \mathsf{l}_{2f} \tag{20c}$$

$$(\mathsf{n}_1, \mathbf{B}_1) + (\mathsf{n}_2, \mathbf{B}_2) = 0 \tag{20d}$$

The equations of continuity for charge are

$$\delta \operatorname{div} \mathbf{E} + j\omega\varrho_f = 0 \tag{21a}$$

$$\operatorname{div} \{\mathsf{l}_{1f} + \mathsf{l}_{2f}\} + j\omega\{\mathsf{n}_{1f} + \mathsf{n}_{2f}\} - \delta_1(\mathsf{n}_1, \mathbf{E}_1) - \delta_2(\mathsf{n}_2, \mathbf{E}_2) = 0 \tag{21b}$$

By eliminating ρ_f between (19a) and (21a) and $(n_{1f} + n_{2f})$ between (20a) and (21b),

$$\text{div } \mathsf{E} = 0 \tag{22a}$$

$$\{\eth_1 + j\omega\epsilon_1\}(\hbar_1, \mathsf{E}_1) + \{\eth_2 + j\omega\epsilon_2\}(\hbar_2, \mathsf{E}_2) = \text{div } \{\mathsf{I}_{1f} + \mathsf{I}_{2f}\} \tag{22b}$$

in all simple media with time lag.

The complex factor appearing in (19c) and (22b) can be expanded as follows:

$$\eth + j\omega\epsilon = \eth + j\omega\epsilon_0\epsilon_r = (\sigma' - j\sigma'') + j\omega\epsilon_0(\epsilon_r' - j\epsilon_r'') \tag{23a}$$

Collecting real and imaginary parts,

$$\eth + j\omega\epsilon = (\sigma' + \omega\epsilon_0\epsilon_r'') + j(\omega\epsilon_0\epsilon_r' - \sigma'') \tag{23b}$$

Let a real, effective conductivity σ_e and a real, effective *relative* dielectric constant ϵ_{er} be defined as follows

$$\sigma_e = \sigma' + \omega\epsilon_0\epsilon_r'' = \sigma' + \omega\epsilon'' \tag{24a}$$

$$\epsilon_{er} = \epsilon_r' - \frac{\sigma''}{\omega\epsilon_0} \tag{24b}$$

It is often convenient to use the shorthand

$$\epsilon_e = \epsilon_0\epsilon_{er} = \epsilon' - \frac{\sigma''}{\omega} \tag{24c}$$

where ϵ_e is called the real effective *absolute* dielectric constant. With this notation

$$\eth + j\omega\epsilon = \sigma_e + j\omega\epsilon_e \tag{25a}$$

The ratio

$$h_e = \frac{\sigma_e}{\omega\epsilon_e} \tag{25b}$$

is called the *power factor* in dielectrics where it is small. Its general significance is discussed in Sec. 15. The effective conductivity (24) includes a term σ' due to free charge moving in an imperfect conductor and a term $\omega\epsilon''$ due to time lag in polarization in a dielectric. In good conductors, the latter is negligible; in good dielectrics operated at frequencies near those leading to molecular or atomic resonances, the term in $\omega\epsilon''$ is large and σ' usually small. The effective relative dielectric constant ϵ_{er} involves a term ϵ_r' due to ordinary polarization and a

term $-\sigma''/\omega\epsilon_0$ due to a time lag in convection current response. In good dielectrics, the term $-\sigma''/\omega\epsilon_0$ is negligible compared with ϵ'_r so that the power factor becomes $h_o = \omega\epsilon''/\omega\epsilon' = \epsilon''_r/\epsilon'_r$, which is independent of the frequency unless ϵ'_r and ϵ''_r are themselves functions of the frequency. In ionic solutions, both ϵ'_r and $-\sigma''/\omega\epsilon_0$ may be significant owing to the time of ion formation; in space-charged regions containing a rarified atmosphere of electrons, the time lag in convection current is so great that the only significant term in (24) and (25) is the term $-\sigma''/\omega\epsilon_0$. Thus an effective dielectric constant $\epsilon_{er} = -\sigma''/\omega\epsilon_0$ may be defined for a region containing only free charges.

A useful approximate expression for σ is readily derived for regions that can be described entirely in terms of a very small volume density of free charge ρ_f such as the ionized upper atmosphere (ionosphere) or the interelectrode spaces in a vacuum tube in certain simple cases. If the volume density ρ_f is due entirely to a rarified, slowly varying distribution of electrons with mean free paths that are sufficiently great so that the interactions between electrons (so-called collisions) may be neglected, all electrons in a volume element $\Delta\tau$ move with the same nonrandom velocity u if exposed to a periodically varying electric field E that is sensibly constant in amplitude throughout $\Delta\tau$. Each electron satisfies the dynamical equation (II.13.5), so that all the charges in $\Delta\tau$ satisfy the equation

$$\rho_f\Delta\tau E_{inst.} = D_f\Delta\tau \frac{du_{inst.}}{dt}. \tag{26a}$$

where $\rho_f\Delta\tau$ is the total charge in $\Delta\tau$, $D_f\Delta\tau$ is the total mass in $\Delta\tau$, and u is the mean nonrandom velocity of the charges in $\Delta\tau$. In complex notation with $E_{inst.} = Ee^{j\omega t}$, $u_{inst.} = ue^{j\omega t}$, and ρ_f and D_f assumed constant in $\Delta\tau$, the solution of (26a) for u is

$$u = \frac{\rho_f}{j\omega D_f} E \tag{26b}$$

Upon multiplying (26b) by ρ_f the volume density of moving free charge i_f is obtained.

$$i_f = \frac{-j\rho_f^2}{\omega D_f} E = \sigma E \tag{26c}$$

It follows directly that with $\delta = \sigma' - j\sigma''$

$$\sigma' = 0; \qquad \sigma'' = \frac{\rho_f^2}{\omega D_f} \tag{27a}$$

It is often convenient to express the densities ρ_f and D_f in terms of the total number of electrons N in $\Delta\tau$ or in terms of the number of electrons per unit volume $n = N/\Delta\tau$. Since $\rho_f = eN/\Delta\tau = en$ and $D_f = mN/\Delta\tau = mn$, where e is the charge (negative), m the mass of an electron, it is possible to write (27a) in the alternative form

$$\sigma'' = \frac{e^2 n}{\omega m} \tag{27b}$$

Since $\mathsf{P} = 0$ in the space-charged region, this may be represented by the following effective constants:

$$\sigma_e = 0; \qquad \epsilon_{er} = 1 - \frac{e^2 n}{\omega^2 \epsilon_0 m} \tag{27c}$$

This is a value less than unity. If desired, a mathematically equivalent representation of the space-charged region using a fictitious volume density of polarization current $\dot{\mathsf{P}} = j\omega\mathsf{P}$ instead of i_f may be constructed in the manner described in general terms in Chapter I.

For many purposes involving imperfect conductors or slightly conducting dielectric media, it is convenient to introduce a *complex dielectric factor* defined by

$$\xi = \epsilon_0 \left(\epsilon_{er} - j\frac{\sigma_e}{\omega\epsilon_0} \right) = \varepsilon - j\frac{\delta}{\omega} = \epsilon_e - j\frac{\sigma_e}{\omega} \tag{28}$$

This may be substituted directly in (19c) and (22b). Since i_f is required only for perfect conductors, the following simplified relations are true in the interior and at boundaries between two imperfectly conducting media. For simplicity in writing, the following shorthand is used:

$$\beta^2 \equiv \frac{\omega^2 \xi}{v} = \omega^2 \xi \mu \tag{29a}$$

Since

$$\beta_0^2 = \frac{\omega^2}{v_0^2}; \qquad v_0 = \sqrt{\frac{\nu_0}{\epsilon_0}}$$

it is sometimes convenient to define a *complex* velocity v such that

$$\beta^2 = \frac{\omega^2}{v^2}; \qquad v = \sqrt{\frac{\nu}{\xi}} \tag{29b}$$

Similarly, by analogy with the characteristic resistance

$$\zeta_0 \equiv \frac{1}{\sqrt{\nu_0 \epsilon_0}}$$

it is possible to define a complex characteristic impedance ζ of a simple medium such that

$$\zeta = \frac{1}{\sqrt{\nu\xi}} \tag{29c}$$

The field equations are

$$\text{div } \mathbf{E} = 0 \tag{30a}$$

$$\text{curl } \mathbf{E} = -j\omega\mathbf{B} \tag{30b}$$

$$\text{curl } \mathbf{B} = j\frac{\beta^2}{\omega}\mathbf{E} \tag{30c}$$

$$\text{div } \mathbf{B} = 0 \tag{30d}$$

The boundary conditions are

$$\xi_1(\mathbf{n}_1, \mathbf{E}_1) + \xi_2(\mathbf{n}_2, \mathbf{E}_2) = 0 \tag{31a}$$

$$[\mathbf{n}_1, \mathbf{E}_1] + [\mathbf{n}_2, \mathbf{E}_2] = 0 \tag{31b}$$

$$\nu_1[\mathbf{n}_1, \mathbf{B}_1] + \nu_2[\mathbf{n}_2, \mathbf{B}_2] = 0 \tag{31c}$$

$$(\mathbf{n}_1, \mathbf{B}_1) + (\mathbf{n}_2, \mathbf{B}_2) = 0 \tag{31d}$$

If a problem involves simple media (excluding perfect conductors) and regions that are not so specialized, the equations (30) are used in the simple media, the equations (13.14) in the unspecialized regions, and the conditions (13.15) at the boundaries. The last may be modified as in (31) for those terms involving the field *in a simple medium.* For example, if medium 2 is a simple medium, (13.15) may be written as follows:

$$\epsilon_0(\mathbf{n}_1, \mathbf{E}_1) + \xi_2(\mathbf{n}_2, \mathbf{E}_2) = -\bar{n}_1 \tag{32a}$$

$$[\mathbf{n}_1, \mathbf{E}_1] + [\mathbf{n}_2, \mathbf{E}_2] = 0 \tag{32b}$$

$$\nu_0[\mathbf{n}_1, \mathbf{B}_1] + \nu_2[\mathbf{n}_2, \mathbf{B}_2] = -\overline{n_m v_1} \tag{32c}$$

$$(\mathbf{n}_1, \mathbf{B}_1) + (\mathbf{n}_2, \mathbf{B}_2) = 0 \tag{32d}$$

If medium 2 is *a* simple medium and medium 1 is a perfect conductor in which $\mathbf{E}$ and $\mathbf{B}$ vanish, (32) reduces to

$$\xi_2(\hbar_2, E_2) = -n_{1f} \tag{33a}$$

$$[\hbar_2, E_2] = 0 \tag{33b}$$

$$v_2[\hbar_2, B_2] = -l_{1f} \tag{33c}$$

$$(\hbar_2, B_2) = 0 \tag{33d}$$

The equations for the potential functions in simple media are

$$\nabla^2 \phi + \beta^2 \phi = 0 \tag{34a}$$

$$\nabla^2 A + \beta^2 A = 0 \tag{34b}$$

The equation of continuity is

$$\text{div } A + j \frac{\beta^2}{\omega} \phi = 0 \tag{34c}$$

The relations for the electromagnetic vectors are

$$E = -\text{grad } \phi - j\omega A = -\frac{j\omega}{\beta^2} \{ \text{grad div } A + \beta^2 A \} \tag{35a}$$

$$B = \text{curl } A \tag{35b}$$

The polarization and magnetization potentials in simple media satisfy the equations

$$\nabla^2 Z + \beta^2 Z = 0 \tag{36a}$$

$$\nabla^2 Y + \beta^2 Y = 0 \tag{36b}$$

They are related as follows to the potential functions and field vectors:

$$\phi = -\text{div } Z \tag{37a}$$

$$A = j \frac{\beta^2}{\omega} Z + \text{curl } Y \tag{37b}$$

$$E = \text{grad div } Z + \beta^2 Z - j\omega \text{ curl } Y \tag{38a}$$

$$B = j \frac{\beta^2}{\omega} \text{ curl } Z + \text{grad div } Y + \beta^2 Y \tag{38b}$$

The last expression was obtained from

$$B = j \frac{\beta^2}{\omega} \text{ curl } Z + \text{curl curl } Y \tag{39}$$

by noting that with (36b)

$$\text{curl curl } Y \equiv \text{grad div } Y - \nabla^2 Y = \text{grad div } Y + \beta^2 Y \tag{40}$$

The telegraphist's equations (8.13, 8.14) in simple media are

$$\nabla^2 E + \beta^2 E = 0 \tag{41a}$$

$$\nabla^2 B + \beta^2 B = 0 \tag{41b}$$

Solutions of these equations must satisfy

$$\operatorname{div} \mathbf{E} = 0 \qquad (42a)$$
$$\operatorname{div} \mathbf{B} = 0 \qquad (42b)$$

Comparison of the general expressions (13.14a,b,c,d), (13.19a, b,c), (13.20a,b), (13.23a,b), (13.24a,b), and (13.25a,b) when written for points in space (*i.e., with all density functions zero*) with the corresponding formulas for simple media (30a,b,c,d), (34a,b,c), (35a,b), (36a,b), (37a,b), and (38a,b) reveals the important fact that all formulas for simple media may be obtained from the corresponding expressions for space by writing ξ for ϵ_0, ν for ν_0, and, therefore, β for β_0. The boundary conditions between an unspecialized medium (1) and space (medium 2) [as given by (13.15) with density functions with subscript 2 set equal to zero] are also the same as the boundary conditions between an unspecialized medium (1) and a simple medium (2) [as obtained from (31)] if these substitutions are made. It follows that the electromagnetic field, the scalar and vector potentials, or the polarization and magnetization potentials due to a distribution of current and charge in bounded regions immersed in a simple medium may be obtained from a solution of the same problem with the simple medium replaced by space if ξ is written for ϵ_0, ν for ν_0, and, hence, β for β_0 and ν for ν_0. It is, therefore, sufficient to solve problems involving antennas, hollow conductors, condensers, etc., using space as the enveloping medium. If any other medium is involved, the simple change from the universal constants ϵ_0, ν_0 to the complex parameters ξ, ν in the solution for space is all that is required. For more ready reference, this conclusion is contained in the following theorem.

Theorem for Simple Media: Formulas for the complex amplitudes of the electromagnetic field, the scalar and vector potentials, and the polarization and magnetization potentials due to an arbitrary distribution of current and charge in regions immersed in space are still correct if space is replaced by a simple medium characterized by ϵ_{er}, ν_r, δ, if the universal constants ϵ_0 and ν_0 in the formulas for space are replaced by the complex parameters $\xi = \epsilon_0(\epsilon_{er} - j\delta_s/\omega\epsilon_0)$, $\nu = \nu_0\nu_r$. It is assumed that all boundaries between the regions in question and space become

boundaries between the same regions and the simple medium and
that no new boundaries are created.

 · **15. Generalized Coefficients for Simple Media.**—The complex
factor β appearing in many of the equations and formulas for
simple media may be separated into real and imaginary parts as
follows.   Let

$$\beta \equiv \beta_0 N \equiv \beta_0(N_s - jX_s) \equiv \beta_s - j\alpha_s \tag{1}$$

where N is called the *generalized complex index of refraction,*
N_s is the real index of refraction, and X_s is the real *extinction
coefficient.*   The *phase constant* $\beta_s = \beta_0 N_s$ and the *attenuation
constant* $\alpha_s = \beta_0 X_s$ are conveniently used as shorthand symbols.
[The subscript s is used to distinguish between the phase and
attenuation constants defined in (1) for an unbounded simple
medium and the phase and attenuation constants defined for
bounded regions, in particular, for transmission circuits.]   A
real phase velocity can be defined as usual

$$v_s \equiv \frac{\omega}{\beta_s} = \frac{v_0}{N_s} \tag{2}$$

By definition (14.29a) and with (14.28) and (14.18b)

$$\beta^2 = \frac{\omega^2 \xi}{v} = \beta_c^2 \left[\frac{1 - jh_e}{1 - jh_m} \right] \tag{3}$$

with

$$h_e \equiv \frac{\sigma_e}{\omega\epsilon_e} = \frac{\sigma_e}{\omega\epsilon_0\epsilon_{er}} \tag{4a}$$

$$h_m \equiv \frac{\nu''}{\nu'} \tag{4b}$$

$$\beta_e^2 \equiv \omega^2 \left(\frac{\epsilon_e}{\nu'} \right) = \beta_0^2 \frac{\epsilon_{er}}{\nu_r'} \tag{4c}$$

Accordingly

$$\beta^2 = \beta_e^2 \left\{ \frac{(1 + h_m h_e) - j(h_e - h_m)}{1 + h_m^2} \right\} \tag{5}$$

or

$$\beta^2 = \beta_e^2 \left\{ \frac{1 + h_m h_e}{1 + h_m^2} \right\} \left\{ 1 - j \left(\frac{h_e - h_m}{1 - h_m h_e} \right) \right\} \tag{6}$$

Since in practice simply magnetizing materials (ν_r = constant)
involve values of ν_r which may be assumed to be real and prac-

tically equal to unity, the complicated formula (6) may be replaced by the simpler formula obtained by writing

$$h_m \doteq 0; \qquad \nu = \nu' = \nu = \nu_0 \nu_r \text{ is real} \tag{7}$$

Then,

$$\mathfrak{g} = \beta_s(1 - jh_s)^{\frac{1}{2}} \tag{8}$$

The real and imaginary parts of the function $(1 - jh)^{\frac{1}{2}}$ have been separated and tabulated in Appendix II in the form[1]

$$(1 - jh)^{\frac{1}{2}} = f(h) - jg(h) \tag{9}$$

with

$$f(h) = \sqrt{\tfrac{1}{2}(\sqrt{1 + h^2} + 1)} = \cosh\left(\tfrac{1}{2}\sinh^{-1} h\right) \tag{10}$$

$$g(h) = \sqrt{\tfrac{1}{2}(\sqrt{1 + h^2} - 1)} = \sinh\left(\tfrac{1}{2}\sinh^{-1} h\right) \tag{11}$$

Accordingly,

$$\beta_s = \beta_s f(h_s) = \beta_0 N_s \tag{12}$$

$$\alpha_s = \beta_s g(h_s) = \beta_0 X_s \tag{13}$$

$$v_s = \frac{\omega}{\beta_s} = \frac{v_s}{f(h_s)} = \frac{v_0}{N_s} \tag{14}$$

Note that

$$\frac{\alpha_s}{\beta_s} = \frac{X_s}{N_s} = \frac{g(h_r)}{f(h_r)} \leq 1 \tag{15}$$

For some purposes it is convenient to define a characteristic velocity v_s and a characteristic impedance ζ_r for a simple medium. Thus by analogy with the universal constants

$$v_0 = \sqrt{\frac{\nu_0}{\epsilon_0}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ meters/second}$$

$$\zeta_0 = \frac{1}{\sqrt{\nu_0 \epsilon_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.7 \text{ ohms}$$

it is possible to define

$$v_s \equiv \sqrt{\frac{\nu}{\epsilon_s}} = \frac{1}{\sqrt{\mu \epsilon_s}} = v_0 \sqrt{\frac{\nu_r}{\epsilon_{er}}} \tag{16a}$$

$$\zeta_s = \frac{1}{\sqrt{\nu \epsilon_s}} = \sqrt{\frac{\mu}{\epsilon_s}} = \frac{\zeta_0}{\sqrt{\nu_r \epsilon_{er}}} \tag{16b}$$

These constants will be shown to play the same part in describing

[1] Due to G. W. Pierce.

electromagnetic phenomena in simple perfect or poorly conducting dielectrics as do v_0 and ζ_0 in space.

The complex velocity v is given by

$$v = \frac{\omega}{\beta} = \frac{\omega}{\beta_s - j\alpha_s} = \frac{v_s}{\sqrt{1 + \frac{\alpha_s^2}{\beta_s^2}}} e^{j \tan^{-1} (\alpha_s/\beta_s)} \tag{17}$$

Poor conductors are defined by

$$h_s^2 << 1 \tag{18}$$

so that

$$f(h_s) \doteq 1; \qquad g(h_s) \doteq \frac{h_s}{2} \tag{19}$$

and

$$\beta_s \doteq \beta_s = \frac{\omega}{v_s}; \qquad v_s \doteq v_s \tag{20}$$

$$\alpha_s \doteq \beta_s \left(\frac{\sigma_s}{2\omega\epsilon_s} \right) = \frac{1}{2} \sigma_s \zeta_s \tag{21}$$

Note that

$$h_s = \frac{2\alpha_s}{\beta_s} \tag{22}$$

so that

$$\left(\frac{\alpha_s}{\beta_s} \right)^2 << 1 \tag{23}$$

Also,

$$v \doteq v_s e^{j(\alpha_s/\beta_s)} \tag{24}$$

Good conductors are defined by

$$h_s >> 1 \tag{25}$$

and

$$\epsilon_r = 1 \tag{26}$$

so that

$$f(h_s) \doteq g(h_s) \doteq \sqrt{\frac{h_s}{2}} \tag{27}$$

and

$$\beta_s \doteq \alpha_s \doteq \beta_s \sqrt{\frac{h_s}{2}} = \sqrt{\frac{\omega\sigma_s}{2v}} = \sqrt{\frac{1}{2} \omega\mu\sigma_s} \tag{28}$$

$$v_s = \sqrt{\frac{2\omega\bar{v}}{\sigma_s}} = \sqrt{\frac{2\omega}{\mu\sigma_s}} \tag{29}$$

$$v = \frac{v_s}{\sqrt{2}} e^{j\frac{\pi}{4}} \tag{30}$$

tically equal to unity, the complicated formula (6) may be replaced by the simpler formula obtained by writing

$$h_m \doteq 0; \qquad \nu = \nu' = \nu = \nu_0 \nu_r \text{ is real} \tag{7}$$

Then,

$$\mathfrak{g} = \beta_s(1 - jh_s)^{\frac{1}{2}} \tag{8}$$

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$$\beta_s = \beta_s f(h_c) = \beta_0 N_s \tag{12}$$

$$\alpha_s = \beta_s g(h_s) = \beta_0 X_s \tag{13}$$

$$v_s = \frac{\omega}{\beta_s} = \frac{v_s}{f(h_c)} = \frac{v_0}{N_s} \tag{14}$$

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it is possible to define

$$v_s \equiv \sqrt{\frac{\nu}{\epsilon_s}} = \frac{1}{\sqrt{\mu \epsilon_s}} = v_0 \sqrt{\frac{\nu_r}{\epsilon_{cr}}} \tag{16a}$$

$$\zeta_s = \frac{1}{\sqrt{\nu \epsilon_s}} = \sqrt{\frac{\mu}{\epsilon_s}} = \frac{\zeta_0}{\sqrt{\nu_r \epsilon_{cr}}} \tag{16b}$$

These constants will be shown to play the same part in describing

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electromagnetic phenomena in simple perfect or poorly conducting dielectrics as do v_0 and ζ_0 in space.

The complex velocity v is given by

$$v = \frac{\omega}{\beta} = \frac{\omega}{\beta_s - j\alpha_s} = \frac{v_s}{\sqrt{1 + \frac{\alpha_s^2}{\beta_s^2}}} e^{j \tan^{-1}(\alpha_s/\beta_s)} \tag{17}$$

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$$\alpha_s \doteq \beta_e \left(\frac{\sigma_e}{2\omega\epsilon_e}\right) = \frac{1}{2}\sigma_e\zeta_e \tag{21}$$

Note that

$$h_e = \frac{2\alpha_s}{\beta_s} \tag{22}$$

so that

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$$h_e >> 1 \tag{25}$$

and

$$\epsilon_r = 1 \tag{26}$$

so that

$$f(h_e) \doteq g(h_e) \doteq \sqrt{\frac{h_e}{2}} \tag{27}$$

and

$$\beta_s \doteq \alpha_s \doteq \beta_e \sqrt{\frac{h_e}{2}} = \sqrt{\frac{\omega\sigma_e}{2\nu}} = \sqrt{\frac{1}{2}\omega\mu\sigma_e} \tag{28}$$

$$v_s = \sqrt{\frac{2\omega\nu}{\sigma_e}} = \sqrt{\frac{2\omega}{\mu\sigma_e}} \tag{29}$$

$$v = \frac{v_s}{\sqrt{2}} e^{j\frac{\pi}{4}} \tag{30}$$

The reciprocal of α_s is dimensionally a length that is called the *skin depth* or *skin thickness* in good conductors.   Its physical significance is discussed in Chapter V.   It is

$$d_s \equiv \sqrt{\frac{2\nu}{\omega\sigma_c}} = \sqrt{\frac{2}{\omega\mu\sigma_e}} \qquad (31)$$

In terms of the skin depth

$$\beta_s = \alpha_s = \frac{1}{d_s}; \qquad v_s = \omega d_s \qquad (32)$$

An important and subsequently useful relation which is true in simple media may be derived directly from the general field equation (13.14c)

$$\nu_0 \text{ curl } \mathbf{B} = \overline{\varrho_m \mathbf{v}} + j\omega\epsilon_0\mathbf{E} \qquad (33)$$

and its equivalent (14.30c) in simple media

$$\nu_o\nu_r \text{ curl } \mathbf{B} = j\omega\xi\mathbf{E} = (\sigma_o + j\omega\epsilon_{rr}\epsilon_0)\mathbf{E} \qquad (34)$$

Since both relations are true in a simple medium, it is possible to eliminate $\mathbf{B}$ and obtain the following linear complex relation between $\overline{\varrho_m \mathbf{v}}$ and $\mathbf{E}$:

$$\overline{\varrho_m \mathbf{v}} = \frac{1}{\nu_r} \{\sigma_o + j\omega\epsilon_0(\epsilon_{rr} - \nu_r)\}\mathbf{E} = K\mathbf{E} \qquad (35)$$

with

$$\begin{aligned} \mathfrak{y}_r &= \frac{1}{\nu_r} \\ K &= \mathfrak{y}_r\{\sigma_o + j\omega\epsilon_0(\epsilon_{rr} - \nu_r)\} \end{aligned} \qquad (36)$$

The complex parameter K is called the *generalized conduction parameter* because it is the formal analogue of the conductivity σ in the constitutive relation

$$\mathbf{i}_f = \sigma\mathbf{E} \qquad (37)$$

K can be defined in all simple media, so that it is correct to conclude from (35) that the complex amplitude $\overline{\varrho_m \mathbf{v}}$ of the essential density of moving charge is linearly related to the complex amplitude of the electric field at all points in a simple medium.

If it is desired to consider separately an impressed electric field as discussed in Sec. II.18, it is possible to write

$$\mathbf{i}_f = \sigma(\mathbf{E} + \mathbf{E}^e) \qquad (38)$$

in place of (37).   Instead of giving (34) this leads to

$$\nu_0 \nu_r \text{ curl } \mathbf{B} = \delta \mathbf{E}^\circ + (\sigma_s + j\omega\epsilon_{sr}\epsilon_0)\mathbf{E} \tag{39}$$

Upon eliminating ν_0 curl $\mathbf{B}$ using (33) as before

$$\overline{\varrho_m \mathbf{v}} = K\mathbf{E} + \mathbf{u}_r \delta \mathbf{E}^\circ \tag{40}$$

16. Fields of Electric and Magnetic Types in General Curvilinear Coordinates in Simple Media.—The symmetrical forms (14.38a,b) make it possible to write the independent electric and magnetic type fields as follows:

Electric type

$$\mathbf{E} = \text{grad div } \mathbf{Z} + \beta^2 \mathbf{Z} \tag{1a}$$

$$\mathbf{B} = j\frac{\beta^2}{\omega} \text{ curl } \mathbf{Z} \tag{1b}$$

Magnetic type

$$\mathbf{B} = \text{grad div } \mathbf{Y} + \beta^2 \mathbf{Y} \tag{2a}$$
$$\mathbf{E} = -j\omega \text{ curl } \mathbf{Y} \tag{2b}$$

It is also possible to define a magnetic type scalar function ψ by analogy with (14.37a) for the electric type scalar potential ϕ. Thus

$$\text{(3a)} \qquad \phi = -\text{div } \mathbf{Z} \qquad \psi = -\text{div } \mathbf{Y} \tag{3b}$$

Both ϕ and ψ satisfy equations of the form (13.36)

$$\text{(4a)} \qquad \nabla^2\phi + \beta^2\phi = 0 \qquad \nabla^2\psi + \beta^2\psi = 0 \tag{4b}$$

The separate formulation of fields of the electric type derived from $\mathbf{Z}$ and fields of the magnetic type derived from $\mathbf{Y}$ is convenient in satisfying boundary conditions involving simple media; it is not fundamentally necessary.   In the most general case all problems can be set up in terms of $\mathbf{Z}$ alone, although the solution may be analytically much more difficult than if $\mathbf{Y}$ is used.

If $\mathbf{Z}$ and $\mathbf{Y}$ are both restricted so that

$$\text{(5a)} \qquad \mathbf{Z} = \hat{w}Z_w \qquad \mathbf{Y} = \hat{w}Y_w \tag{5b}$$

where $\hat{w}$ is a unit vector in the direction of the coordinate w in the orthogonal system u, v, w, it is no longer possible to formulate all problems in terms of $\mathbf{Z}$ alone.   In fact only problems in which the motion of charge is in the w direction can be

analyzed using Z.   On the other hand, problems in which the motion of charge is confined to closed loops in a surface at right angles to w are especially adapted to solution in terms of Y.   By an appropriate choice of coordinate system and of the direction w, a large variety of practically important problems can be analyzed if the restrictions regarding the coordinate system can be satisfied.

In an orthogonal system of coordinates u, v, w, the distance ds_u measured between two surfaces for which u differs by du is

$$ds_u = c_1 du \qquad (6)$$

where, in general, c_1 is a function of u, v, w.   Similarly

$$ds_v = c_2 dv \qquad (7)$$
$$ds_w = c_3 dw \qquad (8)$$

The quantities c_1, c_2, c_3 have the following values:

Coordinates u, v, w in this order	c_1	c_2	c_3	ds_u	ds_v	ds_w
Rectangular x, y, z	1	1	1	dx	dy	dz
Cylindrical r, θ, z	1	r	1	dr	$r\,d\theta$	dz
Spherical O, Φ, R	R	$R\sin O$	1	$R\,dO$	$R\sin O\,d\Phi$	dR

The following conditions will be imposed on the coordinates u, v, w to be used in solving problems in terms of Z and Y as defined in $(5a,b)$

$$c_3 = 1 \qquad (9)$$

$$\frac{c_1}{c_2} \text{ is independent of } w \qquad (10)$$

These conditions are seen to be true for the three most commonly used systems of coordinates.   The reasons for requiring (9) and (10) are given below.

In general orthogonal coordinates with $(5a)$ the three coordinate equations corresponding to $(14.36a)$ are

$$(\hat{u}, \nabla^2 \hat{w} Z_w) = 0 \qquad (11)$$
$$(\hat{v}, \nabla^2 \hat{w} Z_w) = 0 \qquad (12)$$
$$(\hat{w}, \nabla^2 \hat{w} Z_w) + \beta^2 Z_w = 0 \qquad (13)$$

The operator ∇^2 operating on a vector can be evaluated only

from

$$\nabla^2 \mathsf{Z} = \text{grad div } \mathsf{Z} - \text{curl curl } \mathsf{Z} \tag{14}$$

Using expressions for the gradient and divergence given in Appendix I, the u and w components of (14) are

$$(\hat{u}, \nabla^2 \hat{w} Z_w) = \frac{1}{e_1} \frac{\partial}{\partial w} \left\{ \frac{1}{e_1 e_2 e_3} \frac{\partial}{\partial w} (e_1 e_2 Z_w) \right\} - (\hat{u}, \text{curl curl } \hat{w} Z_w) \tag{15}$$

$$(\hat{w}, \nabla^2 \hat{w} Z_w) = \frac{1}{e_3} \frac{\partial}{\partial w} \left\{ \frac{1}{e_1 e_2 e_3} \frac{\partial}{\partial w} (e_1 e_2 Z_w) \right\} - (\hat{w}, \text{curl curl } \hat{w} Z_w) \tag{16}$$

Using the general expression for the components of the curl in orthogonal coordinates (Appendix I), the three equations (11) to (13) become

$$\frac{1}{e_1} \frac{\partial}{\partial u} \left\{ \frac{1}{e_1 e_2 e_3} \frac{\partial}{\partial w} (e_2 c_1 Z_w) \right\} - \frac{1}{c_2 c_3} \frac{\partial}{\partial w} \left\{ \frac{e_2}{c_3 c_1} \frac{\partial}{\partial u} (c_3 Z_w) \right\} = 0 \tag{17}$$

$$\frac{1}{e_2} \frac{\partial}{\partial v} \left\{ \frac{1}{e_2 e_3 e_1} \frac{\partial}{\partial w} (e_1 c_2 Z_w) \right\} - \frac{1}{c_3 c_1} \frac{\partial}{\partial w} \left\{ \frac{c_1}{c_2 c_3} \frac{\partial}{\partial v} (c_3 Z_w) \right\} = 0 \tag{18}$$

$$\frac{1}{e_3} \frac{\partial}{\partial w} \left\{ \frac{1}{e_3 e_2 e_1} \frac{\partial}{\partial w} (e_1 e_2 Z_w) \right\} + \frac{1}{c_1 c_2} \frac{\partial}{\partial u} \left\{ \frac{e_2}{c_3 c_1} \frac{\partial}{\partial u} (c_3 Z_w) \right\}$$
$$+ \frac{1}{e_1 e_2} \frac{\partial}{\partial v} \left\{ \frac{e_1}{c_2 c_3} \frac{\partial}{\partial v} (e_3 Z_w) \right\} + \beta^2 Z_w = 0 \tag{19}$$

Subject to (9) and (10), the following simplification is achieved. Using (17) with e_2/e_1 written in front of $\partial/\partial w$ and $c_3 = 1$,

$$\frac{1}{c_1} \frac{\partial}{\partial u} \left\{ \frac{1}{c_1 c_2} \frac{\partial}{\partial w} (c_2 c_1 Z_w) \right\} = \frac{1}{c_1} \frac{\partial}{\partial w} \frac{\partial}{\partial u} (Z_w) = \frac{1}{c_1} \frac{\partial}{\partial u} \frac{\partial}{\partial w} (Z_w) \tag{20}$$

A sufficient condition to satisfy (20) is

$$\frac{1}{c_1 c_2} \frac{\partial}{\partial w} (c_2 c_1 Z_w) = \frac{\partial Z_w}{\partial w} \tag{21}$$

The same result may be obtained from (19). With (9), (10), and (21), (20) reduces to

$$\frac{\partial^2 Z_w}{\partial w^2} + \nabla^2_{u,v} Z_w + \beta^2 Z_w = 0 \tag{22}$$

where from Appendix I

$$\nabla^2_{u,v} = \frac{1}{c_1 c_2} \left\{ \frac{\partial}{\partial u} \left(\frac{c_2}{c_1} \frac{\partial}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{c_1}{c_2} \frac{\partial}{\partial v} \right) \right\} \tag{23}$$

Equations like (22) are obtained for Y_w from (14.36b), and also for ϕ and ψ using (3a), (5a), (21b), and (18) in Appendix I, to give

$$\phi = -\frac{1}{e_1 e_2}\frac{\partial}{\partial w}(e_1 e_2 Z_w) = -\frac{\partial Z_w}{\partial w} \tag{24}$$

This is substituted in (22), differentiated with respect to z.

The electric and magnetic types may be formulated as follows:
Electric type

$$\frac{\partial^2 Z_w}{\partial w_2} + \nabla^2_{u,v} Z_w + \beta^2 Z_w = 0 \tag{25a}$$

$$\frac{\partial^2 \phi}{\partial w^2} + \nabla^2_{u,v}\phi + \beta^2 \phi = 0 \tag{26a}$$

$$\phi = -\frac{\partial Z_w}{\partial w} \tag{27a}$$

Magnetic type

$$\frac{\partial^2 Y_w}{\partial w^2} + \nabla^2_{u,v} Y_w + \beta^2 Y_w = 0 \tag{25b}$$

$$\frac{\partial^2 \psi}{\partial w^2} + \nabla^2_{u,v}\psi + \beta^2 \psi = 0 \tag{26b}$$

$$\psi = -\frac{\partial Y_w}{\partial w} \tag{27b}$$

Using (1a,b) and (2a,b) with appropriate formulas from Appendix I

Electric Type	*Magnetic Type*	
$(28a)$ $E_u = \dfrac{1}{e_1}\dfrac{\partial^2 Z_w}{\partial u \partial w}$	$B_u = \dfrac{1}{c_1}\dfrac{\partial^2 Y_w}{\partial u \partial w}$	$(28b)$
$(29a)$ $E_v = \dfrac{1}{e_2}\dfrac{\partial^2 Z_w}{\partial v \partial w}$	$B_v = \dfrac{1}{c_2}\dfrac{\partial^2 Y_w}{\partial v \partial w}$	$(29b)$
$(30a)$ $E_w = \dfrac{\partial^2 Z_w}{\partial w^2} + \beta^2 Z_w$	$B_w = \dfrac{\partial^2 Y_w}{\partial w^2} + \beta^2 Y_w$	$(30b)$
$(31a)$ $B_u = \dfrac{j\beta^2}{\omega c_2}\dfrac{\partial Z_w}{\partial v}$	$E_u = \dfrac{-j\omega}{c_2}\dfrac{\partial Y_w}{\partial v}$	$(31b)$
$(32a)$ $B_v = \dfrac{-j\beta^2}{\omega e_1}\dfrac{\partial Z_w}{\partial u}$	$E_v = \dfrac{j\omega}{c_1}\dfrac{\partial Y_w}{\partial u}$	$(32b)$
$(33a)$ $B_w = 0$	$E_w = 0$	$(33b)$

It follows from (33a,b) that the

electric or E-type field	magnetic or H-type field

has no component of

B		E
in the direction of w. Since the		
magnetic field		electric field

is entirely transverse to the direction of

Z		Y
at every point, the		
electric type		magnetic type

is called

transverse magnetic, TM		transverse electric, TE

17. Transmission-line Form for Simple Media.—In order to obtain the desired transmission-line form it is necessary to require the coordinate w in the general formulation in Sec. 15 to be the coordinate z of an otherwise unrestricted orthogonal system.   This means that u, v lie in a plane perpendicular to z which is the direction of

$$\mathbf{Z} = \imath Z_z; \qquad \mathbf{Y} = \imath Y_z \tag{1}$$

In this case e_1 and e_2 are not functions of z so that the variables in (16.25a,b), (16.26a,b) are easily separated by writing, e.g.,

$$Z_w = f_B(z) F_B(u,v) \tag{2}$$

where $f_B(z)$ is a function of z alone, $F_B(u,v)$ a function of u, v alone. Substituting (2) in (16.25a) leads to

$$\frac{1}{f_B} \frac{\partial^2 f_B}{\partial z^2} + \beta^2 = -\frac{1}{F_B} \nabla_{u,v}^2 F_B \tag{3}$$

This equation can be true for all values of u, v, z only if both sides are equal to a constant.   Let this be k_B^2.   Then

$$\frac{\partial^2 f_B}{\partial z^2} + (\beta^2 - k_B^2) f_B = 0 \tag{4}$$

$$\nabla_{u,v}^2 F_B + k_B^2 F_B = 0 \tag{5}$$

with

$$\gamma_{0B}^2 = k_B^2 - \beta^2 \tag{6}$$

and after multiplying (4) by F_B, (5) by f_B, (16.25a,b) to (16.27a,b) become, with appropriate superscripts E and M,

$$(7a) \qquad \frac{\partial^2 Z_s}{\partial z^2} - \gamma_{0B}^2 Z_s = 0 \qquad \frac{\partial^2 Y_s}{\partial z^2} - \gamma_{0M}^2 Y_s = 0 \qquad (7b)$$

$$(8a) \qquad \frac{\partial^2 \phi}{\partial z^2} - \gamma_{0B}^2 \phi = 0 \qquad \frac{\partial^2 \psi}{\partial z^2} - \gamma_{0M}^2 \psi = 0 \qquad (8b)$$

with

$$(9a) \qquad \nabla_{u,v}^2 Z_s + k_B^2 Z_s = 0 \qquad \nabla_{u,v}^2 Y_s + k_M^2 Y_s = 0 \qquad (9b)$$

$$(10a) \qquad \phi = -\frac{\partial Z_s}{\partial z} \qquad \psi = -\frac{\partial Y_s}{\partial z} \qquad (10b)$$

Differentiating (10*a,b*) with respect to *z* and using (7*a,b*), the following equations are obtained:

$$(11a) \qquad \frac{\partial \phi}{\partial z} = -\gamma_{0B}^2 Z_s \qquad \frac{\partial \psi}{\partial z} = -\gamma_{0M}^2 Y_s \qquad (11b)$$

Equations (7*a,b*) and (8*a,b*) are like the second-order transmission-line equations; (10*a,b*) and (11*a,b*) are like the first-order transmission-line equations derived in Sec. VI.23. The latter can be put into precisely the same form as the first-order transmission-line equations if the following quantities are used instead of **Z, Y,** and ψ.

Let

$$(12a) \qquad \mathbf{U} \equiv j\omega\xi\mathbf{Z} \qquad \mathbf{V} \equiv j\omega\mathbf{Y} \qquad (12b)$$

$$\Psi \equiv v\psi \equiv \psi/\mathfrak{y} \qquad (12c)$$

Also let

$$(13a) \qquad y_B \equiv j\omega\xi \qquad z_M \equiv j\omega\mathfrak{y} \qquad (13b)$$

$$(14a) \qquad z_B \equiv \frac{\gamma_{0B}^2}{j\omega\xi} \qquad y_M \equiv \frac{\gamma_{0M}^2}{j\omega\mathfrak{y}} \qquad (14b)$$

then

$$(15a) \qquad -\frac{\partial U_s}{\partial z} = y_B\phi \qquad -\frac{\partial V_s}{\partial z} = z_M\Psi \qquad (15b)$$

$$(16a) \qquad -\frac{\partial \phi}{\partial z} = z_B U_s \qquad -\frac{\partial \Psi}{\partial z} = y_M V_s \qquad (16b)$$

In form these are precisely the transmission-line equations. Moreover U_s and Ψ are measured in amperes; ϕ and V_s in volts; y_B and y_M are in mhos per meter; z_B and z_M are in ohms per meter. The propagation constants are

$$(17a) \qquad \gamma_{0B} = \sqrt{z_B y_B} \qquad \gamma_{0M} = \sqrt{z_M y_M} \qquad (17b)$$

A characteristic impedance can be defined formally as in line theory (Sec. VI.23; also Volume III).

$$(18a) \qquad Z_{CE} = \sqrt{\frac{z_E}{y_E}} \qquad Z_{CM} = \sqrt{\frac{z_M}{y_M}} \qquad (18b)$$

The *formal* equivalence between (15) to (18) and transmission-line expressions must *not* be misinterpreted to be an *actual* equivalence. Although the scalar potential function ϕ in volts can be specialized to give the potential difference between the conductors of a transmission line, the vector component U_z in amperes *cannot be specialized to give the total current in one of the conductors*. Accordingly, the characteristic impedance Z_{CE} cannot be specialized to give the characteristic impedance of a conventional transmission line. It can be shown (Volume III) that for two- or four-wire open lines and for conventional coaxial lines with air as dielectric Z_{CE} is always equal to $\zeta_0 = 376.7$ ohms regardless of wire size or separation.

Electric and magnetic fields may be expressed in terms of the four functions U_z, ϕ, V_z, Ψ using (16.28–16.33) and (12–18). Note that $w = z$, $e_3 = 1$, and c_1 and c_2 are independent of z.

Also $\beta^2 = \omega^2 \xi/\nu = \omega^2 \xi y$.

	Electric Type		*Magnetic Type*	
$(19a)$	$E_u = -\dfrac{1}{c_1}\dfrac{\partial \phi}{\partial u}$	$B_u = -\dfrac{1}{c_1 \nu}\dfrac{\partial \Psi}{\partial u}$		$(19b)$
$(20a)$	$E_v = -\dfrac{1}{c_2}\dfrac{\partial \phi}{\partial v}$	$B_v = -\dfrac{1}{c_2 \nu}\dfrac{\partial \Psi}{\partial v}$		$(20b)$
$(21a)$	$E_z = \dfrac{k_E^2}{y_E}U_z$	$B_z = \dfrac{k_M^2}{\nu z_M}V_z$		$(21b)$
$(22a)$	$B_u = \dfrac{1}{c_2 \nu}\dfrac{\partial U_z}{\partial v}$	$E_u = -\dfrac{1}{c_2}\dfrac{\partial V_z}{\partial v}$		$(22b)$
$(23a)$	$B_v = -\dfrac{1}{c_1 \nu}\dfrac{\partial U_z}{\partial u}$	$E_v = \dfrac{1}{c_1}\dfrac{\partial V_z}{\partial u}$		$(23b)$
$(24a)$	$B_z = 0$	$E_z = 0$		$(24b)$

with

$$(25a) \qquad \phi = -\frac{1}{y_E}\frac{\partial U_z}{\partial z} \qquad\qquad \Psi = -\frac{1}{z_M}\frac{\partial V_z}{\partial z} \qquad (25b)$$

Slightly more symmetrical formulas are obtained if the magnetic H field is used instead of the magnetic B field. In simple media their complex amplitudes are related by the formula

$$\mathsf{H} = \nu\mathsf{B} \qquad (26)$$

If the factors k have the special values

$$k_E = 0 = k_M \tag{27}$$

E_z vanishes in the electric type, B_z in the magnetic type.   In this important special case, electric and magnetic fields are both transverse and are said to be of the *transverse-electromagnetic*, or TEM, type.

18. Complex Energy Functions.—The specialization of the energy formulation to complex harmonic functions of time must begin with the power equation (10.2)

$$\frac{dW}{dt} - A_q \frac{dQ}{dt} - \frac{dU}{dt} - T = 0 \tag{1}$$

This expresses the relation between the *instantaneous* time rates of change of mechanical, thermal, and electromagnetic energy functions and the instantaneous electromagnetic energy transfer function T defined for a thermally isolated region τ and its closed boundary Σ_τ.   If the field vectors E and B and the auxiliary vectors D and H vary harmonically in time, the energy functions and the transfer function, which are defined in terms of E, B, D, and H, must also have periodic time variations.   Interest is not primarily in the instantaneous time rates of change of the energy functions, but in their average values over a longer interval. Since such an interval includes many periods T_p (not to be confused with the transfer function T without subscript p) and since each period is like every other one, a time average over a single period is the same as the time average over a great many whole periods.   A fraction of a period in a sufficiently long time contributes only a negligible amount to the average.

The time average of any function X_{inst} over a period is given by the integral

$$\bar{\bar{X}} = \frac{1}{T_p} \int_0^{T_p} X_{inst}\, dt \tag{2}$$

The double superscript bar is used to indicate time-average values.   Thus, for

$$X_{inst} = X \cos(\omega t + \theta_x) \tag{3}$$

$$\bar{X} = \frac{1}{T_p} \int_0^{T_p} X \cos(\omega t + \theta_x) dt$$

$$= \frac{1}{\omega T_p} \int_0^{2\pi} X \cos(x + \theta_x) dx = 0 \quad (4)$$

where $x = \omega t$.

$$\bar{X}^2 = \frac{1}{T_p} \int_0^{T_p} X^2 \cos^2(\omega t + \theta_x) dt = \frac{1}{\omega T_p} \int_0^{2\pi} X^2 \cos^2(x + \theta_x) dx$$

$$= \frac{4}{2\pi} \int_0^{\pi/2} X^2 \cos^2(x + \theta_x) dx = \frac{X^2}{2} \quad (5)$$

$$\frac{d\bar{\bar{X}}^2}{dt} = 0 \quad (6)$$

If $Y_{\text{inst}} = Y \cos(\omega t + \theta_Y)$, the time-average product $\overline{XY}$ becomes

$$\overline{XY} = \frac{1}{T_p} \int_0^{T_p} XY \cos(\omega t + \theta_Y) \cos(\omega t + \theta_x) dt$$

$$= \frac{XY}{T_p} \int_0^{T_p} \{\cos \theta_x \cos \theta_Y \cos^2 \omega t + \sin \theta_x \sin \theta_Y \sin^2 \omega t$$

$$- \tfrac{1}{2}(\cos \theta_x \sin \theta_Y + \sin \theta_x \sin \theta_Y) \sin 2\omega t\} dt$$

$$= \tfrac{1}{2} XY \cos(\theta_x - \theta_Y) \quad (7)$$

If the complex notation is used with the instantaneous value of the desired real function X_{inst} or Y_{inst} given by the real part of the complex instantaneous function

$$X_{\text{inst}} = Xe^{j\omega t} = Xe^{j\theta_x}e^{j\omega t} \quad (8a)$$

$$Y_{\text{inst}} = Ye^{j\omega t} = Ye^{j\theta_Y}e^{j\omega t} \quad (8b)$$

the correct values for the real parts of products and squared terms may be obtained after first defining complex conjugate functions by changing the sign of the imaginary parts of (8).

$$X_{\text{inst}}^* = X^*e^{-j\omega t} = Xe^{-j\theta_x}e^{-j\omega t} \quad (9a)$$

$$Y_{\text{inst}}^* = Y^*e^{-j\omega t} = Ye^{-j\theta_Y}e^{-j\omega t} \quad (9b)$$

It follows at once that

$$\bar{X}^2 = \tfrac{1}{2}X^2 = \tfrac{1}{2}X_{\text{inst}}X_{\text{inst}}^* = \tfrac{1}{2}XX^* \quad (10)$$

$$\overline{XY} = \tfrac{1}{2}XY = \text{real part } \tfrac{1}{2}X_{\text{inst}}Y_{\text{inst}}^* = \text{real part } \tfrac{1}{2}XY^* \quad (11)$$

A time-average power equation may be derived from

$$\int_\tau \overline{(\mathbf{i}_f,\mathbf{E})}_{\text{inst}} d\tau + \int_\Sigma \overline{(\mathbf{I}_f,\mathbf{E})}_{\text{inst}} d\sigma = \text{real part } \frac{1}{2}\left\{\int_\tau (\mathbf{i}_f^*,\mathbf{E}) d\tau \right.$$
$$\left. + \int_\Sigma (\mathbf{I}_f^*,\mathbf{E}) d\sigma\right\} \quad (12)$$

In order to expand the right side of (12), it is necessary to substitute for $\mathbf{i}_f^*$ and $\mathbf{I}_f^*$ from the equations

$$\text{curl } \mathbf{H}^* = \mathbf{i}_f^* - j\omega\mathbf{D}^* \quad (13)$$
$$[\mathbf{n},\mathbf{H}^*] = -\mathbf{I}_f^* \quad (14)$$

to obtain

$$\frac{1}{2}\int_\tau \{(\mathbf{E}, \text{curl } \mathbf{H}^*) + j\omega(\mathbf{E},\mathbf{D}^*)\} d\tau - \sum_i \frac{1}{2}\int_{S_i} ([\mathbf{n},\mathbf{H}^*],\mathbf{E}) d\sigma_i \quad (15)$$

with the vector identity

$$\text{div } [\mathbf{E},\mathbf{H}^*] = (\mathbf{H}^*, \text{curl } \mathbf{E}) - (\mathbf{E}, \text{curl } \mathbf{H}^*) \quad (16)$$

and

$$\text{curl } \mathbf{E} = -j\omega\mathbf{B} \quad (17)$$

(15) becomes

$$j\frac{\omega}{2}\int_\tau \{(\mathbf{E},\mathbf{D}^*) - (\mathbf{B},\mathbf{H}^*)\} d\tau - \frac{1}{2}\int_\tau \text{div } [\mathbf{E},\mathbf{H}^*] d\tau$$
$$+ \sum_i \frac{1}{2}\int_{S_i} (\mathbf{n},[\mathbf{E},\mathbf{H}^*]) d\sigma_i \quad (18)$$

After applying the divergence theorem to the second integral (if necessary expanded into a sum over separate regions where the functions $\mathbf{E}$ and $\mathbf{H}^*$ are continuous), the integrals over surfaces of discontinuity cancel the corresponding integrals from the last term in (18) and only the integral over the enclosing surface Σ_r remains.

$$\frac{1}{2}\left\{\int_\tau (\mathbf{i}_f^*,\mathbf{E}) d\tau + \int_{\Sigma_r} (\mathbf{I}_f^*,\mathbf{E}) d\sigma\right\}$$
$$= -\frac{1}{2}\left[j\omega \int_\tau \{(\mathbf{B},\mathbf{H}^*) - (\mathbf{E},\mathbf{D}^*)\} d\tau + \int_{\Sigma_r} (\mathbf{n},[\mathbf{E},\mathbf{H}^*]) d\sigma\right] \quad (19)$$

This is a complex power equation that can be separated into two real equations. The equation given by the real parts in (19)

is the desired time-average power equation; that given by the imaginary parts is an additional true equation.

As in the general case of unrestricted time dependence, the complex power equation assumes an attractive form particularly in space and in simple media with *no time lags* in polarization, magnetization, or current responses. In such media, by definition,

$$i_f = \sigma(E + E^e); \qquad D = \epsilon E; \qquad H = \nu B \qquad (20a)$$

with σ, ϵ, ν real.   Also

$$I_f = 0 \qquad (20b)$$

$$(i_f^*, E) = \sigma(E^* + E^{e*}, E) = \frac{i_f^2}{\sigma} - (i_f^*, E^e) \qquad (21a)$$

$$(B, H^*) = \nu B^2 \qquad (21b)$$

$$(E, D^*) = \epsilon E^2 \qquad (21c)$$

The complex Poynting vector is

$$S = \tfrac{1}{2}[E, H^*] = \tfrac{1}{2}\nu[E, B^*] = Se^{j\theta_S} = S(\cos \theta_S + j \sin \theta_S) \qquad (21d)$$

with

$$S = \tfrac{1}{2}[E, H] = \tfrac{1}{2}\nu[E, B]; \qquad \theta_s = \theta_E - \theta_H \qquad (21e)$$

Upon substituting from (21) in (19), assumming i_f in phase with E^e

$$\frac{1}{2}\int_\tau \frac{i_f^2}{\sigma} d\tau - \frac{1}{2}\int_\tau (i_f, E^e)d\tau = -j\frac{\omega}{2}\int_\tau \{\nu B^2 - \epsilon E^2\} d\tau$$
$$- \int_{\Sigma_\tau} (\mathbf{n}, S)e^{j\theta_S}d\sigma \qquad (22)$$

Upon separating real and imaginary parts, two equations are obtained.   The real power equation is

$$\frac{1}{2}\int_\tau (i_f, E^e)d\tau - \frac{1}{2}\int_\tau \frac{i_f^2}{\sigma} d\tau - \int_{\Sigma_\tau} (\mathbf{n}, S) \cos \theta_S \, d\sigma = 0 \qquad (23)$$

An auxiliary equation, which may be called the reactive power equation, is

$$\frac{\omega}{2}\int_\tau \{\nu B^2 - \epsilon E^2\}d\tau + \int_{\Sigma_\tau} (\mathbf{n}, S) \sin \theta_S \, d\sigma = 0 \qquad \cdot \quad (24)$$

It is possible to introduce the notation (11.6), (11.7) by defining time-average electric and magnetic energy functions $\bar{U}_E$ and $\bar{U}_M$ and a complex energy-transfer function the real part of which is the time-average energy-transfer function

$$T = \bar{T}_r + jT_i$$

Thus

$$\bar{U}_M = \frac{\nu}{4} \int_\tau B^2 d\tau; \qquad \bar{U}_E = \frac{\epsilon}{4} \int_\tau E^2 d\tau \qquad (25)$$

$$\bar{T}_r = \text{real part} \left\{ \int_{\Sigma_\tau} (\mathsf{A,S}) d\sigma \right\} = \int_{\Sigma_\tau} (\mathsf{A,S}) \cos \theta_N d\sigma \qquad (26a)$$

$$T_i = \text{imaginary part} \left\{ \int_{\Sigma_\tau} (\mathsf{A,S}) d\sigma \right\} = \int_{\Sigma_\tau} (\mathsf{A,S}) \sin \theta_N d\sigma \qquad (26b)$$

In terms of time-average rates of change of mechanical, thermal, and electromagnetic energy functions, the power equations (23) and the auxiliary equation (24) may be written as follows:

$$\frac{d\overline{W}}{dt} - A_Q \frac{d\overline{Q}}{dt} - \bar{T}_r = 0 \qquad (27)$$

$$2\omega(\bar{U}_M - \bar{U}_E) + T_i = 0 \qquad (28)$$

The terms in (27) are identified with those in (23) in the same order; the terms in (28) are identified with those in (24). Since the power equation does not involve the time-average electric and magnetic energy functions, it must be concluded that

$$\frac{d\bar{U}}{dt} = 0$$

so that

$$\bar{U} = \ddot{U}_M + \ddot{U}_E = \text{const.} \qquad (29)$$

Whenever

$$T_i = 0 \qquad (30)$$

which is certainly true when *one* or both of the following conditions are satisfied at *all points* on the surface Σ_τ,

$$\theta_N = \theta_H - \theta_U = 0 \qquad (31a)$$

$$(\mathsf{A,S}) = 0 \qquad (31b)$$

it follows from (28b) that

$$\ddot{U}_M = \ddot{U}_E \qquad (32)$$

and

$$\ddot{U} = 2\ddot{U}_M = 2\ddot{U}_E \qquad (33)$$

It is found that (30) is true in many important problems.

It is possible to take account of time lags in polarization and current response in sufficiently specialized physical models by noting that if conductivity and dielectric constant are complex (13) may be written as follows using (14.19c) with (14.25a)

$$\text{curl } \mathbf{H}^* = (\delta^* - j\omega\epsilon^*)\mathbf{E}^* = (\sigma_e - j\omega\epsilon_e)\mathbf{E}^* \qquad (34)$$

The expression on the right is identically the same as that for no time lags except that the real effective conductivity and dielectric constant replace σ and ϵ. *If it is assumed that the forces required to maintain a periodically varying complex polarization current* $j\omega\mathbf{P}e^{j\omega t}$ *are the same as those required to maintain an equivalent complex current of free charge* $\mathbf{i}e^{j\omega t}$, *so that* $\sigma_e\mathbf{E}^*$ *is assumed to be equivalent in all respects involving the equilibrium of forces to* $\sigma\mathbf{E}^*$ *with* $\sigma_e = \sigma$, then (23), (24), and (25) may be modified by writing a subscript e on both σ and ϵ. Since from (13.24a), $\sigma_e = \sigma' + \omega\epsilon''$, it follows, if $\mathbf{E} + \mathbf{E}^e$ is maintained constant with $\mathbf{E}^e$ directed opposite to $\mathbf{E}$ in charge-separating regions, that the rate of change of the average heat-energy function

$$A_q \frac{d\overline{Q}}{dt} = \int_\tau \frac{\overline{i_f^2}}{\sigma} \, d\tau = \int_\tau \sigma(E - E^e)^2 d\tau \qquad (35)$$

is increased by $\int_\tau \omega\epsilon''(E - E^e)^2 d\tau$, owing to polarization lag when $\sigma_e = (\sigma' + \omega\epsilon'')$ is written for σ. The time-average electric-energy function

$$\overline{U}_e = \frac{\epsilon}{4} \int_\tau E^2 d\tau \qquad (36)$$

is decreased when according to (26) $\epsilon_e = \epsilon' - \sigma''/\omega$ is written for ϵ.

ELECTROMAGNETIC WAVES IN UNBOUNDED REGIONS

WAVES OF POTENTIAL

1. Helmholtz's Integrals.—The general equations for the complex amplitudes of the potential functions are sometimes called Helmholtz's equations. They are

$$\nabla^2 \phi + \beta_0^2 \phi = -\frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla^2 A + \beta_0^2 A = -\frac{\rho_m v}{\nu_0} \tag{2}$$

The equation of continuity for the potential functions is

$$\operatorname{div} A + j\frac{\beta_0^2}{\omega}\phi = 0 \tag{3}$$

As usual

$$\beta_0^2 = \frac{\omega^2}{v_0^2} = \frac{\omega^2 \epsilon_0}{\nu_0} = \omega^2 \epsilon_0 \mu_0 \tag{4}$$

Solutions of (1) and (2) may be written in many forms, but all solutions that are consistent with physically reasonable conditions, and hence are useful in determining analogues of experimentally measurable quantities, automatically satisfy the equation of continuity (3). This can be proved in very general terms.[1] The form of solution that is particularly useful in a general analysis of electric circuits, including especially antennas and open-wire transmission lines that are not in a bounded region, is a particular integral called the Helmholtz integral. Helmholtz's integrals for (1) and (2) are derived as follows. As a first step, consider the equation

$$\nabla^2 v + \beta_0^2 v = 0 \tag{5}$$

where v is a scalar point function. In order to obtain a par-

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[1] See, for example, M. MASON and W. WEAVER, "The Electromagnetic Field," pp. 281, 285, or J. A. STRATTON, "Electromagnetic Theory," p. 429.

ticular integral, let (5) be written in spherical coordinates R, Θ, Φ, under the assumption that v is independent of Θ and Φ. Using Appendix I, (5) reduces to

$$\frac{d^2(Rv)}{dR^2} + \beta_0^2(Rv) = 0 \tag{6}$$

A particular integral of (6) and, hence, also of (5) is

$$v = \frac{e^{-i\beta_0 R}}{R} \tag{7}$$

as may be verified by direct substitution.

As a second step, let ϕ and v be two scalar point functions that are finite and continuous with their first and second derivatives in a volume τ enclosed in a surface Σ_r. According to Green's symmetrical theorem (III.2.16)

$$\int_\tau (\phi \nabla^2 v - v \nabla^2 \phi) d\tau = \int_{\Sigma_\tau} \left(\phi \frac{\partial v}{\partial n} - v \frac{\partial \phi}{\partial n} \right) d\sigma \tag{8}$$

Let v in (8) be identified with (7) with the understanding that R is the distance from a fixed point $P(x,y,z)$, where ϕ is to be determined, to a variable point of integration $P'(x',y',z')$. Both points are within τ. If (7) is to be substituted in (8), it is necessary to exclude temporarily the point P, because as P' is moved throughout τ, $1/R$ becomes infinite at $R = 0$. Therefore, let the point P be enclosed in a small sphere of radius a, and let (8) be formed for the region τ_a which is the same as τ excluding the volume of the small sphere. The surface of this sphere is $\Sigma_a = 4\pi a^2$. The surface integrals in (8) must be evaluated over the complete boundary Σ_τ of τ_a, and hence over its outer surface Σ_0 and the inner boundary Σ_a. Thus, using (7) with (5), and denoting by a prime quantities that are functions only of the point $P'(x',y',z')$ which locates the element of integration as this is moved throughout τ_a and on its boundaries, (8) becomes

$$-\int_{\tau_a} \frac{e^{-i\beta_0 R}}{R} (\nabla^2 \phi' + \beta_0^2 \phi') d\tau' = \int_{\Sigma_0} \left\{ \phi' \frac{\partial}{\partial n} \left(\frac{e^{-i\beta_0 R}}{R} \right) - \frac{e^{-i\beta_0 R}}{R} \frac{\partial \phi'}{\partial n} \right\} d\sigma'$$
$$+ \int_{\Sigma_a} \left\{ \phi' \frac{\partial}{\partial n} \left(\frac{e^{-i\beta_0 R}}{R} \right) - \frac{e^{-i\beta_0 R}}{R} \frac{\partial \phi'}{\partial n} \right\} d\sigma' \tag{9}$$

The distance $R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ is a function of both primed and unprimed coordinates. On the surface

$\Sigma_a = 4\pi a^2$, the external normal to the volume τ_a is directed inward along the radius. Hence, on this surface $\partial/\partial n = -\partial/\partial R$ and $d\sigma = a^2 d\Omega$ where $d\Omega$ is an element of solid angle. The integral over Σ_a in (9) becomes

$$\int_{\Sigma_a} \left\{ \phi' \frac{\partial}{\partial n}\left(\frac{e^{-j\beta_0 R}}{R}\right) - \frac{e^{-j\beta_0 R}}{R}\frac{\partial\phi'}{\partial n} \right\} d\sigma'$$

$$= \int_{\Sigma_a} \left\{ \phi'\left(\frac{1}{R^2} + \frac{j\beta_0}{R}\right)e^{-j\beta_0 R} + \frac{e^{-j\beta_0 R}}{R}\frac{\partial\phi'}{\partial R} \right\}_{R=a} a^2 d\Omega' \quad (10)$$

Since $\partial\phi'/\partial R$ is finite in τ, (10) reduces to the following simple form in the limit as a is allowed to approach zero:

$$\lim_{a\to 0} \int_{\Sigma_a}\left\{ \phi'\frac{\partial}{\partial n}\left(\frac{e^{-j\beta_0 R}}{R}\right) - \frac{e^{-j\beta_0 R}}{R}\frac{\partial\phi'}{\partial n}\right\} d\sigma' = \lim_{a\to 0}\int_{\Sigma_a} \phi' d\Omega'$$
$$= 4\pi\phi \quad (11)$$

where $d\Omega'$ is an element of solid angle. The function ϕ depends only on the coordinates x, y, z of the point P. With (11) in (9),

$$4\pi\phi = -\int_\tau \frac{e^{-j\beta_0 R}}{R}(\nabla^2\phi' + \beta_0^2\phi')d\tau'$$

$$-\int_{\Sigma_0}\left\{ \phi'\frac{\partial}{\partial n}\left(\frac{e^{-j\beta_0 R}}{R}\right) - \frac{e^{-j\beta_0 R}}{R}\frac{\partial\phi'}{\partial n}\right\} d\sigma' \quad (12)$$

Using (1) in the volume integral in (12) and expanding the integrand of the surface integral, (12) becomes

$$\phi = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho'}{R}e^{-j\beta_0 R}d\tau'$$

$$+ \frac{1}{4\pi}\int_{\Sigma_0}\left\{ R\left(\frac{\partial\phi'}{\partial n} + j\beta_0\phi'\frac{\partial R}{\partial n}\right) + \phi'\frac{\partial R}{\partial n}\right\}\frac{e^{-j\beta_0 R}}{R^2}d\sigma' \quad (13)$$

This is the complete expression for the scalar potential if, as has been assumed, ϕ and its first derivative are continuous throughout the region τ. Within the scope of the present formulation in terms of the densities $\bar\rho$, $\bar\pi$ (but not if a surface density of polarization $\mathbf{k}$ is defined), the boundary conditions on ϕ and its derivatives, *viz.*, (III.5.24) to (III.5.26), require ϕ to be continuous but its normal derivative at a boundary characterized by a nonvanishing surface charge to be discontinuous. If such boundaries exist within τ, they must be excluded in the integration. This is done by enclosing such a boundary between two

parallel surfaces each at a constant distance δ (atomic dimension) from the boundary. The volume integral in (13) is then carried out over all volumes excluding only the narrow slices of thickness 2δ between parallel surfaces Σ_i enclosing boundaries where $\partial\phi/\partial n$ is discontinuous. Additional surface integrals then appear on the right in (13). They have the form

$$\int_{\Sigma_i} \left\{ \left(\frac{\partial\phi'}{\partial n_1}\right)_1 + \left(\frac{\partial\phi'}{\partial n_2}\right)_2 \right\} \frac{e^{-j\beta_0 R}}{R}\, d\sigma' \qquad (14)$$

where the subscripts 1 and 2 are used to designate the regions on each side of the boundary; $\hat{n}_1$ and $\hat{n}_2$ are external normals to the regions indicated by the subscript at the boundary. The normal derivatives are taken on each side of the boundary. Using (III.5.25) written for complex amplitudes

$$\left(\frac{\partial\phi}{\partial n_1}\right)_1 + \left(\frac{\partial\phi}{\partial n_2}\right)_2 = -\frac{\hat{n}}{\epsilon_0} \qquad (15)$$

With (15) substituted in (14), and (14) included in (13), the resulting expression for ϕ is

$$\phi = \frac{1}{4\pi\epsilon_0} \int_\tau \left\{ \frac{\varrho'}{R} e^{-j\beta_0 R}\, d\tau' + \int_\Sigma \frac{\hat{n}'}{R} e^{-j\beta_0 R} d\sigma' \right\} + \frac{1}{4\pi} \int_{\Sigma_0} u'd\sigma' \qquad (16)$$

For convenience in writing, only a single volume integral and a single surface integral are shown. In general, each represents a sum of integrals, and it is understood that the volume integral is to be evaluated over all regions in which ϱ differs from zero, the surface integral over all boundaries where $\hat{n}$ is defined. The following shorthand is used:

$$u' \equiv \left\{ R \left(\frac{\partial\phi'}{\partial n} + j\beta_0\phi' \frac{\partial R}{\partial n}\right) + \phi' \frac{\partial R}{\partial n} \right\} \frac{e^{-j\beta_0 R}}{R^2} \qquad (17)$$

(If surfaces on which a surface density of polarization $\mathbf{k}$ is defined are within Σ_0, integrals involving $\mathbf{k}$ must be included. They are readily obtained if it is noted that ϕ is discontinuous by $\mathbf{k}$ at such a surface. No such density is required in most practical problems.)

Since each of the Cartesian components of $\mathbf{A}$ satisfies an equation like (1), each must have a solution of the form (16). If these are combined vectorially and use is made of the boundary

conditions (III.5.27) to (III.5.31) written for complex amplitudes, the integral solution of (2) is

$$\mathbf{A} = \frac{1}{4\pi\nu_0} \left\{ \int_\tau \frac{\overline{\varrho_m \mathbf{v}'}}{R} e^{-j\beta_0 R} d\tau' + \int_\Sigma \frac{\overline{n_m \mathbf{v}'}}{R} e^{-j\beta_0 R} d\sigma' \right\} + \frac{1}{4\pi} \int_{\Sigma_0} \mathbf{u}' d\sigma' \quad (18)$$

with

$$\mathbf{u}' = \left\{ R \left(\frac{\partial \mathbf{A}'}{\partial n} + j\beta_0 \mathbf{A}' \frac{\partial R}{\partial n} \right) + \mathbf{A}' \frac{\partial R}{\partial n} \right\} \frac{e^{-j\beta_0 R}}{R^2} \quad (19)$$

[Surface densities of magnetization are not included in (18).]

If discrete regions in which ϱ, $\bar{n}$, $\overline{\varrho_m \mathbf{v}}$, $\overline{n_m \mathbf{v}}$ are defined are immersed in a homogeneous simple medium that exists everywhere within Σ_0 and is characterized by the complex constants δ, ε_r, ν_r, formulas like (16) and (18) are obtained directly by applying the theorem for simple media formulated in III.14. This involves writing

$$\xi = \epsilon_0 \left(\epsilon_r - \frac{j\delta}{\omega\epsilon_0} \right) = \epsilon - \frac{j\delta}{\omega} \quad (20)$$

for ϵ_0 and

$$\nu = \nu_0 \nu_r \quad (21)$$

for ν_0, so that the complex propagation constant

$$\beta = \omega \sqrt{\frac{\xi}{\nu}} \quad (22)$$

replaces $\beta_0 = \omega \sqrt{\epsilon_0/\nu_0}$. The formulas for ϕ and $\mathbf{A}$ in a simple medium are[1]

$$\phi = \frac{1}{4\pi\xi} \left\{ \int_\tau \frac{\varrho'}{R} \exp(-j\beta R) d\tau' + \int_\Sigma \frac{\bar{n}'}{R} \exp(-j\beta R) d\sigma' \right\} + \frac{1}{4\pi} \int_{\Sigma_0} u' d\sigma' \quad (23)$$

with

$$u' = \left\{ R \left(\frac{\partial \phi'}{\partial n} + j\beta\phi' \frac{\partial R}{\partial n} \right) + \phi' \frac{\partial R}{\partial n} \right\} \frac{\exp(-j\beta R)}{R^2} \quad (24)$$

[1] For typographical reasons, the notation "exp $(-j\beta R)$" is used with some types of letters instead of the usual exponential form. The conventional form is preferred and retained wherever possible.

and

$$\mathsf{A} = \frac{1}{4\pi v} \left\{ \int_\tau \frac{\overline{\varrho_m v'}}{R} \exp(-j\beta R) d\tau' + \int_\Sigma \frac{\overline{n_m v'}}{R} \exp(-j\beta R) d\sigma' \right\}$$
$$+ \frac{1}{4\pi} \int_{\Sigma_0} \mathsf{u}' d\sigma' \quad (25)$$

with

$$\mathsf{u}' = \left\{ R \left(\frac{\partial \mathsf{A}'}{\partial n} + j\beta \mathsf{A}' \frac{\partial R}{\partial n} \right) + \mathsf{A}' \frac{\partial R}{\partial n} \right\} \frac{\exp(-j\beta R)}{R^2} \quad (26)$$

It is understood that ϱ, $\bar{n}$, $\overline{\varrho_m v'}$, $\overline{n_m v'}$ do not include the distributions of charge and current in the simple medium but only distributions in discrete regions immersed in this medium. The effect of the charges and currents in the medium are taken into account in the parameters ξ, v.

In applying (16) and (18) or (23) and (25) to the solution of electromagnetic problems, it is necessary to know u' and u' on the envelope Σ_0 that encloses the regions in which periodically varying distributions of charge and of current are maintained. The location of this envelope is quite arbitrary and may be chosen wherever convenient. If it is possible to locate it in such a way that both u' and u' vanish everywhere on its surface, the last integral in (16), (18), (23), and (25) vanishes and the calculation of scalar and vector potentials reduces to integrations over such volumes and surfaces within Σ_0 in and on which nonvanishing densities of periodically varying charge and current exist. This is true whenever the surface Σ_0 can be chosen in such a way that on it both ϕ and A are zero at all points. Both ϕ and A vanish with E and B, and the amplitude of a periodically varying electromagnetic field reduces to negligibly small values in the interior of a sufficiently good conductor. Therefore, if a periodically varying distribution of charge is maintained in a cavity in a good conductor, it may be assumed that u' and u' are both zero if Σ_0 is entirely in the conductor. Since current and charge on metal walls are not easily found using (23) and (26), other solutions of (1) and (2) are often more useful. (See Chapter V, also Volume III.)

In the case of antennas or other circuits erected on or near the surface of the earth, a surface Σ_0 can be imagined to enclose the entire earth and all transmitting and receiving antennas. But since such a surface is not in a good conductor but in space, there

is no reason to suppose that on it either ϕ or $\mathbf{A}$ is zero. On the contrary, the first two integrals in (16) and (18) must certainly and in general have nonvanishing values on any such surface in space. Suppose the surface Σ_0 is a great sphere about the earth and this is made infinite in radius. Then the potentials on this surface due to the distributions of charge in a finite universe including the earth as obtained from (16) and (18) are

$$\phi = \lim_{R \to \infty} \frac{1}{4\pi\epsilon_0 R_0} \left\{ \int_\tau \varrho' e^{-j\beta_0 R} d\tau' + \int_\Sigma \bar{n}' e^{-j\beta_0 R} d\sigma' \right\} \tag{27}$$

$$\mathbf{A} = \lim_{R \to \infty} \frac{1}{4\pi\nu_0 R_0} \left\{ \int_\tau \overline{\varrho_m v'} e^{-j\beta_0 R} d\tau' + \int_\Sigma \overline{n_m v'} e^{-j\beta_0 R} d\sigma' \right\} \tag{28}$$

where R_0 is measured to a convenient origin. The potentials clearly vanish in the limit since τ is finite, but only as $1/R_0$. If it is assumed that (27) and (28) are the complete solution, $\int_{\Sigma_0} u' d\sigma'$ and $\int_{\Sigma_0} \mathbf{u}' d\sigma'$ in (16) and (18) must vanish. However, if (27) is substituted in (17), the leading term is

$$u' = \lim_{R_0 \to \infty} \frac{-j\beta_0}{4\pi\epsilon_0} \left\{ \int_\tau \varrho' e^{-j\beta_0 R} d\tau' + \int_\Sigma \bar{n}' e^{-j\beta_0 R} d\sigma' \right\} \frac{e^{-j\beta_0 R}}{R_0^2} \tag{29}$$

On the great sphere Σ_0, an element of surface is $d\sigma' = R_0^2 d\Omega$ where $d\Omega$ is an element of solid angle. Hence, if (27) and (28) are the complete solution, the following integral must vanish:

$$\lim_{R_0 \to \infty} \frac{1}{4\pi} \int_{\Sigma_0} u' d\sigma'$$
$$= \lim_{R \to \infty} \frac{-j\beta_0}{4\pi\epsilon_0} \int_{\Sigma_0} \left\{ \int_\tau \varrho' e^{-j\beta_0 R} d\tau' + \int_\Sigma \bar{n}' e^{-j\beta_0 R} d\sigma' \right\} e^{-j\beta_0 R} d\Omega \tag{30}$$

The complete expression on the right is obviously finite in the limit. However, since β_0 is real, it is not zero unless the distributions, amplitudes, and locations of the periodically varying densities are such as to make (30) vanish. This may be true in special cases, but there is no evident reason why it should be postulated in general.

If it is assumed that the distant reaches of the universe are not completely empty space but contain sufficient matter so that they may be considered to approximate an extremely rarefied simple medium, β_0 in (30) must be replaced by $\beta = \beta_s - j\alpha_s$. The integrand in (30) then contains a factor $e^{-2\alpha_s R}$ that approaches zero as R becomes infinite. If such a medium exists in the

distant universe, the vanishing of an expression like (30) may be explained physically in terms of the following sequence. In order to maintain a periodically varying current in a conductor on the earth, similarly periodic motions also have to be maintained in the charges of the medium, although, as is shown later, at a later time. The fact that this medium is imperfect or slightly conducting means that such nonrandom motions of charge are possible only at the expense of an increase in random or heat motions as discussed in Sec. II.16. It is because random motions are increased in the medium that α_s differs from zero. Surely from the point of view of maintaining a periodic motion of charge on the earth and of the effects observed as a result, it is immaterial whether an increase in random motion in the most distant parts of the universe occurs in a continuous medium or in more or less discrete lumps of matter distributed at random. It is reasonable, therefore, to assume that the last integral in (23) and (25) is zero in the limit as Σ_0 approaches a sphere of infinite radius.

Even though the above argument is not to be regarded as a proof of the conclusion, but rather as a means of making it plausible, one difficulty must be pointed out. A periodic motion of charge at a single frequency implies a beginning at an infinite time earlier. Although from the point of view of the transient observed on the earth, it is immaterial whether the alternations were begun an infinite time earlier or only a few seconds or minutes earlier, this is not true of the effect at great distances from the earth. As will be described in great detail later in this chapter, the effect of a particular motion of charge at one point is observed at another distant point only after the lapse of an interval of time that increases with the distance between the points in question. It follows that a truly periodic motion at a single frequency can be established in an infinite universe only in an infinite time. Therefore, insofar as all currents maintained for finite intervals of time are concerned, the above discussion is not directly applicable. In this case, however, it is readily argued that a surface Σ_0 can always be chosen so far away that on it both ϕ and A are zero in the entire duration of the periodic motion and as long after as desired. In fact, by moving Σ_0 to infinity, ϕ and A would be zero on it during all finite intervals including the entire span of existence

of the human race, so that the last integrals in (16) and (18) could contribute nothing to the potentials. This reasoning is not entirely satisfactory because (16) and (18) actually assume a truly periodic phenomenon and, hence, an infinite duration in a universe of infinite extent.

Whether because of the one or the other of the above arguments, it seems reasonable in any case to assume that for all purposes involved in the solution of electromagnetic problems in air above the earth the potential functions may be computed accurately from the first two integrals alone in (16) and (18) with β_0 real or, if simple media are involved as in the interior of the earth, from the first two integrals alone in (23) and (25). Analytically, the requirement that the last integral in (16) and in (18) vanish may be contained in a so-called "radiation condition" which is imposed as a postulate. This is

$$\lim_{R \to \infty} R\left(\frac{\partial \phi}{\partial n} + j\beta_0 \phi \frac{\partial R}{\partial n}\right) = 0 \tag{31}$$

and a similar relation for the vector potential. The radiation condition is usually combined with the "condition for regularity at infinity,"

$$\lim_{R \to \infty} (R\phi) \text{ is finite} \tag{32}$$

and a similar condition for the vector potential.

Subject to (31) and (32) *the solutions of* (1) *and* (2) *can be shown to be unique.*[1] They are given by

$$\phi = \frac{1}{4\pi\epsilon_0}\left\{\int_\tau \frac{\varrho'}{R}\, e^{-j\beta_0 R}d\tau' + \int_\Sigma \frac{\bar{n}'}{R}\, e^{-j\beta_0 R}d\sigma'\right\} \tag{33}$$

$$\mathbf{A} = \frac{1}{4\pi\nu_0}\left\{\int_\tau \frac{\overline{\varrho_m \mathbf{v}'}}{R}\, e^{-j\beta_0 R}d\tau' + \int_\Sigma \frac{\overline{\mathbf{n}_m \mathbf{v}'}}{R}\, e^{-j\beta_0 R}d\sigma'\right\} \tag{34}$$

The integration is to be carried out over all regions and surfaces where nonvanishing density functions are defined. Only a single integral is written in each case; actually, many integrals over many regions separated in space may be involved. Strictly, (33) and (34) are not general solutions, respectively, of (1) and (2) since any solution of the homogeneous equations [(1) and

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[1] See, for example, A. Sommerfeld in P. Frank and R. von Mises, "Differentialgleichungen der Physik," Vol. II, p. 807.

(2) with zero written on the right] may be added. However, since the electromagnetic vectors and the potential functions are by definition computed *only* from charges and moving charges, all such solutions that are physically significant must be due to distributions of charge and current *somewhere* in the universe. Hence if the integrations in (33) and (34) are assumed to be taken over all regions with nonvanishing values of the density functions, (33) and (34) are the general solutions. In practice, (33) and (34) are used to determine ϕ and A due to charges and currents in a particular region such as an antenna, or in several antennas, and it is these values that are of practical importance.

The element of integration $d\tau'$ or $d\sigma'$ is located at P' by a set of primed coordinates, e.g., x', y', z'. The point P at which ϕ or A is calculated is located by unprimed coordinates such as x, y, z. The distance between the points P and P' is

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

It is not difficult, and is of considerable importance, to show under what conditions (33) and (34) satisfy the general equation of continuity for the potential functions (3). Because the volume integrals in (33) and (34) are improper if the point P where ϕ and A are calculated is within a region of nonvanishing densities of charge or moving charge, differentiation under the sign of integration is defined by enclosing P in a small sphere of radius a which is excluded from the region of integration. This is not necessary in the surface integral in which the element $d\sigma'$ is not in the thin layer where the surface density is defined, but on its boundary. Thus, using (34) and writing τ_a for the volume τ excluding the small sphere of radius a,

$$\text{div } \mathsf{A} = \frac{1}{4\pi\nu_0} \lim_{a \to 0} \int_{\tau_a} \text{div} \left\{ \overline{\varrho_m \mathsf{v}'} \, \frac{e^{-j\beta_0 R}}{R} \right\} d\tau' \\ + \frac{1}{4\pi\nu_0} \int_{\Sigma} \text{div} \left\{ \overline{n_m \mathsf{v}'} \, \frac{e^{-j\beta_0 R}}{R} \right\} d\sigma' \quad (35)$$

Since the operation div is with respect to the unprimed coordinates of the point P, the integrands in (35) have the form div $\mathsf{C}\psi$ with C a constant vector (either $\overline{\varrho_m \mathsf{v}'}$ or $\overline{n_m \mathsf{v}'}$) and $\psi = e^{-j\beta_0 R}/R$ a scalar. Using (II.2.11),

$$\text{div } \mathsf{C}\psi = \psi \text{ div } \mathsf{C} + (\mathsf{C}, \text{grad } \psi) \quad (36)$$

Since C is constant with respect to the variables of differentiation in the case under consideration, div $\mathsf{C} = 0$ and

$$\text{div}\left\{\overline{\varrho_m \mathbf{v}'}\frac{e^{-j\beta_0 R}}{R}\right\} = \left(\overline{\varrho_m \mathbf{v}'}, \text{grad}\,\frac{e^{-j\beta_0 R}}{R}\right) \tag{37}$$

Since $R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ in Cartesian coordinates, it is clear that differentiation of a function involving only R with respect to x, y, z is the negative of differentiation with respect to x', y', z'. Hence, in (37)

$$\text{grad}'\,\frac{e^{-j\beta_0 R}}{R} = -\text{grad}\,\frac{e^{-j\beta_0 R}}{R} \tag{38}$$

The scalar product $(\overline{\varrho_m \mathbf{v}'}, \text{grad}'\, e^{-j\beta_0 R}/R)$ may be expanded using (36) solved for its third term and with all operators primed. Thus

$$\left(\overline{\varrho_m \mathbf{v}'}, \text{grad}'\,\frac{e^{-j\beta_0 R}}{R}\right) = \text{div}'\left\{\overline{\varrho_m \mathbf{v}'}\frac{e^{-j\beta_0 R}}{R}\right\} - \frac{e^{-j\beta_0 R}}{R}\,\text{div}'\overline{\varrho_m \mathbf{v}'} \tag{39}$$

Expressions like (37) and (39) are true with $\mathbf{n}$ written in place of ϱ.

Upon substituting (39) in (35) according to (37) and (38),

$$\text{div }\mathbf{A} = \frac{1}{4\pi\nu_0}\lim_{a\to 0}\left\{\int_{\tau_a}\frac{e^{-j\beta_0 R}}{R}\,\text{div}'\,\overline{\varrho_m \mathbf{v}'}d\tau'\right.$$
$$\left.-\int_{\tau_a}\text{div}'\left(\overline{\varrho_m \mathbf{v}'}\frac{e^{-j\beta_0 R}}{R}\right)d\tau'\right\} + \frac{1}{4\pi\nu_0}\left\{\int_{\Sigma}\frac{e^{-j\beta_0 R}}{R}\,\text{div}'\,\overline{\mathbf{n}_m \mathbf{v}'}d\sigma'\right.$$
$$\left.-\int_{\Sigma}\text{div}'\left(\overline{\mathbf{n}_m \mathbf{v}'}\frac{e^{-j\beta_0 R}}{R}\right)d\sigma'\right\} \tag{40}$$

The divergence theorem (II.6.1) may be applied to the second integral in (40) if $\overline{\varrho_m \mathbf{v}'}$ is continuous in τ_a. In this case,

$$\lim_{a\to 0}\int_{\tau_a}\text{div}'\left(\overline{\varrho_m \mathbf{v}'}\frac{e^{-j\beta_0 R}}{R}\right)d\tau'$$
$$= \int_{\Sigma_r}(\hat{n},\overline{\varrho_m \mathbf{v}'})\frac{e^{-j\beta_0 R}}{R}\,d\sigma' + \lim_{a\to 0}\int_{\Sigma_a}(\hat{n},\overline{\varrho_m \mathbf{v}'})\frac{e^{-j\beta_0 R}}{R}\,d\sigma' \tag{41}$$

Here Σ_r is the external surface of the volume τ_a; Σ_a is the surface of the sphere of radius a enclosing the point P. If the boundary Σ_r is *chosen completely outside* the volume where $\overline{\varrho_m \mathbf{v}'}$ is non-vanishing, so that $\overline{\varrho_m \mathbf{v}'}$ *is zero at all points on* Σ_r, the first integral

on the right in (41) vanishes. In the second integral in (41)
$R = a$, $d\sigma' = a^2 d\Omega$ where $d\Omega$ is an element of solid angle. Since
an uncanceled a remains in the numerator, the integral vanishes
in the limit as a is made equal to zero. Therefore, the entire
second integral in (40) vanishes if $\overline{\varrho_m \mathbf{v}'}$ is continuous within Σ_r
and vanishes on Σ_r. If $\overline{\varrho_m \mathbf{v}'}$ is not continuous across certain
boundaries within Σ_r, these boundaries must be enclosed in sur-
faces Σ. Integrals of the form

$$\int_\Sigma \{(\hbar_1, \overline{\varrho_m \mathbf{v}'_1}) + (\hbar_2, \overline{\varrho_m \mathbf{v}'_2})\} \frac{e^{-j\beta_0 R}}{R} d\sigma' \qquad (42)$$

remain. The shorthand

$$(\hbar, \overline{\varrho_m \mathbf{v}'}) \equiv (\hbar_1, \overline{\varrho_m \mathbf{v}'_1}) + (\hbar_2, \overline{\varrho_m \mathbf{v}'_2}) \qquad (43)$$

may be used as in Sec. I.24.

The last integral in (40) may be shown to be zero using the
divergence theorem in two-dimensional form since $\overline{n_m \mathbf{v}'}$ is con-
fined to thin boundary layers of thickness 2δ.

$$\int_\Sigma \operatorname{div}' \left(\overline{n_m \mathbf{v}'} \frac{e^{-j\beta_0 R}}{R} \right) d\sigma' = \int_s (\hbar, \overline{n_m \mathbf{v}'}) \frac{e^{-j\beta_0 R}}{R} ds \qquad (44)$$

Here s is the enclosing edge of the surface Σ along which the
flow characterized by $\overline{n_m \mathbf{v}'}$ must exist. Since a surface current
can exist only in a surface layer, it follows that either Σ is a
closed surface or $\overline{n_m \mathbf{v}'}$ must vanish at its edge. In either case,
the integral in (44) is zero.

It is now possible to verify (3) using (33) and (34) as modified
by the above discussion. Thus, with $\dfrac{\beta_0^2}{\omega} \dfrac{1}{4\pi\epsilon_0} = \dfrac{\omega}{4\pi\nu_0}$,

$$\operatorname{div} \mathbf{A} + j\frac{\beta_0^2}{\omega}\phi = \frac{1}{4\pi\nu_0} \lim_{a \to 0} \int_{\tau_a} \frac{e^{-j\beta_0 R}}{R} \{\operatorname{div}' \overline{\varrho_m \mathbf{v}'} + j\omega\vartheta\} d\tau'$$

$$+ \frac{1}{4\pi\nu_0} \int_\Sigma \frac{e^{-j\beta_0 R}}{R} \{\operatorname{div}' \overline{n_m \mathbf{v}'} + j\omega\bar{n}' - (\hbar, \overline{\varrho_m \mathbf{v}'})\} d\sigma' \qquad (45)$$

The generalized equations of continuity written for primed
variables are given in (III.13.18a) and (III.13.18b). They are for
the interior

$$\operatorname{div}' \overline{\varrho_m \mathbf{v}'} + j\omega\vartheta' = 0 \qquad (46)$$

for a boundary

$$\operatorname{div}' \overline{n_m v'} + j\omega \overline{n'} - (\overline{n, \rho_m v'}) = 0 \qquad (47)$$

Since (46) and (47) express the conservation of electric charge, a fundamental postulate, they are always true so that

$$\operatorname{div} \mathbf{A} + j \frac{\beta_0^2}{\omega} \phi = 0 \qquad (48)$$

with ϕ and $\mathbf{A}$ given by (33) and (34). The only condition that has been imposed in proving that the integrals (33) and (34), which are solutions of (1) and (2), satisfy (3) or (48) is that the surface Σ_r enclosing the volume containing the periodically varying charges and currents contributing to ϕ and $\mathbf{A}$ *must not be crossed by any moving charges.* That is, everywhere on Σ_r, $\overline{\rho_m v}$ must be zero. This means that the integrations in (33) and (34) must be carried out over an electrically complete and self-contained system if they are to give correct solutions of (1) and (2). An integration extended over a part of a complete system as, for example, over a section of a current-carrying conductor, does not yield a meaningful solution because the surface Σ_r cuts the conductor. The potentials so determined cannot satisfy (3). The conclusion to be drawn from this important fact is of fundamental importance in the study of problems in electromagnetism. It is this: *scalar and vector potential fields are properties of electrically complete systems; they cannot be assigned or parceled out to individual parts of such a system.* The potentials calculated from currents and charges in a *part* of a circuit are not independently significant quantities. In certain special cases, it is convenient to carry out an integration over a complete circuit in parts and to make use of these individually; but it must be remembered that they are actually meaningful only when all parts are combined unless it can be shown that some of the contributions are negligible and the others independent of each other to the desired degree of approximation. A case of this sort is considered in Chapter VI. In general, and unless proved otherwise, it must be assumed that potentials and electromagnetic vectors which may be computed from them are properties of complete systems and not of individual parts.

It was pointed out in Sec. I.26 that a part of a circuit may be made electrically self-contained by *the addition* of equal

and opposite fictitious layers of such surface charge on each side
of the mathematical boundaries between the parts that are to be
treated separately. These modified parts are not individually
equivalent to the originals, though the circuit as a whole may
continue to be. It is in this way possible to have no current
cross the boundaries of the parts so that the potentials calculated
for each part satisfy (1) to (3). For some purposes this artifice
is convenient, but it must be remembered that the parts of the
complete circuit for which calculations are made are *not* the
same as in the actual circuit, so that no transfer of properties
of the parts of one to the parts of the other is legitimate. Only
when the parts are put together and the fictitious layers of equal
and opposite charge cancel is the original condition restored.
For the circuit as a whole, it is immaterial which representation
is used. In other words, this artifice makes possible the analysis
of parts of a modified circuit that is identical with the original
as a whole but not in its parts. It does not make possible the
analysis of parts of the original circuit. It is not correct to
state that potentials which are correctly calculated for the parts
of the modified circuit with boundaries that are not crossed by
currents *are* the potentials for the corresponding parts of the
actual circuit in which the boundaries of the parts are crossed by
current. Whereas the potentials of the former have a meaning,
those of the latter are not and cannot be made meaningful.
Obviously this is equally true whether the modified circuit is
analyzed in terms of densities of current and layers of surface
charge at the boundaries between parts or in terms of an equiva-
lent polarization as explained in Sec. I.26.

The preceding considerations may be summarized as follows:

Theorem of Continuity: Potentials may be calculated from the
Helmholtz integrals (33) and (34) and from any forms derived
from them only if the boundaries of the region of integration
are not crossed by moving charge. Potentials that are calculated
for parts of circuits bounded by surfaces that are crossed by
currents do not satisfy the equation of continuity for potential
functions (3) and are mathematically and physically meaningless.
If this is true in terms of one set of density functions, it is true
for all *equivalent* sets of density functions, where by equivalence
is meant equivalence of the parts as well as of the whole. The
potential of a complete system cannot be assigned or ascribed

piecewise to parts of the system that are enclosed by surfaces crossed by currents.

The equations and integral solutions of the stationary state are obtained from (1), (2) and (33), (34) by setting ω (and hence β_0) equal to zero and interpreting all potential and density functions to be real constants instead of complex amplitudes.

$$\nabla^2 \phi = -\frac{\bar{p}}{\epsilon_0} \tag{49}$$

$$\nabla^2 \mathsf{A} = -\frac{\mathsf{I}}{\nu_0} \tag{50}$$

$$\phi = \frac{1}{4\pi\epsilon_0} \left\{ \int_\tau \frac{\bar{p}'}{R} d\tau' + \int_\Sigma \frac{\bar{\eta}'}{R} d\sigma' \right\} \tag{51}$$

$$\mathsf{A} = \frac{1}{4\pi\nu_0} \left\{ \int_\tau \frac{\mathsf{I}'}{R} d\tau' + \int_\Sigma \frac{\mathsf{I}'}{R} d\sigma' \right\} \tag{52}$$

Two important special cases of Helmholtz's integrals depend, the one upon limiting the magnitude of the product ωR, the other upon restricting the shape of the volume τ in which the charge is moving. They are considered in turn.

2. Helmholtz's Integrals for the Quasi-stationary State.—The angular frequency ω occurs in the Helmholtz integrals in the constant $\beta_0 = \omega/v_0$ where $v_0 = \sqrt{\nu_0/\epsilon_0} = 3 \times 10^8$ meters/second. Since v_0 is extremely large, it is clear that for a wide range of frequencies $f = \omega/2\pi$ of the oscillating charges and of distances R between them and the point where the potentials are computed the product

$$\beta_0 R = \frac{\omega R}{v_0} = \frac{\omega R}{3 \times 10^8} \qquad \text{(radians)} \tag{1}$$

is very small. If $e^{-j\beta_0 R}$ is expanded in the converging power series

$$e^{-j\beta_0 R} = 1 - j\beta_0 R - \frac{(\beta_0 R)^2}{2!} + \frac{j(\beta_0 R)^3}{3!} + \cdots \tag{2}$$

subject to the inequality

$$\beta_0 R << 1 \tag{3}$$

the exponential reduces approximately to unity. Thus

$$e^{-j\beta_0 R} \doteq 1 \tag{4}$$

If (3) is satisfied, the Helmholtz potentials (1.33), (1.34) reduce

to the forms (1.51), (1.52) of the stationary state, but with density functions and potentials complex amplitudes, not real constants. It follows that an approximate solution of the nonstationary state with periodic time dependence may be obtained from the solutions of the stationary state provided (3) is satisfied. Accordingly, the stationary-state model may be extended to include periodic time dependence subject to this condition. This extended stationary state is called the *quasi-stationary state*. It is formally like the stationary state but with densities and field vectors complex amplitudes and not constants.

$$\nabla^2\phi = -\frac{\bar{\varrho}}{\epsilon_0} \tag{5}$$

$$\nabla^2\mathbf{A} = -\frac{\bar{\imath}}{\nu_0} \tag{6}$$

$$\phi = \frac{1}{4\pi\epsilon_0}\left\{\int_\tau \frac{\bar{\varrho}'}{R}\,d\tau' + \int_\Sigma \frac{\bar{n}'}{R}\,d\sigma'\right\} \tag{7}$$

$$\mathbf{A} = \frac{1}{4\pi\nu_0}\left\{\int_\tau \frac{\mathbf{I}'}{R}\,d\tau' + \int_\Sigma \frac{\mathbf{l}'}{R}\,d\sigma'\right\} \tag{8}$$

The inequality (3) may be made specific by writing $\beta_0 = \omega/v_0$. Thus

$$\omega R << 3 \times 10^8 \quad \text{(meters/second)} \tag{9}$$

For an error not greater than 1 per cent

$$\omega R \le 3 \times 10^6 \quad \text{(meters/second)} \tag{10}$$

Unless otherwise indicated it will be assumed that the equations and solutions of the quasi-stationary state are sufficiently accurate for use with periodic time variations whenever (10) is satisfied.

3. Potentials Due to a Distribution of Current and Charge in a Cylindrical Conductor.—The second important special case of the Helmholtz integrals is obtained when the densities of charge and current are nonvanishing only in an isolated, rotationally symmetrical cylinder of length $2h$ and of small radius a in space. The specific conditions are

$$a << R \tag{1}$$
$$\beta_0 a <<< 1 \tag{2}$$

As a result of (1) and (2) it may be assumed that R varies a

negligible amount in carrying out the integration over the cross section $S = \pi a^2$ or around the circumference $s = 2\pi a$ of the cylindrical conductor. The Helmholtz integrals (1.33) and (1.34) may be written as follows for a cylinder with axis along the z axis of a system of coordinates:

$$\phi \doteq \frac{1}{4\pi\epsilon_0} \int_{-h}^{h} \frac{e^{-j\beta_0 R}}{R} \left\{ \int_S \varrho' dS' + \int_s \pi' ds' \right\} dz' \qquad (3)$$

$$\mathbf{A} \doteq \frac{1}{4\pi\nu_0} \int_{-h}^{h} \frac{e^{-j\beta_0 R}}{R} \left\{ \int_S \overline{\varrho_m \mathbf{v}'} dS' + \int_s \overline{n_m \mathbf{v}'} ds' \right\} dz' \qquad (4)$$

where now R is measured to the element dz' *on the axis*. (The notation R_a will be used for distances measured to the axis if distances measured to an element $d\tau'$ also occur.) The first integral in each case is over the length $2h$, the second integral over the cross section πa^2, the third integral around the circumference $2\pi a$ of the conductor.

Before carrying out the integration in (4), the vector $\mathbf{A}$ must be expressed in terms of its three components in cylindrical coordinates r, θ, z.

$$\mathbf{A} = \hat{r}A_r + \hat{\theta}A_\theta + \hat{z}A_z \qquad (5)$$

It is assumed that driving forces are so arranged that

$$\overline{\varrho_m v_\theta} = \overline{n_m v_\theta} = 0 \qquad (6)$$

Since the conductor is in empty space

$$\overline{n_m v_r} = 0 \qquad (7)$$

Hence

$$A_r \doteq \frac{1}{4\pi\nu_0} \int_{-h}^{h} \frac{e^{-j\beta_0 R}}{R} \left\{ \int_S \overline{\varrho_m v_r'} dS' \right\} dz' \qquad (8)$$

$$A_\theta = 0 \qquad (9)$$

$$A_z \doteq \frac{1}{4\pi\nu_0} \int_{-h}^{h} \frac{e^{-j\beta_0 R}}{R} \left\{ \int_S \overline{\varrho_m v_z} dS' + \int_s \overline{n_m v_z} ds' \right\} dz' \qquad (10)$$

Since rotational symmetry is postulated, $\overline{\varrho_m v_r}$ is the same in all radial directions in a given cross section. By pairing elements of the cross section in which equal currents are in opposite directions, it follows that the integral over the cross section in (8) vanishes, and hence

$$A_r \doteq 0 \qquad (11)$$

Thus A_s is the only significant component of $\mathbf{A}$ at distant points. Let

$$q' \equiv \int_S \bar{\varrho}' dS' + \int_s \bar{n}' ds' \tag{12}$$

$$I'_s = \int_S \overline{\varrho_m v'_s} dS' + \int_s \overline{n_m v'_s} ds' \tag{13}$$

q' is the complex amplitude of the total charge per unit length of the conductor.   Since $\bar{\varrho} = 0$ and $\mathbf{P} = 0$ in conductors and the surface charge is rotationally symmetrical

$$q' = \int_{s=2\pi a} n'_f ds' = 2\pi a n'_f \qquad \cdot \tag{14}$$

I'_s is the complex amplitude of the total axial current.   Since $\mathbf{P} = 0$ and from the curl theorem (II.6.7)

$$\iiint \operatorname{curl}_s \mathbf{M} \, dS' dz' = \iint [\hbar, \mathbf{M}]_s ds' dz' \tag{15}$$

(13) reduces to

$$I'_s = \int i'_{fs} dS' + \int l'_{fs} ds' \tag{16}$$

In good conductors the second integral is omitted; in perfect conductors the first integral is zero.

With (5) to (16), the integrals (3) and (4) become

$$\phi = \frac{1}{4\pi\epsilon_0} \int_{-h}^{h} q' \frac{e^{-j\beta_0 R}}{R} \, dz' \tag{17}$$

$$\mathbf{A} = \frac{\hat{z}}{4\pi\nu_0} \int_{-h}^{h} I'_s \frac{e^{-j\beta_0 R}}{R} \, dz' \tag{18}$$

These are the fundamentally important Helmholtz integrals for a cylindrical conductor with periodically varying axial currents. In the quasi-stationary state they reduce to

$$\phi = \frac{1}{4\pi\epsilon_0} \int_{-h}^{h} \frac{q'}{R} \, dz' \tag{19}$$

$$\mathbf{A} = \frac{\hat{z}}{4\pi\nu_0} \int_{-h}^{h} \frac{I'_s}{R} \, dz' \tag{20}$$

In the stationary states

$$\phi = \frac{1}{4\pi\epsilon_0} \int_{-h}^{+h} \frac{q'}{R} \, dz' \tag{21}$$

$$\mathbf{A} = \frac{\hat{z}}{4\pi\nu_0} \int_{-h}^{h} \frac{I'_s}{R} \, dz' \tag{22}$$

4. Potentials Near a Cylindrical Conductor.—The general formulas (3.17) and (3.18) are restricted by (3.1) and (3.2). It can be shown that (3.1) is more severe than necessary, that the formulas are good approximations even at the surface $r = a$,
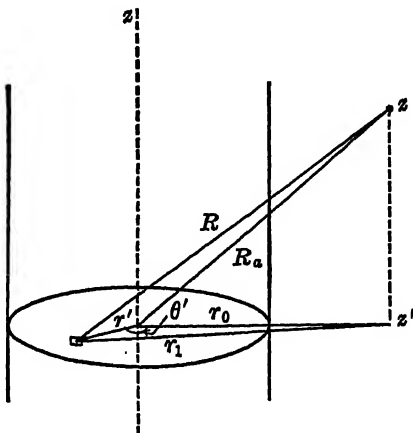

Fig. 4.1.—Cross section of a cylindrical conductor.

if R is measured to the axis of the antenna. The proof is carried out using the notation of Fig. 4.1 where

$$R = \sqrt{(z - z')^2 + r_1^2} = \sqrt{R_a^2 + r'^2 - 2r'r_0 \cos \theta'} \qquad (1)$$

with

$$R_a = \sqrt{(z - z')^2 + r_0^2} \qquad (2)$$

$$r_1 = \sqrt{r'^2 + r_0^2 - 2r'r_0 \cos \theta'} \qquad (3)$$

The proof depends upon demonstrating that the differences between the general integrals

$$\phi = \frac{1}{4\pi\epsilon_0} \left\{ \int_\tau \frac{\varrho'}{R} e^{-j\beta_0 R} d\tau' + \int_\Sigma \frac{\hat{n}'}{R} e^{-j\beta_0 R} d\sigma' \right\} \qquad (4)$$

$$A_s = \frac{1}{4\pi\nu_0} \left\{ \int_\tau \frac{\overline{\varrho_m v'_z}}{R} e^{-j\beta_0 R} d\tau' + \int_\Sigma \frac{\overline{n_m v'_z}}{R} e^{-j\beta_0 R} d\sigma' \right\} \qquad (5)$$

and (3.3), (3.4) is negligible. The differences are

$$D_\phi = \frac{1}{4\pi\epsilon_0} \left\{ \int_{-h}^h dz' \left[\int_0^a \varrho' r' dr' + \hat{n}'a \right] \int_0^{2\pi} \left[\frac{e^{-j\beta_0 R}}{R} - \frac{e^{-j\beta_0 R_a}}{R_a} \right] d\theta' \right\} \qquad (6)$$

$$D_A = \frac{1}{4\pi\nu_0}\left\{\int_{-h}^{h}dz'\left[\int_0^a \overline{\varrho_m v_z'}r'dr' + \overline{n_m v_z'}a\right]\right.$$
$$\left.\int_0^{2\pi}\left[\frac{e^{-j\beta_0 R}}{R} - \frac{e^{-j\beta_0 R_a}}{R_a}\right]d\theta'\right\}\quad (7)$$

Rotational symmetry has, of course, been assumed so that all density functions are independent of θ'. It follows from (1) that R can differ from R_a by magnitudes that in no case exceed the radius a of the conductor. If R is large compared with a, R and R_a differ by a negligible amount as assumed in deriving (3.17) and (3.18). Accordingly, significant contributions to D_ϕ and D_A can come only from that part of the integration with respect to z' for which $|z - z'|$ is not large compared with a and that part of the integration with respect to r' in which r' is not large compared with a. That is, D_ϕ and D_A are always negligible if $|z - z'| >> a$ or if $r_0 >> a$. Since the integration over distant parts of the conductor contributes nothing of significance to D_ϕ or D_A, the cylinder may be assumed to be infinitely long in the integrations in (6) and (7). Since only that part of the integration which involves distances of order of magnitude comparable with small multiples of a contributes significantly to D_ϕ and D_A, it is possible, with $\beta_0 a <<< 1$, to treat $\beta_0 R$ and $\beta_0 R_a$ as small compared with unity in (6) and (7). The parts of the integration where $\beta_0 R << 1$ and $\beta_0 R_a << 1$ are good approximations contribute significantly, the parts where R and R_a are large contribute negligibly in (6) and (7). If it is assumed that all density functions vary axially (*i.e.*, with respect to z') so slowly that they are sensibly constant over distances comparable with small multiples of the radius a (as is actually verified), they may be removed from under the signs of integration with respect to z' in (6) and (7). They are sensibly constant with respect to z' over that part of the integration which can contribute significantly to D_ϕ and D_A, and whether they are constant beyond that range or not is immaterial.

Taking advantage of the simplifications permitted by the several approximations justified above

$$D_\phi = \frac{1}{4\pi\epsilon_0}\left[\int_0^a \varrho'r'dr' + \bar{n}'a\right]\int_0^{2\pi}d\theta'\int_{-\infty}^{+\infty}\left(\frac{1}{R} - \frac{1}{R_a}\right)dz' \quad (8)$$

$$D_A = \frac{1}{4\pi\nu_0}\left[\int_0^a \overline{\varrho_m v_z'}r'dr' + \overline{n_m v_z'}a\right]\int_0^{2\pi}d\theta'\int_{-\infty}^{+\infty}\left(\frac{1}{R} - \frac{1}{R_a}\right)dz' \quad (9)$$

Using standard methods

$$\int_{-\infty}^{\infty} \left(\frac{1}{R} - \frac{1}{R_a}\right) dz' = \ln \left(\frac{r_0}{r_1}\right)^2 \tag{10}$$

$$\int_0^{2\pi} \ln \left(\frac{r_0}{r_1}\right) d\theta' = 0 \tag{11}$$

Hence D_ϕ and D_A both reduce to zero subject to the several approximations in passing from (6) and (7) to (8) and (9). These are reasonable except within distances of the ends of the conductor at $z = \pm h$ which are comparable with small multiples of a. It follows that (3.17) and (3.18) are good approximations even at the surface of the cylinder except perhaps near the ends. But at one end, $z = h$ for example, the upper limit in (10) is zero and the integral is $\ln(r_0/r_1)$ so that (11) is still true. Therefore (3.17) and (3.18) are true at the ends as well, and it is reasonable to assume them to be valid at all points on the cylindrical surface.

5. Waves of Potential.—The Helmholtz integrals for ϕ and A given in (1.33) and (1.34) are solutions of the equations (1.1) and (1.2) for the amplitudes of harmonic functions written in the complex form

$$\phi_{\text{inst}} = \phi e^{j\omega t}; \qquad A_{\text{inst}} = A e^{j\omega t} \tag{1}$$

The complex instantaneous values of ϕ or A are obtained from the Helmholtz integrals defining their complex amplitudes by multiplying these by the exponential time factor $e^{j\omega t}$. The complex instantaneous value of ϕ is

$$\phi_{\text{inst}} = \phi e^{j\omega t} = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\bar{\rho}'}{R} e^{j(\omega t - \beta_0 R)} d\tau' \tag{2}$$

A similar relation may be written for the vector potential A. It is now possible to select either the real part or the imaginary part as a solution that satisfies the general potential equation. If the real part is chosen, corresponding to $\bar{\rho}'_{\text{inst}} = \bar{\rho}' \cos \omega t$,

$$\phi_{\text{inst}} = \phi \cos (\omega t + \theta_\phi) = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{\bar{\rho}'}{R} \cos (\omega t - \beta_0 R) d\tau' \tag{3}$$

An interesting interpretation of the integral (2) may be given if the exponential factor in the integrand is associated

directly with the amplitude of the volume density ϱ'. Let it be assumed that the expression

$$\{\bar{\varrho}'_{\text{inst}}\} = \varrho' c^{j(\omega t - \beta_0 R)} \tag{4}$$

defines the complex instantaneous value of the essential charge density that is associated with the complex instantaneous value of the scalar potential defined in (1). Since $\beta_0 = \omega/v_0$, (4) may be written in the alternative form

$$\{\bar{\varrho}'_{\text{inst}}\} = \varrho' c^{j\omega(t - R/r_0)} = \varrho' c^{j\omega\{t\}} \tag{5}$$

It thus appears that in order to calculate ϕ_{inst} at a point P at the instant t it is necessary to use in the integrand the value of ϱ' characterizing the point P', not at the same instant t, but at the earlier one $\{t\}$, given by

$$\{t\} = t - \frac{R}{v_0} \tag{6}$$

In order to distinguish between the earlier times (and quantities evaluated at them) and the time for which ϕ_{inst} is computed, the former are written in braces as above. In (5) and (6), r is the distance between the point P where ϕ is calculated and the point P' locating the element of integration $d\tau'$. Instead of stating that the instantaneous value of the charge density must be taken at an earlier time than that for which ϕ is to be calculated, the point of view may be reversed by stating that the potential ϕ_{inst} is calculated at a later time compared with that at which the charge is taken for substitution in (2). The potential ϕ_{inst} is called a *retarded potential*, and the integral (2) expresses a retarded or delayed action at a distance for the electromagnetic forces which may be calculated from ϕ_{inst}.

There is nothing peculiar or mysterious about a retarded potential. In the form (2) it is simply a rule for calculating a mathematical function of the time and of position in terms of a supposedly known distribution of density functions. In applying the retarded potentials to the solution of circuit problems, such a rule is all that is needed.

However, it must not be overlooked that it is often both helpful and stimulating for the imagination, especially in dealing with abstract and physically meaningless variables, to construct a picture or a geometrical analogue to describe or interpret the

concise symbolism of a mathematical rule.   No one familiar
with basic arithmetic finds it necessary or even particularly
helpful to visualize a simple addition of abstract numbers in
terms of separately counted piles of pebbles that are moved
together into a single large one and then counted again.   On
the other hand, many find it advantageous in dealing with the
problem of a loop of wire rotating in a magnetic field to interpret
the mathematical operations in terms of a picture of the wire
"cutting" a definite number of imaginary magnetic "lines of
flux" at a certain rate of speed.   In the case at hand, the task
of thinking in terms of a potential function, which is the com-
posite of a retarded integral evaluated at every point in space,
requires a greater intellectual effort than does the mathematical
formulation of a magnetic field varying with respect to a closed
circuit of conductors.   This is true in part because the potential
function is itself only an intermediate step in a complicated
mathematical sequence for relating, for example, the deflection
of an ammeter in one antenna to that in another.   Therefore,
it is of no mean value if a geometrical analogue or picture (to
correspond to the piles of pebbles in the example in addition, or
the "cutting of magnetic lines" in the circuit problem) can be
devised as a simple aid in visualizing the composite summation
of predated density functions.   Because the same differential
equation, the Helmholtz equation, characterizes important parts
of the mathematical models of hydrodynamics, acoustics, and
elastic phenomena, such a geometrical picture is immediately
available.

     For example, it is merely necessary to endow the dependent
variable and the constant parameters occurring in the electro-
dynamical equation for potentials with the properties of the
corresponding variable and constant parameters in the elastic
equation.   Since these latter are closely related to quantities
that have direct mechanical analogues, *viz.*, stresses and
strains in an elastic medium, it is quite a simple mental gym-
nastic to "visualize" the electromagnetic problem by effectively
identifying it with a formally analogous elastic problem.   It
is then entirely natural to transfer the entire terminology
belonging to the stress and strain model to the corresponding
electrodynamical symbols.   Next, it can be argued that it is
manifestly absurd to speak of stresses and strains in sheer

emptiness, so that one is "compelled" to assume that all space is filled with a strange medium that has both the properties of an elastic solid like steel and those of a vacuum. The name given to this physically impossible medium is the ether, and the geo-metrical-mechanical analogue that has been described is nothing other than the ether model of space.

The fact that two entirely independent quantities satisfy the same differential equation makes it inevitable that the mathematical properties of the two quantities will be the same, provided the boundary and initial conditions are alike. But a correspondence between the mathematical properties of two physically unrelated functions does not require, or even make it likely, that a model that is sensible for one function will be correspondingly sensible for the other. It may, in fact, be quite absurd. This does not necessarily make it useless for purposes of visualizing a complicated mathematical operation or for stimulating the imagination. But it must not be forgotten that while using a physically absurd model one is with Alice in the land of the Red Queen. Such is the case when one pictures electromagnetic phenomena in terms of stresses and strains in the ether. Since it is actually only a question of devising a scheme for making a complicated mathematical mechanism more of a game and less of a headache, there seems to be no real reason why any physical significance whatever need be attached to such a model. This point of view takes for granted, of course, that the thesis has been accepted which denies the necessity of provid-ing a physical analogue for every symbol in a mathematical model. It is not at all difficult to describe a readily visualized picture that serves precisely the same purpose as the ether model without mention of stresses or strains. To be sure, it does not have the elaborate (and impossible) physical properties associated with the ether, but since its sole purpose is to aid in under-standing certain mathematical operations, there is no reason why it should have. In fact, a definite advantage is gained by removing the temptation of inventing properties to fit a borrowed model. For the benefit of those who find the concise, mathe-matical rule contained in the retarded potential integral some-what obscure, and perhaps lacking in that color associated with physical pictures, the following geometrical model is offered. It has no experimental counterpart, since it describes a mathe-

matical function, the scalar potential, which has no pointer-reading analogue.

In order to make the model as simple as possible at the outset let it be assumed that the only region in which nonvanishing values of the density functions can be defined consists of a cylindrical conductor of small radius in space. The Helmholtz integral for the complex amplitude ϕ of the scalar potential is given in (3.17). It is

$$\phi = \frac{1}{4\pi\epsilon_0} \int_{-h}^{+h} q' \frac{e^{-j\beta_0 R}}{R} \, ds' \tag{7}$$

The real, instantaneous potential is calculated from

$$\phi_{\text{inst}} = \text{real part } \phi e^{j\omega t} = \frac{1}{4\pi\epsilon_0} \int_{-h}^{h} \frac{q' \cos(\omega t - \beta_0 R)}{R} \, ds' \tag{8}$$

The arbitrary angle θ_q in $q = qe^{j\theta_q}$ has been chosen zero in (8) so that phases are referred to q, and $q = q$ is real. The contribution to the potential ϕ due to the charges in a single element ds' of the cylindrical conductor is $d\phi$. It is given by

$$d\phi = \frac{q' ds'}{4\pi\epsilon_0} \frac{\cos(\omega t - \beta_0 R)}{R} \tag{9}$$

In this expression, $q' ds'$ is the amplitude of the total charge contained in the element ds'; R is the distance from the point P where $d\phi$ is calculated to the center of ds' at P'. The dimensions of the element ds' are assumed to be negligibly small compared with the distance R. Evidently the factor $(q' ds')$ is constant. The factor $(1/R)$ indicates that the magnitude of $d\phi$ diminishes as the reciprocal of the distance between the element ds' and the point P. The factor $\cos(\omega t - \beta_0 R)$ is a trigonometric function that varies periodically between the values $+1$ and -1 as the argument or phase $(\omega t - \beta_0 R)$ varies between odd and even multiples of π. From these statements it is clear that the values of $d\phi$ calculated at a *single instant* for points on a spherical shell of *fixed radius* drawn about the point P' are all the same. The computed value is not the same at different times, but at each instant it is the same for the entire spherical shell. Therefore such a shell is an *equipotential surface*. Because each shell has a constant radius, the argument or phase $(\omega t - \beta_0 R)$ has the same instantaneous value simultaneously at all points on the

spherical shell. Therefore, the shell is an *equiphase surface* as well as an equipotential surface.

Two questions now arise. Is there an infinite group of spherical shells all members of which simultaneously have the same potential? Is there any group of shells of which all members are *simultaneously* characterized by the same phase? The answer to the first question is simple. Owing to the factor $1/R$ in (9), it follows directly that the only potential common to an infinite group is identically zero. Accordingly, there is no infinite group of shells all members of which simultaneously have the same nonvanishing potential. Each spherical shell is itself an equipotential surface, but no two shells are alike; their potential amplitudes diminish continuously as their radii increase.

The answer to the second question also is obtained easily. From the trigonometric relation

$$\cos (\omega t - \beta_0 R) = \cos (\omega t - \beta_0 R - 2n\pi);$$
$$(n = 0,1,2 \cdots) \quad (10)$$

it may be concluded that all surfaces which are characterized by the following discrete set of arguments are in the same phase. The set of arguments is

$$\psi_n \equiv \omega t - \beta_0 R - 2n\pi; \qquad (n = 0,1,2 \cdots) \quad (11)$$

Let a time t_n and a radius R_n be defined as follows:

$$\psi_n = \omega t_n - \beta_0 R_n = \omega t - \beta_0 R - 2n\pi \quad (12)$$

By rearranging terms and noting that by definition $\beta_0 = \omega/v_0$, $\omega = 2\pi f$,

$$(t_n - t) = \frac{1}{v_0} (R_n - R) - \frac{n}{f} \quad (13)$$

In this relation $(R_n - R)$ is the radial distance between two spherical shells that have the same phase, respectively, at times t_n and t. The most important special case is that in which two surfaces of radii R_n and R are characterized by the *same phase* at the *same time*. In this case

$$t_n = t \quad (14)$$

and

$$(R_n - R) = \frac{n v_0}{f}; \qquad (n = 0,1,2 \cdots) \quad (15)$$

This equation defines a family of n spherical surfaces that are characterized simultaneously by the same phase. The minimum distance between two surfaces in the same phase is obtained by setting $n = 1$. The distance so defined is assigned the special symbol λ_0. It is

$$\lambda_0 \equiv R_1 - R = \frac{v_0}{f} \qquad (16)$$

Here $v_0 = 3 \times 10^8$ meters/second is a constant characteristic of free space. It follows that λ_0 varies inversely as the frequency f of the periodically changing charge density in the element ds'. Clearly

$$R_2 - R = \frac{2v_0}{f} = 2\lambda_0 \qquad (17a)$$

$$R_3 - R = \frac{3v_0}{f} = 3\lambda_0, \text{ etc.} \qquad (17b)$$

Hence,

$$R_1 - R = R_2 - R_1 = R_3 - R_2 = \cdots = R_n - R_{n-1} \cdots$$
$$= \lambda_0 \qquad (18)$$

It thus appears that all surfaces which *simultaneously* have the *same phase* differ successively in radius by an amount λ_0. In other words, all points on all spherical surfaces that differ in radius by integral multiples of the characteristic distance λ_0 simultaneously have the same phase. In this way it is possible to distinguish an infinite family of spherical shells of radii $R, R_1, R_2, R_3, \cdots R_n \cdots$ that satisfy the relation (18) and have a given instantaneous phase ψ_1. Since the phase ψ can vary continuously from zero to 2π, there are infinitely many such families continuously distributed between the successive members of any one. A cross section of the values of $d\phi$ associated with the infinite sequence of spherical shells extending from the immediate vicinity of the charge $q'ds'$ to any desired distance may be obtained by determining these values along a single radial line at a given instant. Since $d\phi$ is the same at every point on a spherical shell of fixed radius, it is evidently only necessary to know $d\phi$ for a single point, such as the intersection of a radial line with the shell. The potential at every point along this radial line at the instant $t = 0$ is given by

$$d\phi = \frac{q'ds'}{4\pi\epsilon_0} \frac{\cos \beta_0 R}{R} \qquad (19)$$

In this expression the factor $(\cos \beta_0 R)/R$ defines a cosine curve of amplitude that diminishes as $1/R$ with increasing R. All points along this curve that are separated by a distance λ_0, or an integral multiple of λ_0, belong to a family of equal phase.

The question now arises, what happens to a family of shells that is required to *remain permanently* at the same phase ψ as time passes? That is, a family of shells is to be selected for which

$$\psi = (\omega t - \beta_0 R) = \text{const.} \tag{20}$$

for all values of t and R. Since ω and β_0 are constants for any given frequency, there is no alternative but that R must change as time passes if ψ is to remain constant. Up to the present, attention has been focused on a family of shells with constant radii differing by $n\lambda_0$ which are all in the same phase at the same instant. This phase varies periodically in time. The point of view is now changed to study the behavior of a family of shells characterized by a *permanently constant phase* as defined in (20), but with radii that vary in time. This is equivalent to stating that the family of shells of equal and concurrent phase is moving. The nature of this motion can be determined by differentiating both sides of (20) with respect to the time. Since R is a function of t

$$\frac{d\psi}{dt} = \omega - \beta_0 \frac{dR}{dt} = 0 \tag{21}$$

so that

$$\frac{dR}{dt} = \frac{\omega}{\beta_0} = v_0 \tag{22}$$

It may be concluded that the entire family of shells of constant concurrent phase ψ expands uniformly at the constant radial speed $v_0 = 3 \times 10^8$ meters/second.

A disturbance in which a constant phase is propagated at a uniform speed is called *wave motion*. Each shell of constant phase is called a *wave front*. The characteristic distance λ_0 between successive shells belonging to the same family of which all members are simultaneously in the same phase is called a *wave length*. The Helmholtz equation from which this type of motion is derived is called a *wave equation*.

The complex scalar potential due to a small element of periodically varying charge is given by the integrand in the Helmholtz integral (2). A geometric interpretation or picture of the potential is a spherical wave train originating at the charge and

expanding with uniform radial velocity v_0. By a wave train is meant a continuous sequence of *mathematical shells* of which those at intervals of a constant distance λ_0 are always in the same phase. That is, the quantity $\psi = (\omega t - \beta_0 R)$ remains constant for them. Such a wave train for a single radial line may be visualized by imagining a cosine curve moved parallel to its axis with a velocity v_0 between two guides which have the form of $1/R$ and $-1/R$ curves and which restrict the amplitude.

It is important to bear in mind that the phase waves that govern the amplitude of the potential function ϕ at any point in space have not been endowed with any physical meaning whatsoever. Waves of potential are geometrical pictures of the retarded potential function. Their purpose is to rock no boat, to shake no eardrum, but simply to stimulate the imagination. With their aid, it is possible to compare conveniently phases or relative amplitudes of the scalar potential at different points in space without actually evaluating the Helmholtz integral for each of these points. Just as it is often more convenient to read the values of a trigonometric function from a curve rather than to compute them from an infinite series, so it is frequently advantageous to determine relative phases or amplitudes of the scalar potential at different points in terms of trains of waves emanating from the periodically varying charge. The picture of waves in determining the instantaneous values of a potential function plays much the same part as the picture of a particle rotating in a reference circle does in specifying the instantaneous amplitude of a vibrating mass.

The above discussion has been limited to the scalar potential $d\phi$ due to an element of periodically varying charge $q' ds'$. Since the Helmholtz integral for each rectangular component of the vector potential A is precisely like that for the scalar potential, it follows that the wave representation may be extended to include the element of vector potential dA due to an element of moving charge $I' \mathsf{ds}'$. The picture is in this case somewhat more complicated because the outwardly directed trains of spherical phase waves are characterized both by a magnitude and by a direction.

Each element ds' of the cylindrical thread containing moving charges may be looked upon as the origin of a spherical train of waves as described above. The complex amplitude of the

potential at each point in space due to all such elements in the thread is given by the integral (7). In terms of the wave picture, the instantaneous value of the potential is the resultant of the amplitudes of all the waves from the different elements that pass the point at the particular instant. It evidently depends upon the distribution of charge and of current along the thread. The resultant equiphase surfaces are not spherical. However, at distances R from the thread which are very large compared with its length $2h$ the factor R in the denominator of (7) remains sensibly constant in the integration, so that it may be removed from under the sign. When this is possible, the equiphase surfaces or wave fronts are approximately spherical. The greater the distance from the distributions of charge and current in the thread which contribute to the potential, the more nearly spherical the wave fronts will be.

The geometrical picture illustrating the Helmholtz integral (7) which has just been described involves trains of spherical equiphase surfaces or wave fronts expanding outward in space from each element of charge with a velocity v_0. The amplitude and phase of this train of waves at any instant and at any point is a measure of the contribution to the potential at the chosen instant and point due to the element of charge. The element of charge is said to be a source of diverging waves. An alternative and in some ways more useful interpretation of the integral inverts this picture. Each point P at which the potential is calculated is assumed to be the center of a *converging* train of spherical waves originating at an infinite distance and sweeping inward with a velocity v_0. Any regions in space in which charges vary periodically are imagined to be traversed by the converging spherical waves. The value of q'/R in the case of a thread, or more generally of $\bar{p}'/R$, obtaining anywhere in space, is swept inward by the wave and delivered at the point of convergence P where ϕ is to be calculated. The sum of these values due to all points in space delivered at P at each instant gives the value of the potential at P at that instant when multiplied by $1/4\pi\epsilon_0$. In this second picture the waves of potential are contracting spherical envelopes which continuously collect magnitudes $\bar{p}'/r$ and carry them with velocity v_0 to the point of convergence. From the sum of all the magnitudes that arrive at each instant, the potential at that instant may be determined.

In principle, both geometrical pictures are the same. Each serves to provide a more or less readily visualized mechanism as a substitute for or an interpretation of the mathematical operations defined by integrals like (7) and (2). The integrals can be interpreted directly as expressing a *fundamental law of retarded action at a distance* without such a mechanism. Nevertheless, the use of the wave picture to interpret the Helmholtz integrals is entirely consistent with a retarded action at a distance since no physical reality need be assigned to it. Waves of potential may exist in mathematical space and there serve as a useful means for visualizing an intricate potential field. However, it is contrary to the habits of traditional thought of many minds to accept an interpretation of the potential integrals in terms of action at a distance, whether retarded or instantaneous. Action by direct contact is still considered by many to be the approved way in which natural phenomena must be explained, in spite of the fact that the atomic model has completely eliminated such contacts in the ordinary mechanical sense. The wave picture is consistent with action by direct contact in a medium only if the waves of potential, or the electromagnetic waves derived from them, are shown to have physical significance in the form of direct experimental analogues. This requires a demonstration that space, or a complete vacuum, is a material medium in which a *physically real* wave motion can be propagated.

6. Waves of Potential in Simple Media.—The general integrals (1.33) and (1.34) for the scalar and vector potentials and any of the more special forms derived from them, in particular (3.17) and (3.18) when applied to electric circuits including antennas, are useful for computing the *potential fields at points in space* due to specific distributions of charge and current in bounded regions of any shape. If *unbounded* space is everywhere imagined replaced by a similarly unbounded homogeneous and isotropic simple medium containing the same distributions of charge and current in the same bounded regions, the potential field at points in the simple medium is readily obtained from the field in space by the expedient described at the end of Sec. III.14. This consisted of replacing ϵ_0 by ξ, ν_0 by ν, and β_0 by β just as was done in obtaining (1.23) and (1.25). If this is done, the general integrals (1.33) and (1.34) are replaced by

$$\phi = \frac{1}{4\pi\xi} \left\{ \int_\tau \frac{\varrho'}{R} \exp{(-j\beta R)} d\tau' + \int_\Sigma \frac{\mathfrak{n}'}{R} \exp{(-j\beta R)} d\sigma' \right\} \quad (1)$$

$$\mathbf{A} = \frac{1}{4\pi\nu} \left\{ \int_\tau \frac{\overline{\varrho_m \mathbf{v}'}}{R} \exp{(-j\beta R)} d\tau' + \int_\Sigma \frac{\overline{\mathfrak{n}_m \mathbf{v}'}}{R} \exp{(-j\beta R)} d\sigma' \right\} \quad (2)$$

The volume integrals in (1) and (2) are carried out over all regions where ϱ' and $\overline{\varrho_m \mathbf{v}'}$ are nonvanishing but *not including* charges and currents associated with the simple medium of which account is taken in ξ, ν, and β. The surface integrals are evaluated over all surfaces where $\mathfrak{n}'$ and $\overline{\mathfrak{n}_m \mathbf{v}'}$ are nonvanishing, but they do not include terms due to densities in the simple medium. Integrals corresponding to (3.17) and (3.18) for the potentials due to distributions of current and charge in a cylindrical conductor are

$$\phi = \frac{1}{4\pi\xi} \int_{-h}^{+h} q' \frac{\exp{(-j\beta R)}}{R} \, dz' \quad (3)$$

$$\mathbf{A} = \frac{\hat{z}}{4\pi\nu} \int_{-h}^{+h} I'_z \frac{\exp{(-j\beta R)}}{R} \, dz' \quad (4)$$

If the notation (III.15.1)

$$\beta = \beta_s - j\alpha_s \quad (5)$$

is introduced together with (III.14.28)

$$\xi = \epsilon_s - j\frac{\sigma_s}{\omega} = \epsilon_o(1 - jh_o) = \epsilon_o \sqrt{1 - h_o^2} \, e^{-j \tan^{-1} h_o} \quad (6)$$

and ν is assumed to be real so that

$$\nu = \nu \quad (7)$$

then

$$\phi = \frac{e^{j \tan^{-1} h_o}}{4\pi\epsilon_o \sqrt{1 - h_o^2}} \int_{-h}^{+h} \frac{q'}{R} \, e^{-\alpha_s R} e^{-j\beta_s R} dz' \quad (8)$$

$$\mathbf{A} = \frac{\hat{z}}{4\pi\nu} \int_{-h}^{+h} \frac{I'_z}{R} \, e^{-\alpha_s R} e^{-j\beta_s R} dz' \quad (9)$$

In a poorly conducting simple medium

$$h_o^2 << 1; \qquad \tan^{-1} h_o \doteq h_o \quad (10)$$

and (8) can be simplified accordingly.

The wave picture may be used successfully in interpreting the integral in (8) and (9). The damping term $e^{-\alpha_s R}$ indicates

that the potential amplitudes decrease more rapidly than $1/R$. The amplitude factor now is $e^{-\alpha_e R}/R$. The equiphase surfaces travel with a constant radial phase velocity v_e given by

$$v_e = \frac{\omega}{\beta_e} = \frac{v_0}{N_e} \qquad (11)$$

so that the index of refraction N_e (defined in Sec. III.15) is simply the ratio of the phase velocities of propagation in unbounded space and in an unbounded dielectric medium. The wave length measuring the constant distance between successive equiphase surfaces belonging to the same family is

$$\lambda_e = \frac{\lambda_0}{N_e} \qquad (12)$$

As a result of the finite effective conductivity (due either to the presence of free charges or to a phase lag in polarization), the scalar potential ϕ experiences a constant phase shift $\tan^{-1} h_e$ (with $h_e = \sigma_e/\omega\epsilon_e$) at all points as compared with its phase in a perfect nonconductor. Since this phase shift is small and independent of the coordinates, it is of no practical significance.

ELECTROMAGNETIC WAVES

7. General Electromagnetic Field.—The Helmholtz integrals define the potential field in terms of the essential densities of charge and moving charge. They are

$$\phi = \frac{1}{4\pi\epsilon_0} \left\{ \int_\tau \frac{\varrho'}{R} e^{-j\beta_0 R} d\tau' + \int_\Sigma \frac{\bar{n}'}{R} e^{-j\beta_0 R} d\sigma' \right\} \qquad (1a)$$

$$\mathbf{A} = \frac{1}{4\pi\nu_0} \left\{ \int_\tau \frac{\overline{\varrho_m \mathbf{v}'}}{R} e^{-j\beta_0 R} d\tau' + \int_\Sigma \frac{\overline{n_m \mathbf{v}'}}{R} e^{-j\beta_0 R} d\sigma' \right\} \qquad (1b)$$

The potential functions are defined in terms of the electric and magnetic vectors as follows:

$$\mathbf{E} = -\operatorname{grad} \phi - j\omega\mathbf{A} \qquad (2a)$$
$$\mathbf{B} = \operatorname{curl} \mathbf{A} \qquad (2b)$$

By substituting (1) in (2) and performing the indicated operations, it is possible to derive explicit formulas for $\mathbf{E}$ and $\mathbf{B}$ and so obtain integral solutions for the Maxwell-Lorentz equations that define these vectors.

In order to calculate **E** and **B** from (2) using (1), it is important to note that the potentials and the field vectors are functions of the unprimed coordinates of the point P where they are calculated. In carrying out the operation of taking the gradient of ϕ or the curl of **A**, only the unprimed coordinates are involved, not the primed variables of integration locating the element $d\tau'$ or $d\sigma'$ at P'. (If at any time the operations grad, div, or curl apply to the primed variables, the operators themselves are written with a prime, thus: grad', div', curl'.) The distance R between P and P' is a function of both the primed and the unprimed coordinates; the density functions characterizing the region of integration are functions of the primed variables only. This fact has been indicated consistently by writing both densities and elements of integration with primes. Since the densities are not functions of the unprimed variables, it follows that

$$\text{grad } \delta' = 0 \tag{3a}$$
$$\text{curl } \overline{\varrho_m v'} = 0 \tag{3b}$$

In performing the operations grad and curl on the integrals in (1), it is assumed that the operators may be applied under the sign of integration. This is always possible when the integrals are regular, and this is unquestionably the case if the point P is not in the region of integration τ. If the point P is in τ, the potential amplitudes usually cannot be evaluated because a knowledge of the density functions everywhere in an extended region is generally not available. For practical purposes, therefore, one is limited actually to the case in which P is not in τ, so that there is no need for investigating the validity of operating under the sign when the integral is irregular.

Before proceeding to evaluate grad ϕ and curl **A**, the following considerations are pertinent. If ψ is a scalar function of R,

$$\text{grad } \psi = \frac{\partial \psi}{\partial R} \text{ grad } R = \frac{\partial \psi}{\partial R} \hat{R} \tag{4}$$

Here $\hat{R}$ is a unit vector. The last step follows from the definition of the gradient. It is clear that

$$|\text{grad } R| = \left(\frac{\partial R}{\partial R}\right)_{max} = 1 \tag{5}$$

The direction of the maximum rate of increase of **R** is outward.

It is specified by the unit vector $\hat{R}$. If ψ is any scalar and C any vector, the following vector relation is readily proved:

$$\text{curl } \psi C = \psi \text{ curl } C + [\text{grad } \psi, C]^* \qquad (6)$$

In applying (4) or (6) to (1), let

$$\psi = \frac{e^{-j\beta_0 R}}{R} \qquad (7a)$$

$$C = \overline{\varrho_m v'} \qquad (7b)$$

Then

$$\text{grad } \psi = -\hat{R} e^{-j\beta_0 R} \left(\frac{1}{R^2} + \frac{j\beta_0}{R} \right) \qquad (8)$$

With (3b) and (8)

$$\text{curl } \psi C = [\text{grad } \psi, C] = -[\hat{R}, \overline{\varrho_m v'}] e^{-j\beta_0 R} \left(\frac{1}{R^2} + \frac{j\beta_0}{R} \right) \qquad (9)$$

With (3) to (9) it follows for the volume integrals alone that

$$-\text{grad } \phi = \frac{-1}{4\pi\epsilon_0} \int_\tau \text{grad } (\varrho'\psi) d\tau' = \frac{-1}{4\pi\epsilon_0} \int_\tau \varrho' \text{ grad } \psi \, d\tau' \qquad (10)$$

$$= \frac{1}{4\pi\epsilon_0} \int_\tau \hat{R}\varrho' e^{-j\beta_0 R} \left(\frac{1}{R^2} + \frac{j\beta_0}{R} \right) d\tau'$$

$$\text{curl } A = \frac{1}{4\pi\nu_0} \int_\tau \text{curl } \psi C \, d\tau'$$

$$= \frac{-1}{4\pi\nu_0} \int_\tau [\hat{R}, \overline{\varrho_m v'}] e^{-j\beta_0 R} \left(\frac{1}{R^2} + \frac{j\beta_0}{R} \right) d\tau' \qquad (11)$$

Since, by definition of ν_0 and β_0,

$$\frac{\nu_0}{\epsilon_0} \equiv v_0^2 \qquad (12a)$$

$$\beta_0 \equiv \frac{\omega}{v_0} \qquad (12b)$$

it follows from (1b) that for the volume integral

$$j\omega A = \frac{1}{4\pi\nu_0} \int_\tau \overline{\varrho_m v'} \cdot \frac{j\omega}{R} e^{-j\beta_0 R} d\tau'$$

$$= \frac{1}{4\pi\epsilon_0} \int_\tau \overline{\varrho_m v'} e^{-j\beta_0 R} \frac{j\beta_0}{R v_0} d\tau' \qquad (13)$$

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* This relation is proved most simply in rectangular coordinates. For the x component

$$\text{curl}_x \psi C = \frac{\partial}{\partial y}(\psi C_z) - \frac{\partial}{\partial z}(\psi C_y) = \psi \text{ curl}_x C + C_z \frac{\partial \psi}{\partial y} - C_y \frac{\partial \psi}{\partial z}$$

$$\text{curl}_x \psi C = \psi \text{ curl}_x C + [\text{grad } \psi, C]_x$$

On combining (10), (11), and (13) according to (2), the expressions for $\mathbf{E}$ and $\mathbf{B}$ that define the complex amplitudes of the general electromagnetic field are readily obtained.   They are

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_\tau e^{-j\beta_0 R} \left\{ \hat{\mathbf{R}} \frac{\varrho'}{R^2} + \left(\varrho'\hat{\mathbf{R}} - \frac{\overline{\varrho_m v'}}{v_0} \right) \frac{j\beta_0}{R} \right\} d\tau' + S_{\mathbf{H}} \quad (14a)$$

$$\mathbf{B} = -\frac{1}{4\pi\nu_0} \int_\tau e^{-j\beta_0 R} [\hat{\mathbf{R}}, \overline{\varrho_m v'}] \left(\frac{1}{R^2} + \frac{j\beta_0}{R} \right) d\tau' + S_{\mathbf{B}} \quad (14b)$$

Here $S_{\mathbf{H}}$ and $S_{\mathbf{B}}$ stand for the surface integrals.   These may be obtained from the volume integrals in each case by writing $\mathfrak{n}$ for ϱ and σ for τ.

The expressions defining $\mathbf{E}$ and $\mathbf{B}$ at points in space due to a cylindrical conductor of small cross section, with axis along the z axis of a coordinate system, may be obtained (just as in the case of the potential functions) by carrying out the integrations in (14) over the small cross section and circumference of the conductor assuming R to be sensibly constant.   In this way

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{-h}^{h} e^{-j\beta_0 R} \left\{ \frac{\hat{\mathbf{R}}}{R^2} q' + \frac{j\beta_0}{R} \left(\hat{\mathbf{R}} q' - \frac{\hat{z} I_z'}{v_0} \right) \right\} dz' \quad (15a)$$

$$\mathbf{B} = -\frac{1}{4\pi\nu_0} \int_{-h}^{h} e^{-j\beta_0 R} [\hat{\mathbf{R}}, \hat{z}] I_z' \left(\frac{1}{R^2} + \frac{j\beta_0}{R} \right) dz' \quad (15b)$$

As before,

$$q' = \int_S \varrho' dS' + \int \bar{n}' ds' \quad (16a)$$

$$I_z' = \int \overline{\varrho_m v_z'} dS + \int \overline{n_m v_z'} ds' \quad (16b)$$

Here the first integral in each case is taken over the cross section $S = \pi a^2$ of the conductor, the second integral around its circumference $s = 2\pi a$; q' and I_z' are the amplitudes, respectively, of the total charge per unit length and total axial conduction current.   Contributions to $\mathbf{B}$ due to currents directed radially in the conductor are negligible.

In order to obtain the instantaneous values of $\mathbf{E}$ and $\mathbf{B}$ by introducing the time factor $e^{j\omega t}$ and selecting the real part, it is necessary to refer all phases to a single quantity.   This is conveniently chosen to be the current.   Therefore let

$$I_z' = I_z' \quad (17)$$

The complex amplitude q' may be expressed in terms of I_z' with

the aid of the equation of continuity written for the cylindrical conductor in the following one-dimensional form:

$$\frac{dI'}{dz'} + j\omega q' = 0 \tag{18}$$

Hence with (17)

$$q' = \frac{j}{\omega}\frac{dI'}{dz'} = \frac{j}{\beta_0 v_0}\frac{dI'_z}{dz'} \tag{19}$$

Upon multiplying (15) by $e^{j\omega t}$ on both sides and selecting the real part as the desired solution corresponding to an assumed time dependence

$$I'_{z\,\text{inst}} = \text{real part } (I'_z e^{j\omega t}) = I'_z \cos \omega t \tag{20}$$

$$q'_{\text{inst}} = \text{real part } (q' e^{j\omega t}) = -\frac{1}{\omega}\frac{dI'_z}{dz'} \sin \omega t \tag{21}$$

it follows that

$$\mathbf{E}_{\text{inst}} = -\frac{1}{4\pi\epsilon_0}\int_{-h}^{h}\left\{\frac{\hat{\mathbf{R}}}{Rv_0}\frac{dI'_z}{dz'}\cos(\omega t - \beta_0 R)\right.$$
$$\left. + \left(\frac{\hat{\mathbf{R}}}{\beta_0 v_0 R^2}\frac{dI'_z}{dz'} - \frac{\hat{\mathbf{z}}\beta_0 I'_z}{v_0 R}\right)\sin(\omega t - \beta_0 R)\right\}dz' \tag{22a}$$

$$\mathbf{B}_{\text{inst}} = -\frac{1}{4\pi v_0}\int_{-h}^{h}[\hat{\mathbf{R}},\hat{\mathbf{z}}]I'_z\left\{\frac{1}{R^2}\cos(\omega t - \beta_0 R)\right.$$
$$\left. - \frac{\beta_0}{R}\sin(\omega t - \beta_0 R)\right\}dz' \tag{22b}$$

These are the general expressions defining the instantaneous values of $\mathbf{E}$ and $\mathbf{B}$ at all points outside a cylindrical conductor of small radius which is in space and which is characterized by periodically varying distributions of charge and current. The vector $\mathbf{E}$ has a component along $\mathbf{R}$ and a second component parallel to $\hat{\mathbf{z}}$ and, hence, parallel to the current in the conductor. $\mathbf{B}$, on the other hand, is everywhere perpendicular to both $\mathbf{R}$ and $\hat{\mathbf{z}}$, so that it is directed along tangents to circles drawn with the conducting thread as a common axis. Therefore $\mathbf{E}$ and $\mathbf{B}$ are *everywhere mutually perpendicular*. It is readily seen that the equiphase surfaces of $\mathbf{E}$ and of $\mathbf{B}$ due to the periodically varying charge and current *in an element dz* of the conducting thread are spherical shells. This follows from the fact that at a given instant t the integrands in (22) are constant if R is constant. However, the amplitudes of the values of $d\mathbf{E}$ and $d\mathbf{B}$ associated

with points on such surfaces do not diminish as $1/R$ with increasing R, but in a more complicated way.   In order to determine this amplitude factor and the velocity of equiphase surfaces of E and of B, the expressions $(22a,b)$ may be transformed, using the trigonometric relation

$$M \cos x + N \sin x = \sqrt{M^2 + N^2} \cos \left(x - \arctan \frac{N}{M} \right) \quad (23)$$

or $(15a)$ and $(15b)$ may be written in amplitude-phase-angle form before multiplying by $e^{j\omega t}$ and taking the real part.   In the case of B this can be done directly to obtain

$$B_{\text{inst}} = -\frac{1}{4\pi\nu_0} \int_{-h}^{h} [\hat{R}, \hat{z}] \, I_0' \, \frac{\sqrt{1 + \beta_0^2 R^2}}{R^2}$$
$$\cos(\omega t - \beta_0 R + \arctan \beta_0 R) dz' \quad (24)$$

Because the integrand in the expression $(22a)$ or $(15a)$ for E involves two different vectors, it is necessary to treat each component of E separately in obtaining amplitude and phase angle.   For example, if this is done in Cartesian coordinates an expression is obtained for each component involving amplitudes and phase angles that are functions not only of R and of the constant parameters β_0 and ν_0, but also of I', dI'/dz', and the direction cosines of $\hat{R}$.   An interpretation of dE_{inst} in terms of a simple wave picture is in this case not satisfactory because a different family of equiphase surfaces traveling with different velocities must be defined for each component.

In the case of dB_{inst} as given by the integrand of (24) the phase is a function of R and t alone, so that it is possible to obtain a simple expression for the radial velocity of the spherical equiphase surfaces.   Upon differentiating the expression defining a given phase

$$\omega t - \beta_0 R + \arctan \beta_0 R = \text{const.} \quad (25a)$$

with respect to t, the radial phase velocity is

$$\frac{dR}{dt} = v_0 \left(1 + \frac{1}{\beta_0^2 R^2} \right) \quad (25b)$$

From this relation it appears that the radial velocity of the equiphase surfaces due to the current and charge in *an element dz'* is not constant even in the relatively simple case of dB_{inst}, but depends upon the frequency of the moving charge and upon the

radius of the surface.   It approaches the constant v_0 asymptotically as R increases.   The equiphase surfaces due to the entire conducting thread may be obtained by integrating over its entire length if the current distribution is known.   They have a simple form for all values of R and for both E and B in the important special case of an indefinitely thin center-driven antenna an integral number of half wave lengths long.   This is described in Volume II.

It must be concluded from the above discussion that the general electromagnetic field due to any single element dz or due to an entire conductor of finite length does not permit an interpretation in terms of equiphase surfaces moving with *constant* velocity v_0 in space.   Even when equiphase surfaces can be defined, they expand with a decreasing velocity that approaches v_0 at a sufficiently great distance.   The magnetic vector dB due to the current in an element dz' is always tangent to the spherical equiphase surfaces associated with it, so that it is perpendicular to the direction of motion.   The surfaces of constant phase used to describe dB are therefore called *transverse waves*.   On the other hand, dE has components both tangent to its equiphase surfaces and normal to them.   The electric waves due to an element dz are, therefore, both transverse and longitudinal.

Although the general electromagnetic field is too complex to allow a simple geometrical interpretation, it is possible to consider component parts of the field each of which is readily described in quite transparent form.   These are described below.

8. Induction and Radiation Fields.—It is usually convenient to study the general electromagnetic field defined by (7.14a,b) in two parts.   Let each of the vectors E and B be expressed as the sum of two components as follows:

$$E = E^i + E^r; \qquad B = B^i + B^r \tag{1}$$

The components with the superscript i define what is called the *induction field*.   This is given by

$$E^i = \frac{1}{4\pi\epsilon_0} \int_\tau \hat{R}\, \frac{\varrho'}{R^2} e^{-j\beta_0 R} d\tau' + S_E^i \tag{2a}$$

$$B^i = \frac{1}{4\pi v_0} \int_\tau \frac{[\hat{R}, \overline{\varrho_m v'}]}{R^2}\, e^{-j\beta_0 R} d\tau' + S_B^i \tag{2b}$$

S_E^i and S_B^i stand for surface integrals obtained by writing $\mathbf{n}$ for ϱ and σ for τ in the volume integrals. The components in (1) with the superscript r define the so-called *radiation field*. It is given by

$$\mathbf{E}^r = \frac{1}{4\pi\epsilon_0} \int_\tau c^{-j\beta_0 R} \frac{j\beta_0}{R} \left(\hat{\varrho}'\hat{R} - \frac{\overline{\varrho_m \mathbf{v}'}}{v_0} \right) d\tau' + S_E^r \qquad (3a)$$

$$\mathbf{B}^r = - \frac{1}{4\pi v_0} \int_\tau c^{-j\beta_0 R} \frac{j\beta_0}{R} [\hat{R}, \overline{\varrho_m \mathbf{v}'}] d\tau' + S_B^r \qquad (3b)$$

The induction and radiation fields differ in that the former is proportional to $1/R^2$, the latter to β_0/R. The ratio of these two factors alone is $\beta_0 R$. It follows that with any given β_0 and for sufficiently small values of R (*i.e.*, for $\mathbf{E}$ and $\mathbf{B}$ calculated at points sufficiently *near* the periodically varying charges) the induction field is so large compared with the radiation field that the latter may be neglected. On the other hand, for R very large, the reverse is true; the radiation field predominates and the induction field may be disregarded because it is insignificant.

9. Induction or Near Zone; Coulomb's Law and Biot-Savart Formula. Let the *induction or near zone* be defined to be that region in the immediate vicinity of the volume in and on which charge and current densities are nonvanishing which satisfies the inequality defining the quasi-stationary state.

$$\beta_0 R \ll 1 \qquad (1)$$

As a consequence of (1)

$$E^i \gg E^r; \qquad B^i \gg B^r \qquad (2)$$

Subject to (2),

$$\mathbf{E} \doteq \mathbf{E}^i; \qquad \mathbf{B} \doteq \mathbf{B}^i \qquad (3)$$

As a consequence of (1)

$$c^{-j\beta_0 R} = 1 - j\beta_0 R - \cdots \doteq 1 \qquad (4)$$

This is equivalent to assuming quasi-instantaneous action at a sufficiently short distance. With (4), the induction field is identified with the field of the quasi-stationary state. It is

$$\beta_0 R \ll 1 \begin{cases} \mathbf{E} \doteq \mathbf{E}^i = \dfrac{1}{4\pi\epsilon_0} \left\{ \int_\tau \hat{\mathbf{R}}\, \dfrac{\varrho'}{R^2}\, d\tau' + \int_\Sigma \hat{\mathbf{R}}\, \dfrac{\bar{\eta}'}{R^2}\, d\sigma' \right\} & (5a) \\[2ex] \mathbf{B} \doteq \mathbf{B}^i = -\dfrac{1}{4\pi\nu_0} \left\{ \int_\tau \dfrac{[\hat{\mathbf{R}},\overline{\varrho_m\mathbf{v}'}]}{R^2}\, d\tau' + \int_\Sigma \dfrac{[\hat{\mathbf{R}},\overline{\eta_m\mathbf{v}'}]}{R^2}\, d\sigma' \right\} & (5b) \end{cases}$$

In the stationary state

$$\mathbf{E} = \dfrac{1}{4\pi\epsilon_0} \left\{ \int_\tau \hat{\mathbf{R}}\, \dfrac{\bar{\rho}'}{R^2}\, d\tau' + \int_\Sigma \hat{\mathbf{R}}\, \dfrac{\bar{\eta}'}{R^2}\, d\sigma' \right\} \tag{6a}$$

$$\mathbf{B} = -\dfrac{1}{4\pi\nu_0} \left\{ \int_\tau \dfrac{[\hat{\mathbf{R}},\mathbf{I}']}{R^2}\, d\tau' + \int_\Sigma \dfrac{[\hat{\mathbf{R}},\bar{\mathbf{I}}']}{R^2}\, d\sigma' \right\} \tag{6b}$$

The electric field of the stationary state is called the *electrostatic* field; the magnetic field of the stationary state is called the *magnetostatic* field. Since $\omega = 0$ in the stationary states, the radiation fields vanish identically at all distances so that the entire stationary field is the induction field. At low frequencies, and for all significant distances, $\dfrac{\omega R}{v_0}$ is so small that the quasi-stationary or near-zone field is the entire field.

A fundamentally important static-state relation may be obtained by specializing $(6a)$. Let the distance from any point P where $\mathbf{E}$ is calculated to the center of a volume τ be R_0, where R_0 is large compared with the maximum dimension of τ. Then,

$$\begin{aligned} \mathbf{E} &= \dfrac{1}{4\pi\epsilon_0} \left\{ \int_\tau \hat{\mathbf{R}}\, \dfrac{\bar{\rho}'}{R^2}\, d\tau' + \int_\Sigma \hat{\mathbf{R}}\, \dfrac{\bar{\eta}'}{R^2}\, d\sigma' \right\} \\ &= \dfrac{\hat{\mathbf{R}}_0}{4\pi\epsilon_0 R_0^2} \left\{ \int_\tau \bar{\rho}'d\tau' + \int_\Sigma \bar{\eta}'d\sigma' \right\} = \dfrac{\hat{\mathbf{R}}_0 q}{4\pi\epsilon_0 R_0^2} \end{aligned} \tag{7}$$

where q is the total charge within and on the surface of τ. The electrostatic force (II.12.5) acting on a second volume V characterized by essential densities $\bar{\rho}$ and $\bar{\eta}$ is

$$\mathbf{F} = \int_V \bar{\rho}\mathbf{E}\, dV + \int_S \bar{\eta}\mathbf{E}\, dS \tag{8}$$

If the dimensions of V are small compared with R, $\mathbf{E}$ is sensibly constant over V so that

$$\mathbf{F}_e = \mathbf{E} \left\{ \int_V \bar{\rho}\, dV + \int_S \bar{\eta}\, dS \right\} = \mathbf{E}Q \tag{9}$$

where Q is the total charge within and on V. Upon combining

(7) and (9)

$$\mathbf{F}_e = \hat{\mathbf{R}}_0 \frac{qQ}{4\pi\epsilon_0 R_0^2} \tag{10}$$

In this expression R_0 is the distance between the centers of the two volumes τ and V. $\mathbf{F}_e$ is the electrostatic force acting on V due to τ provided all parts of the two volumes are separated by distances that are large compared with the dimensions of the volumes V and τ. If the two volumes are symmetrically charged spheres, (10) is valid without such a restriction so that V need not be small in terms of R_0. The relation (10) is known as *Coulomb's law*. It is an important special form of the electro-mathematical model that is actually contained in the field and force equations as a particular case. It is customary to use Coulomb's law as a fundamental postulate in a more conventional formulation of electrodynamics.

If the near-zone relations (5a,b) are written for a cylindrical conductor in space, they assume the following form:

$$\beta_0 R << 1 \begin{cases} \mathbf{E} \doteq \mathbf{E}^i = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int_{-h}^{h} \dfrac{q'}{R^2}\,dz' & (11a) \\[3mm] \mathbf{B} \doteq \mathbf{B}^i = -\dfrac{1}{4\pi\nu_0} \displaystyle\int_{-h}^{h} [R,\hat{z}]\,\dfrac{I_s}{\hat{R}^2}\,dz' & (11b) \end{cases}$$

The relation (11b) is commonly called the *Biot and Savart formula* when written in the following equivalent form:

$$d\mathbf{B}^i = \frac{I_s}{4\pi\nu_0}\frac{[d\mathbf{s},\mathbf{R}]}{R^2} \tag{12}$$

The direction of the magnetic field due to the current I_s in the element $d\mathbf{s}$ of a wire is perpendicular to the plane containing R and $d\mathbf{s}$. R is a vector drawn from $d\mathbf{s}$ to the point where the field is computed. The magnitude of the field is

$$dB^i = \frac{I_s}{4\pi\nu_0}\frac{\sin (R,s)}{R^2}\,ds \tag{13}$$

10. Radiation or Far Zone.—The *radiation or far zone* is defined by

$$\beta_0 R >> 1 \tag{1}$$

As a consequence of (1)

$$E^r >> E^i; \qquad B^r >> B^i \tag{2}$$

so that, in the radiation zone,

$$\mathbf{E} \doteq \mathbf{E}^r; \qquad \mathbf{B} \doteq \mathbf{B}^r \tag{3}$$

The radiation-zone field is given by

$$\beta_0 R \gg 1 \begin{cases} \mathbf{E} \doteq \mathbf{E}^r = \dfrac{j\beta_0}{4\pi\epsilon_0} \displaystyle\int_\tau \left\{ \hat{\mathbf{R}} \varrho' - \overline{\dfrac{\varrho_m \mathbf{v}'}{v_0}} \right\} \dfrac{e^{-j\beta_0 R}}{R} \, d\tau' + S_E^r & (4) \\[4mm] \mathbf{B} \doteq \mathbf{B}^r = -\dfrac{j\beta_0}{4\pi v_0} \displaystyle\int_\tau [\hat{\mathbf{R}}, \overline{\varrho_m \mathbf{v}'}] \dfrac{e^{-j\beta_0 R}}{R} \, d\tau' + S_B^r & (5) \end{cases}$$

The corresponding equations for a cylindrical conductor in space are

$$\beta_0 R \gg 1 \begin{cases} \mathbf{E} \doteq \mathbf{E}^r = \dfrac{j\beta_0}{4\pi\epsilon_0} \displaystyle\int_{-h}^{h} \left(\hat{\mathbf{R}} q' - \hat{\mathbf{z}} \dfrac{I_z}{v_0} \right) \dfrac{e^{-j\beta_0 R}}{R} \, dz' & (6) \\[4mm] \mathbf{B} \doteq \mathbf{B}^r = \dfrac{-j\beta_0}{4\pi v_0} \displaystyle\int_{-h}^{h} [\hat{\mathbf{R}}, \hat{\mathbf{z}}] I_z \dfrac{e^{-j\beta_0 R}}{R} \, dz' & (7) \end{cases}$$

These relations show that the electromagnetic field in the radiation zone has precisely the same simple form as the potential field. The wave picture of spherical equiphase surfaces expanding with constant velocity v_0 may be used, therefore, to describe the electromagnetic radiation field due to the periodically varying charges and current in an element dz just as was done in the case of the potential field. The magnetic waves are transverse; the electric waves may be both transverse and longitudinal. An important special case is considered in the next section in which *both* magnetic and electric waves are transverse only.

It has been shown that the general electromagnetic field may be divided into a near- or induction-zone field, and a far- or radiation-zone field. All points in the former are by definition sufficiently near to the moving charges so that an *instantaneous* action at a distance is a good *approximation* to the fundamentally correct *retarded* action at a distance. In this zone it is not convenient or illuminating to use a wave picture because in the vicinity of the periodically varying charges and currents the equiphase surfaces of $\mathbf{B}$ and of the components of $\mathbf{E}$ do not move with a constant velocity. The phase velocity is very high near the moving charges, and the approximation (9.4) is equivalent to assuming an infinite phase velocity. The integrals (9.5a,b) and (9.11a,b), which define the approximate electromagnetic field in the induction or near zone, are usually interpreted using

geometrical pictures that correctly describe exactly only the stationary states in which all densities are constant in time. The formal equivalence between quantities in the stationary states and the corresponding complex amplitudes in the quasi-stationary state makes it possible to use the same geometrical pictures for amplitudes of periodic phenomena as for stationary magnitudes, and this is obviously equivalent to assuming an instantaneous action at a distance.

In the radiation zone, the simplest form of retarded action at a distance obtains. The picture of equiphase surfaces expanding with a *constant* velocity is conveniently used, so that it is customary to speak of electromagnetic waves in this zone. In fact, the radiation zone is frequently called the *wave zone*. Because the induction zone and the radiation zone do not overlap, a so-called *intermediate zone* exists between them. It consists of the region between $\beta_0 R << 1$ and $\beta_0 R >> 1$. The electromagnetic field in this intermediate zone does not have the simple stationary-state properties of the field in the near zone, nor the convenient wave properties of the far zone. An accurate representation is possible only in terms of the integrals (7.14a,b) or (7.15a,b) of the general electromagnetic field. It is well to note that it is illogical to attempt to interpret the properties of the intermediate zone in terms of a combination or superposition of the properties of the near and far zones. In the former, it is assumed as a legitimate approximation that action at a sufficiently short distance is approximately instantaneous; in the latter, it is accepted with equal justification that action at a sufficiently great distance may be pictured as propagated with a constant phase velocity. In the intermediate zone, neither approximation is justified, and consequently no combination of the two is correct.

In order to define the ranges included in the near, intermediate, and far zones more specifically, let the inequalities $\beta_0 R << 1$ and $\beta_0 R >> 1$ be assigned numerical values. Thus, for an error not to exceed 1 per cent

$$\text{Near zone: } R \leq \frac{0.01}{\beta_0} = \frac{3 \times 10^6}{\omega} \text{ (meters)}$$

$$\text{Far zone: } R \geq \frac{100}{\beta_0} = \frac{3 \times 10^{10}}{\omega} \text{ (meters)}$$

Consider the four characteristic angular velocities $\omega = 10^3$, $\omega = 10^6$, $\omega = 10^9$, $\omega = 10^{12}$.

Low frequency: $\quad\begin{cases}\text{Near zone: } R \leq 3 \text{ kilometers} \\ \text{Far zone: } \quad R \geq 30{,}000 \text{ kilometers}\end{cases}$
$\omega = 10^3$

Radio frequency: $\quad\begin{cases}\text{Near zone: } R \leq 3 \text{ meters} \\ \text{Far zone: } \quad R \geq 30 \text{ kilometers}\end{cases}$
$\omega = 10^6$

Ultrahigh frequency: $\quad\begin{cases}\text{Near zone: } R \leq 3 \text{ millimeters} \\ \text{Far zone: } \quad R \geq 30 \text{ meters}\end{cases}$
$\omega = 10^9$

Super frequency: $\quad\begin{cases}\text{Near zone: } R \leq 3 \text{ microns} \\ \text{Far zone: } \quad R \geq 3 \text{ centimeters}\end{cases}$
$\omega = 10^{12}$

From these magnitudes it is clear that low-frequency circuits and most radio-frequency circuits except antennas and long transmission lines are contained entirely within the near zone. Radio transmitters and receivers are generally in the far zone with respect to each other. Multiple antenna arrays, directional systems, and most ultrahigh-frequency circuits include the intermediate zone.

11. Radiation Field of an Antenna in Space.—The general integrals defining the radiation field of a cylindrical conductor of half-length h and small radius a oriented in space along an arbitrarily directed s axis are given by (10.6) and (10.7) with s written for z. An important special application of these integrals is to a straight conductor or antenna that is short compared with the distance from any part of it to the region in which **E** and **B** are to be calculated. If the field is to be calculated at a distance R from an origin O at the center of the antenna, the following inequality must be satisfied:

$$R >> h \tag{1}$$

The condition for the wave zone must also be obeyed. It is

$$\beta_0 R >> 1 \tag{2}$$

In the far or wave zone spherical equiphase surfaces expand with approximately constant radial velocity

$$v_0 = \frac{\omega}{\beta_0} = 3 \times 10^8 \text{ meters/second} \tag{3a}$$

and surfaces in the same phase are separated a radial distance

$$\lambda_0 = \frac{\dot{v_0}}{f} \qquad (3b)$$

Accordingly (2) is equivalent to

$$R >> \frac{\lambda_0}{2\pi} \qquad (4)$$

The vector $\mathbf{R}$ drawn from the point P', locating the element of integration ds' in the conductor to the point P where the field

FIG. 11.1.—Section of antenna with origin of coordinates at O.

is to be calculated, may be written as follows (Fig. 11.1).

$$\mathbf{R} = \mathbf{R}_0 - \mathbf{s}' \qquad (5a)$$
$$R = \sqrt{R_0^2 + s'^2 - 2(\mathbf{R}_0, \mathbf{s}')} \qquad (5b)$$

As a result of (1) it follows that

$$R \doteq R_0 \qquad (6)$$

in all factors that cannot vanish as R increases. These include all quantities except the arguments of the exponential functions. In these the first-order small terms must be retained. That is,

$$R \doteq R_0 - (\hat{\mathbf{R}}_0, \mathbf{s}') = R_0 - s' \cos (R_0, s') \qquad (7)$$

Where $\hat{\mathbf{R}}_0$ is a unit vector pointing from O to P.

Using (6) in amplitude factors and (7) in arguments, the general relations (10.6) and (10.7) written for an arbitrary s axis instead of the z axis become

$$\mathbf{E}^r = \frac{j\beta_0}{4\pi\epsilon_0} \frac{e^{-j\beta_0 R_0}}{R_0} \int_{-h}^{h} \left\{ \hat{\mathbf{R}}_0 q' - \frac{\hat{\mathbf{s}} I_s'}{v_0} \right\} e^{j\beta_0 (\hat{\mathbf{R}}_0, \mathbf{s}')} ds' \qquad (8a)$$

$$B^r = \frac{-j\beta_0}{4\pi v_0} \frac{e^{-j\beta_0 R_0}}{R_0} \int_{-h}^{h} [\hat{\mathbf{R}}_0, \hat{\mathbf{s}}] I_s' e^{j\beta_0 (\hat{\mathbf{R}}_0, \mathbf{s}')} ds' \qquad (8b)$$

Here $\hat{\mathbf{s}}$ is a unit vector along the axis of the antenna. Formulas (8a) and (8b) may be obtained directly from (3.17) and (3.18) using (6) and (7) and the general formulas

$$\mathbf{B} = \text{curl } \mathbf{A}, \qquad \mathbf{E} = -\text{grad } \phi - j\omega\mathbf{A}$$

The expression (8a) can be transformed with the aid of the principle of the conservation of electric charge as contained in

a one-dimensional form of the equation of continuity (7.18)

$$\frac{dI'_s}{ds'} + j\omega q' = 0 \qquad (9)$$

Upon solving this for q' and substituting in (8a),

$$\mathbf{E}^r = \frac{-1}{4\pi\epsilon_0 v_0} \frac{e^{-j\beta_0 R_0}}{R_0} \int_{-h}^{h} \left\{ \hat{\mathbf{R}}_0 \frac{dI'_s}{ds'} + \hat{\mathbf{s}} j\beta_0 I'_s \right\} e^{j\beta_0(\hat{R}_0,s')} ds' \qquad (10)$$

The first term in this integral can be treated by integration by parts.

$$\int_{-h}^{h} e^{j\beta_0(\hat{R}_0,s')} \frac{dI'_s}{ds'} ds' = I'_s e^{j\beta_0(\hat{R}_0,s')} \bigg|_{-h}^{h} -j\beta_0(\hat{R}_0,\hat{s}') \int_{-h}^{h} I'_s e^{j\beta_0(\hat{R}_0,s')} ds' \qquad (11)$$

Since the potential functions as defined by the Helmholtz integrals (1.33) and (1.34) satisfy the equation of continuity for potentials (1.48) *only if no current crosses the surface enclosing the volume of integration,* it is necessary to assume that the conductor is of finite length so that the current vanishes at both ends $s' = \pm h$. In general, a small axial current $I_{\pm h}$ at $s = \pm h$ becomes an equal radial current at the circular edge of the flat end. This radial current vanishes at $r = 0$. Let it be assumed for the present that $I_{\pm h}$ is practically zero. In this case the first term in (11) is zero when the limits are inserted. This leaves for (10)

$$\mathbf{E}^r = \frac{j\beta_0}{4\pi\epsilon_0 v_0} \frac{e^{-j\beta_0 R_0}}{R_0} \int_{-h}^{+h} \{\hat{\mathbf{R}}_0(\hat{R}_0,\hat{s}) - \hat{\mathbf{s}}\} I'_s e^{j\beta_0(\hat{R}_0,\ s')} ds' \qquad (12)$$

I'_s is the complex amplitude of the current directed along s'.

The double vector product may be shown to satisfy the following relation:[1]

[1] In rectangular coordinates,

$$[\mathbf{B},\mathbf{C}] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \hat{x}\{B_y C_z - B_z C_y\} + \hat{y}\{B_z C_x - B_x C_z\} + \hat{z}\{B_x C_y - B_y C_x\}$$

$$[\mathbf{A},[\mathbf{B},\mathbf{C}]] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix}$$

The x component is

$$[\mathbf{A},[\mathbf{B},\mathbf{C}]]_x = \{A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z\}$$

The x component of $\mathbf{B}(\mathbf{A},\mathbf{C}) - \mathbf{C}(\mathbf{A},\mathbf{B})$ is

$$[A,[B,C]] = B(A,C) - C(A,B) \tag{13}$$

Hence,

$$[\hat{R}_0,[\hat{R}_0,\hat{s}]] = \hat{R}_0(\hat{R}_0,\hat{s}) - \hat{s}(\hat{R}_0,\hat{R}_0) \tag{14}$$

However,

$$(\hat{R}_0,\hat{R}_0) = 1 \tag{15}$$

Using (14) with (15), (12) may be written in the following form:

$$E^r = \frac{j\beta_0}{4\pi\epsilon_0 v_0}\frac{e^{-j\beta_0 R_0}}{R_0}\int_{-h}^{h}[\hat{R}_0,[\hat{R}_0,\hat{s}]]I'_s e^{j\beta_0(\hat{R}_0,s')}ds' \tag{16}$$

This formula may be obtained directly from (3.18) using

$$E = -j\frac{\omega}{\beta_0^2}(\text{grad div } A + \beta_0^2 A).$$

Upon comparing (16) with (8b) and recalling that $v_0^2 \equiv \nu_0/\epsilon_0$ it is seen that

$$E^r = v_0[B^r,\hat{R}_0] \tag{17}$$

It thus appears that regardless of the form of the distribution of current in the antenna, the electric and magnetic fields in the radiation or wave zone are *in phase* and mutually *perpendicular*. By writing $e^{-j\beta_0 R_0}$ under the sign of integration in (16) or (8b), it can be recombined with $e^{j\beta_0(\hat{R}_0,s')}$ to give $e^{-j\beta_0 R}$. The form of (16) or (8b) is then essentially the same as that of the potential functions (3.17) and (3.18). Thus,

$$B^r = \frac{-j\beta_0[\hat{R}_0,\hat{s}]}{4\pi\nu_0}\int_{-h}^{h}\frac{I'_s}{R}e^{-j\beta_0 R}ds' \tag{18}$$

$$A = \frac{\hat{s}}{4\pi\nu_0}\int_{-h}^{h}\frac{I'_s}{R}e^{-j\beta_0 R}ds' \tag{19}$$

Each element of current $I'_s ds$ may be associated with a radially expanding spherical train of equiphase surfaces or waves moving with the constant velocity v_0. It is clear from (8b) and (17)

$$B_s(A,C) - C_s(A,B) = B_s\{A_x C_x + A_y C_y + A_z C_z\}$$
$$- C_s\{A_x B_x + A_y B_y + A_z B_z\}$$
$$= A_y B_x C_y + A_z B_x C_z - A_y B_y C_x - A_z B_z C_x$$

This is the same as the x component of $[A,[B,C]]$. The other two components can be proved equal in the same way. If B and C are polar vectors, [B,C] is an axial vector. The vector product of a polar and an axial vector, which is polar, is equal to the difference between two polar vectors.

that both $\mathbf{E}^r$ and $\mathbf{B}^r$ are perpendicular to $\mathbf{R}_0$, so that the electric as well as the magnetic waves are transverse. In other words, at every point P in the wave zone, the periodically varying amplitude $\mathbf{E}^r$ or $\mathbf{B}^r$ due to an element $I_s' ds$ is always pointed at right angles to the radial direction of propagation of the equiphase surface on which it is defined. That is, $\mathbf{E}^r$ and $\mathbf{B}^r$ are tangent to the spherical equiphase surfaces or wave fronts.

If the conductor extends beyond $s = \pm h$, or if it ends at $s = \pm h$ but there are appreciable radial currents on the ends, the axial currents $I_{\pm h}$ do not vanish. In this case the integrated term in (11) is not zero. The complete expression is

$$\mathbf{E}^r = v_0[\mathbf{B}^r,\hat{\mathbf{R}}_0] - \frac{\mathbf{R}_0}{4\pi\epsilon_0 v_0} \frac{e^{-j\beta_0 R_0}}{R_0} \{I_h e^{j\beta_0(\hat{\mathbf{R}}_0,\hat{\mathbf{s}})h} - I_{(-h)} e^{-j\beta_0(\hat{\mathbf{R}}_0,\hat{\mathbf{s}})h}\} \quad (20)$$

Here the first term is perpendicular to $\mathbf{B}^r$, and the second term gives a *radial* component of the *radiation* field.

Consider first the case in which the current-carrying conductor extends beyond $s = \pm h$, so that (20) is applied only to a *section* of conductor and this has nonvanishing currents at the ends. Such a section is not electrically complete and self-contained as required in the theorem of continuity (Sec. 1); consequently (20) cannot be assigned an independent meaning. As pointed out in Sec. I.26, it is possible to make a *section* of a current-carrying conductor electrically complete by the artifice of adding thin fictitious layers of equal and opposite surface charge on the two sides of each of the mathematical boundaries cutting the conductor at $s = \pm h$. The section between $s = +h$ and $s = -h$ is then characterized by the total axial current I_s, by the total charge Q_h at $s = h$, and by the total charge $Q_{(-h)}$ at $s = -h$. These charges and currents must satisfy the surface equation of continuity for charge in the form

$$I_h = j\omega Q_h = j\omega \int_0^a 2\pi r n_h dr \quad (21a)$$

$$I_{(-h)} = -j\omega Q_{(-h)} = -j\omega \int_0^a 2\pi r n_{(-h)} dr \quad (21b)$$

The far-zone electric field of a surface layer of charge confined to a circle of radius a that satisfies (3.1) and (3.2) may be calculated directly from (10.4) in the simplified form

$$\mathbf{E}^r = \frac{\hat{\mathbf{R}} j\beta_0}{4\pi\epsilon_0} \frac{e^{-j\beta_0 R}}{R} \int_{\Sigma} \mathbf{n}' d\sigma' = \hat{\mathbf{R}} \frac{j\beta_0 Q}{4\pi\epsilon_0} \frac{e^{-j\beta_0 R}}{R} \quad (22)$$

R is measured to the center of the circle. For the charges at

$$s = h, \qquad R = R_0 - (\hat{R}_0,\hat{s})h; \qquad Q = Q_h \qquad (23a)$$

For the charges at

$$s = -h, \qquad R = R_0 + (\hat{R}_0,\hat{s})h; \qquad Q = Q_{(-h)} \qquad (23b)$$

In $(23a,b)$ R_0 is measured to the origin $s = 0$. The electric field in the far zone due to the charge Q_h at $s = h$ and the charge $Q_{(-h)}$ at $s = -h$ is

$$\mathbf{E}^r = \hat{R}_0 \frac{j\beta_0}{4\pi\epsilon_0} \frac{e^{-j\beta_0 R_0}}{R_0} \{ Q_h e^{j\beta_0(\hat{R}_0,\hat{s})h} + Q_{(-h)}e^{-j\beta_0(\hat{R}_0,\hat{s})h} \} \qquad (24)$$

These charges contribute nothing to the magnetic field. If (24) is added to (20) and use is made of $(21a,b)$, the electric field in the radiation zone of a section of a current-carrying conductor with added fictitious charges at the boundaries of the section reduces to the first term on the right in (20), *viz.*,

$$\mathbf{E}^r = v_0[\mathbf{B}^r,\hat{R}_0] \qquad (25)$$

If it is desired to consider the contributions to the electromagnetic field of an electrically complete system due to currents in parts that are not individually self-contained, it is usually desirable to add the fictitious surface charges so that the partial fields are due to systems that are at least formally consistent with the theorem of continuity (Sec. 1). As discussed in Sec. I.26, this in no way alters the field of the physically complete system because the fields due to all the added charges cancel. It must be emphasized, however, that the field (25) calculated for a part of a complete system using added surface charges *is not the correct field of that part* any more than is (20) without the use of these added charges. Actually *neither* of the two fields is the correct field of a *part* of a complete system. As stated in conjunction with the theorem of continuity (Sec. 1), *the electromagnetic field of an electrically complete system is a property of that system as a whole and cannot, in general, be ascribed piecemeal to its parts.* It is true, nevertheless, that a measure of the effectiveness of different parts of an antenna in maintaining an electric field in the far zone may be obtained using (20) or (25) and, because it gives the field of a self-consistent system, (25) usually is to be preferred.

If the current-carrying conductor does not extend beyond $s = \pm h$, but I_h and $I_{(-h)}$ do not vanish because they become radial currents of magnitudes $I_{s=h} = I_{r=a}$, $I_{s=-h} = I_{r=a}$ on the circular ends of the antenna, (20) gives the contribution to the field by the axial current and (24) the contribution by the charge $Q_{\pm h}$ deposited on the end surfaces and particularly on its sharp edges by $I_{\pm h}$. Note that in this case $Q_{\pm h}$ is a physically real surface charge, not a fictitious one. It is determined from (21a,b). The resultant field is obtained by combining (20) and (24) using (21a,b), the far-zone electric field reduces to (25) with B^r given by (18). Since the field due to the radial currents on the ends is necessarily small because each circular sheet of current can be divided into equal filaments that are in opposite directions and $\beta_0 a << 1$, (25) is the entire far-zone field. It is the same in form as with $I_{\pm h} = 0$, but the current I_z' on an antenna

FIG. 11.2.—Polar coordinates.

with nonvanishing currents at the ends is not the same as with $I_{\pm h} = 0$, so that B^r and E^r must also be different. Since $I_{\pm h}$ is necessarily very small if $\beta_0 a << 1$ as assumed, it is usually possible to set $I_{\pm h} \doteq 0$ in calculating far-zone fields.

If a system of polar coordinates R_0, Θ, Φ is introduced with origin at the center of the antenna and with Θ measured from its axis which lies along the z axis of a Cartesian system as shown in Fig. 11.2, it follows that in (8b) z may be written for s and

$$(R_0, s') = z' \cos \Theta \tag{26}$$

Also

$$-[\hat{R}_0, \hat{s}] = [\hat{s}, \hat{R}_0] = \hat{\Phi} \sin \Theta \tag{27}$$

and

$$[\hat{R}_0, [\hat{R}_0, \hat{s}]] = -[\hat{R}_0, \hat{\Phi}] \sin \Theta = \hat{\Theta} \sin \Theta \tag{28}$$

Accordingly, using (8b) and (17)

$$B^r = \hat{\Phi} B_\Phi{}^r; \qquad E^r = \hat{\Theta} E_\Theta{}^r \tag{29}$$

The components of the far-zone or radiation field of an antenna oriented along the z axis may be written as follows:
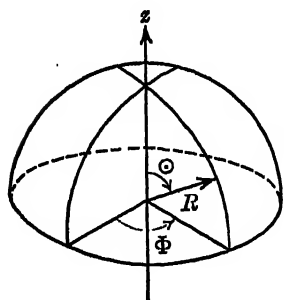
$$E_R^r = 0; \qquad\qquad B_R^r = 0$$

$$E_\Theta^r = f(h,\Theta)\frac{e^{-j\beta_0 R_0}}{R_0}; \qquad B_\Theta^r = 0 \qquad (30)$$

$$E_\Phi^r = 0; \qquad\qquad B_\Phi^r = \frac{f(h,\Theta)}{v_0}\frac{e^{-j\beta_0 R_0}}{R_0}$$

with

$$f(h,\Theta) \equiv \frac{j\omega}{4\pi v_0}\int_{-h}^{h} I_z' e^{j\beta_0 z'\cos\Theta}\sin\Theta\, dz' \qquad (31)$$

The wave-zone field of an antenna in space is defined by (30) with (31).

Further study of the electromagnetic field of antennas depends upon the determination of the axial distribution of current in the conductor. This constitutes the initial problem in Volume II. One special case of historical interest is considered in the next section.

It is interesting to note that

FIG. 11.3.—Far-zone or radiation field.

$$\frac{E_\Theta^r}{B_\Phi^r} = v_0 = 3 \times 10^8 \text{ meters/second} \qquad (32)$$

In space,

$$H_\Phi = v_0 B_\Phi \qquad (33)$$

and

$$\frac{E_\Theta^r}{H_\Phi^r} = \frac{v_0}{v_0} = \frac{1}{\sqrt{v_0\epsilon_0}} = \zeta_0 = 376.7 \text{ ohms} \qquad (34)$$

ζ_0 may be looked upon as a resistance, v_0 as a velocity characteristic of the propagation of electromagnetic waves in space. The following relations obtain in the far zone:

$$E_0^r = v_0 B_\Phi^r; \qquad H_\Phi^r = v_0 D_\Phi^r \qquad (35)$$

$$E_0^r = \zeta_0 H_\Phi^r; \qquad B_\Phi^r = \zeta_0 D_\Phi^r \qquad (36)$$

12. Hertzian Dipole.—A discussion of the electromagnetic field of current-carrying conductors is not complete, at least for historical reasons, without mention of the classical radiator, the Hertzian dipole. It is considered here instead of in Volume II

because it is physically rather than technically interesting. In macroscopic terms, it is equivalent to a pair of small metal spheres or circular metal disks each of radius b that are connected by a short wire of length $2h$ and radius a. The dimensions satisfy the inequalities

$$a << b << h <<< \lambda \tag{1}$$

The two spheres or disks are alternately charged with a total charge $+Q$ and $-Q$ by a periodically varying current of uniform amplitude I maintained in the wire. The scalar potential ϕ
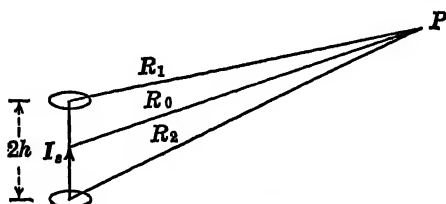

Fig. 12.1.—Hertzian dipole.

due to the two charged spheres or disks is obtained from the Helmholtz integral (1.33). At distances R_1 and R_2 from the centers of the spheres or disks (Fig. 12.1) that are large compared with the radii b of the disks,

$$\phi = \frac{Q}{4\pi\epsilon_0} \left\{ \frac{e^{-j\beta_0 R_1}}{R_1} - \frac{e^{-j\beta_0 R_2}}{R_2} \right\} \tag{2}$$

The vector potential due to the uniform current in the wire is

$$\mathbf{A} = \hat{s}A_s; \qquad A_s = \frac{I_s}{4\pi\nu_0} \int_{-h}^{+h} \frac{e^{-j\beta_0 R}}{R} \, ds' \tag{3}$$

If R_1, R_2, and R are sufficiently great so that terms of the form h^2/R^2 are negligible compared with unity, but not necessarily terms in h/R, it is a good approximation to write

$$R_1 \doteq R_0 - h \cos 0 = R_0 - (\hat{R}_0, \hat{s})h \tag{4}$$

$$R_2 \doteq R_0 + h \cos 0 = R_0 + (\hat{R}_0, \hat{s})h \tag{5}$$

$$R \doteq R_0 - s' \cos 0 = R_0 - (\hat{R}_0, s') \tag{6}$$

If (4) and (5) are substituted in (2) in both denominator and exponent, and terms in h^2/R_0^2 are neglected, the following expression is obtained:

$$\phi = \frac{2hQ}{4\pi\epsilon_0} e^{-j\beta_0 R_0} \left\{ \frac{j\beta_0}{R_0} + \frac{1}{R_0^2} \right\} (\hat{R}_0, \hat{s}) \tag{7}$$

In terms of the polarization vector defined by

$$\mathbf{p} \equiv s2hQ \tag{8}$$

this becomes

$$\phi = \frac{1}{4\pi\epsilon_0} \, e^{-j\beta_0 R_0} \left\{ \frac{j\beta_0}{R_0} + \frac{1}{R_0^2} \right\} (\hat{R}_0, \mathbf{p}) \tag{9}$$

If (6) is substituted in (3) in both denominator and exponent and the integration is carried out neglecting terms in h^2/R_0^2, the result is

$$\mathbf{A} = s A_s; \qquad A_s = \frac{2h I_s}{4\pi\nu_0} \frac{e^{-j\beta_0 R_0}}{R_0} \tag{10}$$

This may be expressed in terms of the polarization vector defined in (8) by noting that the surface equation of continuity when written at the boundary between the upper end of the wire and the attached disk may be written in the integrated form

$$j\omega Q = I_s \tag{11}$$

Using (8),

$$s2h I_s = j\omega \mathbf{p} \tag{12}$$

so that

$$\mathbf{A} = \frac{j\omega \mathbf{p}}{4\pi\nu_0} \frac{e^{-j\beta_0 R_0}}{R_0} \tag{13}$$

Thus (9) and (13) express the potential functions in terms of a *dipole* of complex moment $p = 2hQ$.

The electromagnetic field may be calculated directly from (9) and (13) using

$$\mathbf{E} = -\text{grad } \phi - j\omega \mathbf{A} \tag{14}$$
$$\mathbf{B} = \text{curl } \mathbf{A} \tag{15}$$

It is convenient to write the gradient in spherical coordinates R_0, Θ, Φ with the scalar potential ϕ independent of Φ. That is,

$$\text{grad } \phi = \hat{R}_0 \frac{\partial\phi}{\partial R_0} + \hat{\Theta} \frac{1}{R_0} \frac{\partial\phi}{\partial\Theta} \tag{16}$$

Upon carrying out the differentiation and combining terms, the electromagnetic field of a dipole is

$$\mathbf{E} = \frac{e^{-j\beta_0 R_0}}{4\pi\epsilon_0} \left[\frac{-\beta_0^2}{R_0} [\hat{R}_0, [\hat{R}_0, \mathbf{p}]] + \frac{j\beta_0}{R_0^2} \{3\hat{R}_0(\hat{R}_0, \mathbf{p}) - \mathbf{p}\} \right. \\ \left. + \frac{1}{R_0^3} \{3\hat{R}_0(\hat{R}_0, \mathbf{p}) - \mathbf{p}\} \right] \tag{17}$$

$$\mathbf{B} = \frac{-j\omega e^{-j\beta_0 R_0}}{4\pi \nu_0} [\hat{\mathbf{R}}_0, \mathbf{p}] \left\{ \frac{1}{R_0^2} + \frac{j\beta_0}{R_0} \right\} \tag{18}$$

subject to $h^2 << R_0^2$. (19)

If each element of length ds of a current-carrying conductor is provided with added fictitious surface charges $+Q$ and $-Q$ at its ends in order to make it electrically complete, it is electrically equivalent to a Hertzian dipole such that

$$\text{s}I, \, ds = j\omega\mathbf{p}; \qquad \mathbf{p} = Q\text{ds} \tag{20}$$

By integrating over the length of the conductor, the same expressions for the electromagnetic field of the conductor as a whole are obtained as using (7.15a,b). Direct comparison shows that (18) with (19) is the same as $d\mathbf{B}$ in (7.15b), but that (17) is very much more complicated than $d\mathbf{E}$ in (7.15a). The added terms in (17), including the entire $1/R_0^3$ term, are due to the fictitious charges at the ends of each element ds. These all vanish when the integration is carried out over a conductor with zero currents at the ends. Clearly, the analysis of current-carrying conductors by integrating a fictitious distribution of dipoles instead of the actual distributions of current and charge is unnecessarily complicated in general. On the other hand, at points in the far zone (17) and (18) reduce to the simple forms

$$\mathbf{E}^r = \nu_0 [\mathbf{B}^r, \hat{\mathbf{R}}_0] \tag{21}$$

$$\mathbf{B}^r = \frac{\beta_0^2 \nu_0}{4\pi \nu_0} [\hat{\mathbf{R}}_0, \mathbf{p}] \frac{e^{-j\beta_0 R_0}}{R_0} \tag{22}$$

These are in complete agreement with (11.25) and (11.18) if $d\mathbf{E}^r$ and $d\mathbf{B}^r$ are written for $\mathbf{E}^r$ and $\mathbf{B}^r$ in (21) and (22) and $\mathbf{p}$ is replaced by $\text{s}I, \, ds/j\omega$.

13. Spherical and Plane Electromagnetic Waves.—The complex field equations that define the complex amplitudes of the vectors $\mathbf{E}$ and $\mathbf{B}$ at points in space are

$$\begin{array}{ll} \text{div } \mathbf{E} = 0; & \text{curl } \mathbf{B} = j\frac{\beta_0^2}{\omega}\mathbf{E} \\ \text{curl } \mathbf{E} = -j\omega\mathbf{B}; & \text{div } \mathbf{B} = 0 \end{array} \tag{1}$$

By forming the curl of both sides of one of the curl equations and substituting from the other to separate the variables, the following equations are obtained:

$$\text{curl curl } \mathbf{E} - \beta_0^2 \mathbf{E} = 0; \qquad \text{curl curl } \mathbf{B} - \beta_0^2 \mathbf{B} = 0 \qquad (2)$$

As before,

$$\beta_0^2 = \frac{\omega^2 \epsilon_0}{v_0} = \frac{\omega^2}{v_0^2} \qquad (3)$$

The solutions of these equations for the electromagnetic field in the far zone of an antenna have been obtained in the form

$$\beta_0 R_0 >> 1 \begin{cases} E_R^r = 0 & B_R^r = 0 \\ E_\Theta^r = F(\Theta, R_0) & B_\Theta^r = 0 \\ E_\Phi^r = 0 & B_\Phi^r = \dfrac{F(\Theta, R_0)}{v_0} \\ \qquad F(\Theta, R_0) = f(\Theta, h) \dfrac{e^{-j\beta_0 R_0}}{R_0} \end{cases} \qquad (4)$$

The relatively simple field defined by (4) necessarily satisfies less intricate differential equations than the general vector relations (2). These simpler equations are easily obtained by direct evaluation of the curl curl in spherical coordinates using Appendix I and (4). The subscript 0 will be omitted temporarily.

$$\text{curl}_R \mathbf{E} \equiv \mathbf{G}_R = 0; \qquad \text{curl}_R \mathbf{B} \equiv \mathbf{C}_R = \frac{1}{v_0 R^2 \sin \Theta} \frac{\partial}{\partial \Theta} (RF \sin \Theta)$$

$$\text{curl}_\Theta \mathbf{E} \equiv \mathbf{G}_0 = 0; \qquad \text{curl}_\Theta \mathbf{B} \equiv \mathbf{C}_0 = \frac{-1}{v_0 R} \frac{\partial (RF)}{\partial R} \qquad (5)$$

$$\text{curl}_\Phi \mathbf{E} \equiv \mathbf{G}_\Phi = \frac{1}{R} \frac{\partial (RF)}{\partial R}; \qquad \text{curl}_\Phi \mathbf{B} \equiv \mathbf{C}_\Phi = 0$$

Repeating the process

$$\text{curl}_R \mathbf{G} = \frac{1}{R \sin \Theta} \frac{\partial}{\partial \Theta} (G_\Phi \sin \Theta); \qquad \text{curl}_R \mathbf{C} = 0$$

$$\text{curl}_\Theta \mathbf{G} = -\frac{1}{R} \frac{\partial (R G_\Phi)}{\partial R}; \qquad \text{curl}_0 \mathbf{C} = 0 \qquad (6)$$

$$\text{curl}_\Phi \mathbf{G} = 0; \qquad \text{curl}_\Phi \mathbf{C} = \frac{1}{R} \left(\frac{\partial (R C_0)}{\partial R} - \frac{\partial C_R}{\partial \Theta} \right)$$

Upon combining (5) and (6)

$$\text{curl}_R \text{ curl } \mathbf{E} = \frac{1}{R^2} g(\Theta, RF); \qquad \text{curl}_R \text{ curl } \mathbf{B} = 0$$

$$\text{curl}_\Theta \text{ curl } \mathbf{E} = -\frac{1}{R} \frac{\partial^2 (RF)}{\partial R^2}; \qquad \text{curl}_0 \text{ curl } \mathbf{B} = 0 \qquad (7)$$

$$\text{curl}_\Phi \text{ curl } \mathbf{E} = 0; \qquad \text{curl}_\Phi \text{ curl } \mathbf{B} = -\frac{1}{v_0 R} \frac{\partial^2 (RF)}{\partial R^2}$$
$$- \frac{1}{v_0 R^2} h(\Theta, RF)$$

$g(\Theta, RF)$ and $h(\Theta, RF)$ are functions of Θ and RF which it is not necessary to evaluate explicitly because terms in $1/R^2$ are to be neglected compared with those in $1/R$ in accordance with the conditions of the far zone assumed in deriving (4). Without the $1/R^2$ terms, (2) reduces to

$$\frac{\partial^2 (RF)}{\partial R^2} + \beta_0^2 (RF) = 0 \qquad (8)$$

This is satisfied by

$$RF = f(\Theta) e^{-j\beta_0 R} \qquad (9)$$

or, adding the subscript 0, which was omitted in (5) to (9) for convenience in writing,

$$F = f(\Theta) \frac{e^{-j\beta_0 R_0}}{R_0} \qquad (10)$$

Here $f(\Theta)$ is a constant of integration with respect to R_0. Clearly (10) is a solution of the same form as (4).[1] It is conveniently interpreted in terms of transverse spherical waves expanding radially with a constant velocity v_0. Equiphase surfaces are separated by a constant distance λ_0. The radial equation (8) which is a specialized form of (2) is called a spherical wave equation.

In a large proportion of the problems involving the far-zone field of an antenna (or any other arrangement containing periodically varying charges and currents), interest lies primarily in the field in a relatively small part of space. That is, it is frequently necessary to deal only with **E** and **B** in a region that is small compared with the distance R_0 from a point P_0 at its

---

[1] It is to be noted that a solution of the form $e^{+j\beta_0 R_0}/R_0$ also satisfies (8), just as integrals like (1.33) and (1.34), but written with $+j\beta_0$ in the exponents instead of with $-j\beta_0$, also satisfy the general potential equations (1.1) and (1.2). These mathematically possible solutions are discarded because they serve no useful purpose in the electromathematical model which seeks to predict theoretical analogues of experimental pointer readings. Expressions with $+j\beta_0$ instead of $-j\beta_0$ in the arguments of the exponential functions would have to be interpreted in *unbounded* space as advanced functions rather than as retarded ones. That is, an effect at a distance would be observed *before* rather than *after* the motion of charge took place which is supposed to produce it. Effects that precede their causes are not observed experimentally, so that they need not be formulated theoretically. If the region is *not* unbounded, solutions involving terms in $e^{+j\beta_0 R}$ may be required to satisfy the boundary conditions.

center to the origin O in the distant antenna. In terms of the spherical coordinates R, Θ, Φ, with origin at O, such a region may be identified with the volume

$$V = (R'' - R')R_0(\Theta'' - \Theta')R_0(\Phi'' - \Phi')\sin\theta \qquad (11)$$

Here $(R'' - R')$ is the radial thickness of V; $R_0(\Theta'' - \Theta')$ and $R_0(\Phi'' - \Phi')$ are the mean arcs subtended by it at its radial

Fig. 13.1.—Volume $V = (R'' - R')\ R_0(\Theta'' - \Theta')\ R_0(\Phi'' - \Phi')\ \sin\Theta$. P_0 is at the center of the shaded area.

center (Fig. 13.1). Let it be required that the radial thickness of the volume satisfy the following inequality:

$$(R'' - R') << R_0 \qquad (12)$$

Subject to (12) the distance $R = OP$ to any point P in V is approximately given by

$$R \doteq R_0 \qquad (13)$$

in amplitude factors. In arguments of periodic functions, (13) may not be used, and no restrictions whatsoever need be imposed upon the quantity

$$\beta_0(R'' - R') = \frac{2\pi(R'' - R')}{\lambda_0} \qquad (14)$$

other than those implied in (12) and the condition for the far zone

$$\beta_0 R_0 = \frac{2\pi R_0}{\lambda_0} >> 1 \qquad (15)$$

If R_0 is sufficiently great, $R'' - R'$ may be very large compared with λ_0. Let the mean lateral dimensions of the volume V be required to be no greater than its radial thickness. That is,

$$R_0(\Theta'' - \Theta') \le (R'' - R') \qquad (16a)$$
$$R_0(\Phi'' - \Phi') \sin \theta \le (R'' - R') \qquad (16b)$$

Hence, with (12),

$$(\Theta'' - \Theta') \le \left(\frac{R'' - R'}{R_0}\right) << 1 \qquad (16c)$$

$$(\Phi'' - \Phi') \sin \theta \le \left(\frac{R'' - R'}{R_0}\right) << 1 \qquad (16d)$$

If (16c) and (16d) are satisfied, the volume V does not differ greatly from a parallelepiped of sides $(R'' - R')$, $R_0(\Theta'' - \Theta')$, $R_0(\Phi'' - \Phi') \sin \theta$.

Let the direction cosines of the radial line $\mathsf{R}_0 = OP_0$ be expressed in terms of a system of rectangular coordinates x, y, z, with origin also at O. The direction cosines of R_0 are

$$l_0 = \cos (R_0,x); \qquad m_0 = \cos (R_0,y); \qquad n_0 = \cos (R_0,z) \quad (17a)$$

The components of R_0 along the coordinate axes are x_0, y_0, z_0. They are

$$x_0 = R_0 l_0; \qquad y_0 = R_0 m_0; \qquad z_0 = R_0 n_0 \qquad (17b)$$

From the Pythagorean theorem and (17a,b) it follows that

$$\frac{x_0^2 + y_0^2 + z_0^2}{R_0^2} = l_0^2 + m_0^2 + n_0^2 = 1 \qquad (18)$$

By substituting for l_0, m_0, n_0 from (17b) in the right side of (18),

$$R_0 = x_0 l_0 + y_0 m_0 + z_0 n_0 \qquad (19)$$

Clearly any line s lying parallel to OP_0 has the same direction cosines as R_0. If the coordinates of its end point are x, y, z,

$$s = x l_0 + y m_0 + z n_0 \qquad (20)$$

As a consequence of (16) which restricts $(\Theta'' - \Theta')$ and $(\Phi'' - \Phi')$ to very small angles, it may be assumed that the *direction* of any radial vector $\mathsf{R} = OP$ terminating at P in V is sensibly the

same as that of $R_0 = OP_0$. If the coordinates of P are x, y, z, then

$$R \doteq s = xl_0 + ym_0 + zn_0 \tag{21}$$

With (13) in amplitude factors and (21) in arguments, (10) may be written

$$F \doteq Ke^{-j\beta_0 s} \tag{22}$$

with

$$K = \frac{f(\Theta)}{R} \doteq \frac{f(\Theta)}{R_0} = \text{const.} \tag{23}$$

If the time factor is included,

$$Fe^{j\omega t} \doteq Ke^{j(\omega t - \beta_0 s)} \tag{24}$$

This expression characterizes the approximate distribution of both **E** and **B** in the volume V; it is readily interpreted in terms of a simple wave picture. The relation (20) defines a plane at right angles to s and at a distance s from the origin. Consequently, (24) may be described using a picture of *plane* equiphase surfaces at *right angles to s* and traveling along s with a constant velocity v_0. That is, the arcs of radially expanding spherical equiphase surfaces defined by (10) which pass through the volume V may be assumed to be approximately plane and of constant amplitude provided the distance of V from the source is sufficiently great, and the solid angle subtended by it is small. The relation (24) satisfies the plane wave equation

$$\frac{d^2F}{ds^2} + \beta_0^2 F = 0 \tag{25}$$

This is readily obtained by specializing (8) according to (13) and (21). The former permits the assumption that $R \doteq R_0$ is sensibly constant in the differentiation; the latter identifies R approximately with s.

In problems involving specifically the field in a volume V in the wave zone, and not directly the antenna or its location, it is convenient to use a set of rectangular coordinates x, y, z defined as follows: the x axis lies along the line OP_0 so that it coincides with the spherical coordinate R_0; the y axis is tangent to and in the direction of the coordinate Φ_0 at P_0; the z axis is tangent to and in the direction $-\Theta_0$ at P_0 (Fig. 13.1). With this choice, x, y, z are a right-handed system, and it follows that

the electromagnetic field at P_0, and approximately throughout V, is given by

$$E_\Theta{}^r \doteq -E_z{}^r; \qquad B_\Phi{}^r \doteq B_y{}^r \qquad (26)$$

Also,

$$s = x \qquad (27)$$

With this notation, (24) reduces to the following simple form defining the far-zone field in a volume V:

$$-E_{z\text{inst}}^r = v_0 B_{y\text{inst}}^r = Ke^{j(\omega t - \beta_0 x)} \qquad (28)$$

This expression may be interpreted in terms of plane transverse waves normal to the x axis and traveling along this with a constant velocity v_0.

14. Polarized Electromagnetic Waves.—The electromagnetic field in a sufficiently small region V in the wave zone of an antenna is defined approximately by the following relations:

$$\mathbf{E}_{\text{inst}} = -\hat{z}E_z{}^r e^{j\omega t} = -\hat{z}Ke^{j(\omega t - \beta_0 x)} \qquad (1a)$$

$$\mathbf{B}_{\text{inst}} = \hat{y}B_y{}^r e^{j\omega t} = \frac{\hat{y}Ke^{j(\omega t - \beta_0 x)}}{v_0} \qquad (1b)$$

The positive x axis lies along the radial vector $\mathbf{R}_0 = OP_0$ locating the center P_0 of the region V. From the real part of (1) it can be concluded that $\mathbf{E}_{\text{inst}}$ and $\mathbf{B}_{\text{inst}}$ at every point change continuously and periodically in length as time passes, but that they are always directed parallel, respectively, to the z and the y axes at the point P_0 and approximately parallel to these axes throughout the volume V. The $\mathbf{E}$ field is therefore said to be linearly polarized[1] along the z axis; the $\mathbf{B}$ field is linearly polarized along the y axis. Both fields are *polarized in the plane at right angles to the direction of propagation of the equiphase surfaces or wave fronts*. It is important to note that all statements which have been made thus far are valid regardless of the location of the region V relative to the antenna provided only that it is in the far zone. The amplitude factor K in (1) depends both upon the spherical angle Θ_0 locating the vector $\mathbf{R}_0 = OP_0$ and upon the length of R_0. But for all values of Θ and of R_0 (which satisfy $\beta_0 R_0 \gg 1$) the electromagnetic

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[1] Care must be exercised not to confuse the polarization of an electromagnetic vector with the volume density of polarization $\mathbf{P}$ describing the orientation of dipoles. They are entirely different concepts which are not related.

field at P_0, and approximately throughout V, always lies in a plane at right angles to OP_0. Consequently, a change in the orientation of the antenna at O can at most alter the amplitude K or rotate the axes of polarization of $\mathbf{E}$ and $\mathbf{B}$ in this plane. Therefore, the electric and magnetic fields in the neighborhood of any point in the far zone of a straight antenna are always linearly polarized along mutually perpendicular axes that lie in a plane at right angles to the line joining the point with the center of the antenna.

If the electromagnetic field in V is not due to a single antenna but to a combination of antennas grouped at O and oriented relative to each other in any way whatsoever (*e.g.*, two antennas at right angles to each other), the resultant electromagnetic field in V due to the entire array is still in a plane at right angles to OP_0. If the x axis is along OP_0 as before, and the y and z axes are fixed in any positions that make a right-handed system, then, in the most general case, $\mathbf{E}_{\text{inst}}$ and $\mathbf{B}_{\text{inst}}$ may have components with different complex amplitudes along the y and z axes. Assuming that only a single frequency is involved, $\mathbf{E}_{\text{inst}}$ and $\mathbf{B}_{\text{inst}}$ may be expressed as follows:

$$\mathbf{E}_{\text{inst}} \doteq -(\hat{z}E_z^r + \hat{y}E_y^r)e^{j\omega t} \tag{2a}$$

$$\mathbf{B}_{\text{inst}} \doteq (\hat{y}B_y^r + \hat{z}B_z^r)e^{j\omega t} \tag{2b}$$

where

$$-E_z^r = v_0 B_y^r = Ke^{-j\beta_0 x} \tag{3a}$$

$$-E_y^r = v_0 B_z^r = Ne^{-j\beta_0 x} \tag{3b}$$

The complex amplitudes K and N may be written in terms of real amplitudes (a and b) and phase angles (g and p). Thus

$$K = ae^{-jg}; \qquad N = be^{-jp} \tag{4}$$

With (3) and (4) the real parts of (2a,b) become

$$-E_{z\text{inst}}^r = v_0 B_{y\text{inst}}^r = a \cos (\omega t - \beta_0 x - g) \equiv a \cos \psi \tag{5a}$$

$$-E_{y\text{inst}}^r = v_0 B_{z\text{inst}}^r = b \cos (\omega t - \beta_0 x - p) \equiv b \cos (\psi - \delta) \tag{5b}$$

The symbols ψ and δ are introduced as a shorthand for

$$\psi \equiv \omega t - \beta_0 x - g \tag{6}$$

$$\delta \equiv p - g \tag{7}$$

It is readily verified that the following expressions are true:

$$\left(\frac{E_z^2}{a^2} + \frac{E_y^2}{b^2} - \frac{2E_y E_z \cos \delta}{ab}\right)_{\text{inst}} = \sin^2 \delta \qquad (8a)$$

$$\left(\frac{B_y^2}{a^2} + \frac{B_z^2}{b^2} - \frac{2B_y B_z \cos \delta}{ab}\right)_{\text{inst}} = \frac{\sin^2 \delta}{v_0^2} \qquad (8b)$$

These equations define the loci of the ends of the vectors E_{inst} and B_{inst} as ψ varies. At a given distance x from the source, the argument ψ can change only in time. Hence (8) gives the time variation in length and orientation of E_{inst} or B_{inst} at all points in V that lie in the yz plane. The relations (8) are the equations of ellipses in the general case in which the major and minor axes do not lie along the coordinate axes as shown in Fig. 14.1. The major axis of the ellipse characterizing E_{inst} is along the minor axis of the ellipse belonging to B_{inst}. Under all conditions, the ellipses lie in the yz plane if the x axis is chosen to pass through the origin of the periodically varying charge in the antennas and through the center of the volume V in which the field is calculated. The orientation of the ellipses in the yz plane and their eccentricity depend upon the magnitudes of the amplitudes a and b of the components of E_{inst} or B_{inst} along the coordinate axes and upon the phase difference δ between these components. The most general electro-magnetic field in a small region in the wave zone of a group of antennas operating at a single frequency is elliptically polarized in a plane at right angles to the line joining the source with the region.

FIG. 14.1.—Instantaneous electric and magnetic vectors and loci of their end points in an elliptically polarised field.

Since (8) gives the locus of the ends of E_{inst} and B_{inst} in general, it must include the case of linear polarization which has been shown to characterize the far-zone field of a *single* antenna. When the components of E_{inst} (or B_{inst}) are mutually in phase with $\delta = 0$, or exactly opposite in phase with $\delta = \pi$, (8a) becomes

$$\frac{E_{z\text{inst}}}{a} = \pm \frac{E_{y\text{inst}}}{b} \qquad (9)$$

This is a linear relation between $E_{y\text{inst}}$ and $E_{z\text{inst}}$, so that the ellipse

is degenerate and a straight line with an orientation that depends upon the amplitudes a and b of the components. If $b = 0$, $E_{y\text{inst}} = 0$ and the axis of polarization is the z axis. If $a = b$, the axis of polarization is the 45° line.

As a result of the fact that E_{inst} and B_{inst} are elliptically polarized in general, there is always a preferred direction for E_{inst} and one at right angles to this for B_{inst} unless the ellipse degenerates into a circle. This is true when the components of E_{inst} (or those of B_{inst}) have the same amplitudes and differ in phase by $\pi/2$. Thus, if $a = b$ and $\delta = \pm\pi/2$, (8) reduces to

$$E_{y\text{inst}}^2 + E_{z\text{inst}}^2 = b^2 \tag{10a}$$

$$B_{y\text{inst}}^2 + B_{z\text{inst}}^2 = \frac{b^2}{v_0^2} \tag{10b}$$

These are the equations of circles in the yz plane.

Summarizing, the electric and the magnetic fields in a sufficiently small volume V in the far zone of a monochromatic source are in general elliptically polarized; in special cases they are circularly or linearly polarized. The ellipse, the circle, or the straight line always lies in the plane at right angles to the direction of propagation of the approximately plane waves that may be pictured to pass through V. It is to be noted that the electromagnetic field associated with natural sunlight is *unpolarized*. With the aid of special devices, however, it is possible to produce an electromagnetic light field that is elliptically, circularly, or linearly polarized. On the other hand, it is not possible to produce an unpolarized electromagnetic field using a monochromatic source such as a group of antennas operating at a single frequency.

15. Electromagnetic Waves in Unbounded Simple Media. The general expression (7.1·1a,b) and all special formulas derived from them for the electromagnetic field at points in empty space may be modified to apply to a simple medium that replaces empty space merely by writing ξ for ϵ_0, v for v_0, β for β_0, and v for v_0. For example, the important case of a cylindrical conductor immersed in a simple medium may be obtained from (11.30) and (11.31). Thus

$$\mathbf{E}^r = \hat{\Theta}\, \frac{j\omega}{4\pi v}\, \frac{\exp\,(-j\beta R_0)}{R_0} \int_{-h}^{h} I_z' \exp\,(j\beta z'\cos\Theta)\sin\Theta\, dz' \tag{1}$$

$$\mathbf{B^r} = \hat{\Phi} \frac{j\beta}{4\pi v} \frac{\exp(-j\beta R_0)}{R_0} \int_{-h}^{h} I'_s \exp(j\beta z' \cos \Theta) \sin \Theta \, dz' \quad (2)$$

Let it be assumed that v is real. Also

$$\beta = \beta_s - j\alpha_s \quad (3)$$

Hence

$$\mathbf{E^r} = \hat{\Theta} \frac{j\omega}{4\pi v} \frac{e^{-(\alpha_s + j\beta_s)R_0}}{R_0} \int_{-h}^{h} I'_s e^{(\alpha_s + j\beta_s)z' \cos \Theta} \sin \Theta \, dz' \quad (4)$$

$$\mathbf{B^r} = \hat{\Phi} \frac{j\beta_s \sqrt{1 + \left(\frac{\alpha_s}{\beta_s}\right)^2}}{4\pi v} \frac{e^{-(\alpha_s + j\beta_s)R_0 - j \tan^{-1}(\alpha_s/\beta_s)}}{R_0}$$
$$\int_{-h}^{h} I'_s e^{(\alpha_s + j\beta_s)z' \cos \Theta} \sin \Theta \, dz' \quad (5)$$

It is to be noted that $\mathbf{E^r}$ and $\mathbf{B^r}$ are not in phase. Thus the relation (11.17) using (III.15.17) becomes

$$\mathbf{E^r} = v[\mathbf{B^r}, \mathbf{R}_0] = \frac{v_s}{\sqrt{1 + \frac{\alpha_s^2}{\beta_s^2}}} e^{j \tan^{-1}(\alpha_s/\beta_s)} [\mathbf{B^r}, \mathbf{R}_0^r] \quad (6)$$

If the simple medium is a very poor conductor in which $\alpha_s << \beta_s$, the phase angle $\tan^{-1}(\alpha_s/\beta_s) \doteq \alpha_s/\beta_s$ is negligible and $\mathbf{E^r}$ and $\mathbf{B^r}$ are practically in phase. They are exactly in phase in a perfectly nonconducting dielectric.

In a good conductor $\beta_s = \alpha_s$ so that the phase angle

$$\tan^{-1}\left(\frac{\alpha_s}{\beta_s}\right) = \frac{\pi}{4}$$

Thus in a good conductor the electric field leads the magnetic field by an eighth of a period. This is considered in greater detail in Chapter V.

The wave picture of spherical equiphase surfaces expanding with a constant radial velocity may be used to describe the far-zone field due to current and charge *in an element dz'* much as in space. In an unbounded simple medium, as assumed in (4) and (5), the surfaces of constant phase or wave fronts travel with a radial velocity $v_s = v_c/f(h_s) = v_0/N_s$ which is not independent of frequency. (v_s/v_c) is shown as a function of $h_s = \sigma_r/\omega \epsilon_s$ in Fig. 15.1. In the special case of a poorly conducting medium, $v_s \doteq v_c = v_0 \sqrt{v_r/\epsilon_{cr}}$ which depends on frequency in ϵ_{cr} and is

smaller than $v_0 = 3 \times 10^8$ meters/second. In a highly conducting medium, $v_s = \sqrt{2\omega\nu/\sigma_e}$, a quantity that varies greatly with frequency and is very much less than v_0 as shown in Fig. 15.1. In a medium that is not a perfect nonconductor, the amplitude associated with the expanding surfaces of constant phase decreases more rapidly than $1/R_0$ owing to the exponential attenuating factor $e^{-\alpha_s R_0}$.

Fig. 15.1.—The ratio v_s/v_e as a function of $h_e = \sigma_e/\omega\epsilon_e$.

Although the wave interpretation for the radiation-zone field in a simple medium is formally like that for empty space, the two cases are, nevertheless, quite different from the point of view of the atomic model. In the case of space, each wave front may be associated entirely with the current and charge in the conductor from which it travels outward with a velocity that approaches the constant value v_0 in the wave zone. In a simple medium, itself composed of groups of charges in the form of atoms and molecules, the periodically varying charge and current in the conductor produce similar periodic variations in all the charges and groups of charges forming the simple medium. That is, the value of **E** or **B** at any point in the medium must be interpreted in terms of an integral over the cylindrical conductor and another integral over the entire medium in which it is immersed. Clearly, if an equiphase surface is associated with each element of periodically varying charge, then there must be infinitely many such surfaces expanding from all elements (both of the cylindrical conductor and of the medium) with a velocity that approaches v_0 at a sufficiently great distance from each element. It must be concluded from (4) that the composite

retarded effect of the periodically varying charges in the antenna combined with that of the similarly varying charges in the medium is formally equivalent to a retarded effect due to the charges *in the antenna alone*, but with a change in the time factor from $\left(t - \dfrac{R}{v_0}\right)$ to $\left(t - \dfrac{RN_s}{v_0}\right)$.

In order to explain this effect qualitatively in terms of the atomic model, let it be assumed that polarization and magnetization responses are instantaneous in the dielectric medium and that this is perfectly nonconducting. The field at any point in the medium is the resultant of the field $\mathbf{E}_a$, $\mathbf{B}_a$ due to the antenna and the fields $\mathbf{E}_i$, $\mathbf{B}_i$ due to each volume element $\Delta\tau_i$. The former is determined by the condition of charge at a time $t - (R/v_0)$ where R is the distance from the point P to the center of the antenna (in the wave zone), the latter by the condition of polarization and magnetization of the ith volume element at a time $t - (R_i/v_0)$. If it is assumed for simplicity that volume densities of polarization and magnetization are instantly adjusted to be proportional, respectively, to the average values of $\mathbf{E}$ and $\mathbf{B}$ at the center of each volume element, it is clear that the values of $\mathbf{E}$ or $\mathbf{B}$ due to the antenna and that due to a single volume element are in phase at points that lie beyond the element along a line drawn from the antenna through its center. At points that do not lie along this line, the field due to the element lags behind that due to the antenna because the total distance from the antenna to the element to P is greater than that directly from the antenna to P. It follows, therefore, that the resultant field at any point due to a distant antenna and all volume elements in the polarizable and magnetizable medium, which are near enough to contribute significantly to the field, must lag behind that due to the antenna alone. The amount of this lag depends on the relative magnitude of the field due to the antenna and the field due to the medium. Since the latter depends upon $\mathbf{P}$ and $-\mathbf{M}$, and these in turn are proportional, respectively, to $\mathbf{E}$ and $\mathbf{B}$, it is clear that the amount of lag must depend upon the magnitude of the proportionality constants $(\epsilon_r - 1)$ and $(\nu_r - 1)$ in the relations

$$\mathbf{P} = (\epsilon_r - 1)\epsilon_0\mathbf{E}; \qquad -\mathbf{M} = (\nu_r - 1)\nu_0\mathbf{B} \tag{7}$$

As a consequence of the lag in the resultant value of the field at

any point behind that due to the antenna alone, the polarization and magnetization, which adjust themselves instantly to the resultant field, also lag behind the field due to the antenna. In the final steady state, the field at any point in the medium lags behind that due to the antenna alone by an amount that depends upon the relative magnitudes of the contributions to the field by the antenna and by the medium. Although the field and, hence, the polarization and magnetization at every point are thus determined by the earlier conditions of polarization and magnetization of all volume elements in all directions, it is clear that the contributions from atoms in the straight line between P and the antenna must be greatest because the total distance is shortest. If it is assumed, quite arbitrarily for example, that the time lag δ at any distance R from the antenna is proportional to that distance, then it is possible to write $\left(t - \dfrac{R}{v_0} - \delta\right) = \left(t - \dfrac{R}{v_0} - kR\right)$ instead of $\left(t - \dfrac{R}{v_0}\right)$ in a dielectric medium with k a constant. This may be written in the form $t - \dfrac{R}{v_0}(1 + kv_0) = t - \dfrac{rN_s}{v_0}$. Here $N_s = (1 + kv_0)$ is a constant. It is thus seen in a qualitative way how a polarizable and magnetizable medium can produce a progressively increasing phase lag in the field of an antenna in the case of an instantaneous polarization and magnetization response. If the response is not instantaneous or if the medium is conducting, an additional lag is introduced.

RADIATION OF ENERGY

16. Electromagnetic Field and Vector Potential in the Radiation Zone of an Antenna.—The vector potential at points in space in the far zone of a cylindrical antenna is readily obtained from (11.19) using (11.6) and (11.7) subject to (11.1) and (11.2). It is

$$\mathbf{A}^r = \frac{\$}{4\pi v_0}\frac{e^{-j\beta_0 R_0}}{R_0}\int_{-h}^{h} I_s' e^{j\beta_0(\hat{\mathbf{R}}_0, \mathbf{s}')}ds' \tag{1}$$

Comparison of (1) with (11.8b) shows that

$$\mathbf{B}^r = -j\beta_0[\hat{\mathbf{R}}_0, \mathbf{A}^r] \tag{2}$$

Using (11.17) with $\beta_0 = \omega/v_0$

$$\mathbf{E}^r = v_0[\mathbf{B}^r, \hat{\mathbf{R}}_0] = -j\omega[[\hat{\mathbf{R}}_0, \mathbf{A}^r], \hat{\mathbf{R}}_0] = j\omega[\hat{\mathbf{R}}_0, [\hat{\mathbf{R}}_0, \mathbf{A}^r]] \tag{3}$$

The entire electromagnetic field in the far zone is thus readily computed from the vector potential. This is especially convenient in analyzing arrays of antennas, particularly if the individual elements are not parallel.

Written in polar coordinates R_0, Θ, Φ, with origin at the center of the antenna,

$$\mathbf{A}^r = \hat{\mathbf{R}}_0 A_R^r + \hat{\mathbf{\Theta}} A_\Theta{}^r + \hat{\mathbf{\Phi}} A_\Phi{}^r \tag{4}$$

Since

$$[\hat{\mathbf{R}}_0, \hat{\mathbf{R}}_0] = 0; \qquad [\hat{\mathbf{R}}_0, \hat{\mathbf{\Theta}}] = \hat{\mathbf{\Phi}}; \qquad [\hat{\mathbf{R}}_0, \hat{\mathbf{\Phi}}] = -\hat{\mathbf{\Theta}} \tag{5}$$

$$\mathbf{B}^r = -j\beta_0 \{\hat{\mathbf{\Phi}} A_\Theta{}^r - \hat{\mathbf{\Theta}} A_\Phi{}^r\} \tag{6}$$

$$\mathbf{E}^r = -j\omega \{\hat{\mathbf{\Theta}} A_\Theta{}^r + \hat{\mathbf{\Phi}} A_\Phi{}^r\} \tag{7}$$

If the axis of the antenna lies along the z axis so that

$$\mathbf{A}^r = \hat{z} A_z^r \tag{8}$$

$$A_\Phi{}^r = 0; \qquad A_\Theta{}^r = -A_z^r \sin\Theta \tag{9}$$

In this case

$$\mathbf{B}^r = -j\beta_0 A_\Theta{}^r \hat{\mathbf{\Phi}} \tag{10}$$

$$\mathbf{E}^r = -j\omega A_\Theta{}^r \hat{\mathbf{\Theta}} \tag{11}$$

The above relations also can be derived from the fundamental equations

$$\mathbf{B} = \text{curl } \mathbf{A} \tag{12}$$

$$\mathbf{E} = -\text{grad } \phi - j\omega\mathbf{A} = -\frac{j\omega}{\beta_0^2}\{\text{grad div }\mathbf{A} + \beta_0^2 \mathbf{A}\} \tag{13}$$

if terms in $1/R^2$ and higher powers of $1/R$ are neglected and (1) is used. Thus, since $\mathbf{A}^r$ involves only first-power terms in $1/R_0$, it follows that

$$B_R^r = \text{curl}_R \mathbf{A}^r = \frac{1}{R_0 \sin\Theta}\left\{\frac{\partial}{\partial\Theta}(A_\Phi{}^r \sin\Theta) - \frac{\partial A_\Theta{}^r}{\partial\Phi}\right\} \doteq 0 \tag{14}$$

B_R^r is set equal to zero because all terms are of order $1/R_0^2$ or higher.

$$B_\Theta{}^r = \text{curl}_\Theta \mathbf{A}^r = \frac{1}{R_0 \sin\Theta}\frac{\partial A_R^r}{\partial\Phi} - \frac{1}{R_0}\frac{\partial}{\partial R_0}(R_0 A_\Phi{}^r)$$
$$\doteq -\frac{1}{R_0}\frac{\partial}{\partial R_0}(R_0 A_\Phi{}^r) \tag{15}$$

$$B_\Phi{}^r = \text{curl}_\Phi \mathbf{A}^r = \frac{1}{R_0}\left[\frac{\partial}{\partial R_0}(R_0 A_\Theta) - \frac{\partial A_R}{\partial\Theta}\right] \doteq \frac{1}{R_0}\frac{\partial}{\partial R_0}(R_0 A_\Theta{}^r) \tag{16}$$

From (1),

$$\frac{1}{R_0} \frac{\partial}{\partial R_0} (R_0 A^r) = -j\beta_0 A^r \tag{17}$$

so that with (14) through (17)

$$B_R^r \doteq 0; \qquad B_\Theta^r \doteq j\beta_0 A_\Phi^r; \qquad B_\Phi^r \doteq -j\beta_0 A_\Theta^r \tag{18}$$

in agreement with (6). The expression (7) for E^r can be derived in a similar way using (13), but it is in any case simpler to use (3) once B^r has been determined.

It is interesting to interpret (13) in terms of the far-zone field (11) for an antenna along the z axis. In coordinate form (11) is

$$E_R^r = -\frac{\partial \phi^r}{\partial R_0} - j\omega A_R^r \tag{19}$$

$$E_\Theta^r = -\frac{1}{R_0} \frac{\partial \phi^r}{\partial \Theta} - j\omega A_\Theta^r \tag{20}$$

$$E_\Phi^r = -\frac{1}{R_0 \sin \Theta} \frac{\partial \phi^r}{\partial \Phi} - j\omega A_\Phi^r \tag{21}$$

As a result of rotational symmetry and the assumed condition that no current circulates around the circumference of the conductor, both $\partial \phi^r / \partial \Phi$ and A_Φ^r are zero, so that E_Φ^r vanishes. Since the integral for ϕ^r is of order $1/R_0$, it follows directly that $\frac{1}{R_0} \frac{\partial \phi^r}{\partial \Theta}$ is of order $1/R_0^2$ or higher so that it is negligible in the far zone where, in agreement with (11), $E_\Theta^r = -j\omega A_\Theta^r$. Both $\partial \phi^r / \partial R_0$ and A_R^r involve terms of magnitude $1/R_0$ so that neither is negligible. Since E_R^r is negligible in the far zone, it must be concluded that $\partial \phi^r / \partial R_0$ and $j\omega A_R^r$ can differ at most by terms in $1/R_0^2$ and not by terms in $1/R_0$. That is, in the far zone neglecting terms in $1/R_0^2$ and higher,

$$-\frac{\partial \phi^r}{\partial R_0} \doteq j\omega A_R^r \tag{22}$$

This can be verified from the integrals for ϕ^r and A_R^r using the equation of continuity. Although A_R^r is not negligible in the far zone, it is not required in computing the far-zone electromagnetic field from the vector potential.

Because the vector potential due to a current-carrying conductor is always in the direction of the current at all points

in space, it is usually simpler to combine the vector potentials due to a number of nonparallel antennas and compute the electromagnetic field from their resultant rather than combine the fields computed separately. This is illustrated in the analysis of the rhombic antenna in Volume II.

17. Radiation Functions of the Far-zone Field of Antennas. The general expression (16.1) for the vector potential in the radiation zone of a cylindrical antenna that has its axis along the direction $\hat{s}$ may be written as follows, noting that

$$\zeta_0 = \frac{1}{\sqrt{\nu_0\epsilon_0}} = \frac{v_0}{\nu_0}$$

$$\mathbf{A}^r = \hat{s}A_s^r; \qquad A_s^r = \left(\frac{\zeta_0 I_p}{4\pi\omega}\frac{e^{-j\beta_0 R_0}}{R}\right)\left(\beta_0 \int_{-h}^{h} f(s')e^{j\beta_0(\hat{R}_0,\hat{s}')}ds'\right) \quad (1)$$

I_p is the complex amplitude of current at a convenient but quite arbitrary reference point p in the antenna. Reference currents usually selected are either the input current at the origin $s' = 0$, or the maximum current wherever that may be. $f(s')$ is a dimensionless distribution function which is complex in general. Specifically

$$I_s' = I_p f(s') \tag{2}$$

The components A_Θ^r and A_Φ^r, which are alone important in computing the electromagnetic field in the far zone, may be referred to any conveniently oriented set of polar coordinates R_0, Θ, Φ.

$$A_\Theta^r = (\hat{\Theta},\hat{s})A_s^r; \qquad A_\Phi^r = (\hat{\Phi},\hat{s})A_s^r \tag{3}$$

If an array consists of a number of antennas oriented in arbitrary ways, a vector potential like (1) is defined for each straight section. Then Θ and Φ components *referred to a single set of polar coordinates* are obtained as in (3). The values of A_Θ^r and A_Φ^r due to the entire array are the sums of the several components. Thus for n units or sections,

$$A_\Theta^r = \sum_{i=1}^{n} A_{0i}^r; \qquad A_\Phi^r = \sum_{i=1}^{n} A_{\Phi i}^r \tag{4}$$

The complex Poynting vector is defined in space by

$$\mathbf{S} = \tfrac{1}{2}\nu_0[\mathbf{E},\mathbf{B}^*] \tag{5}$$

Here $\mathbf{B}^*$ is the complex conjugate of $\mathbf{B}$. Using (16.6), (16.7), and

$$\beta_0 = \omega/v_0 = \omega \sqrt{\epsilon_0/\nu_0}; \quad \zeta_0 = 1/\sqrt{\nu_0 \epsilon_0}; \quad [\hat{\Theta}, \hat{\Phi}] = \hat{R}_0,$$

$$\mathbf{S} = \frac{\omega^2}{2\zeta_0} \{A_\Theta{}^r A_\Theta{}^{r*} + A_\Phi{}^r A_\Phi{}^{r*}\} \hat{R}_0 = \frac{\omega^2}{2\zeta_0} \{A_\Theta{}^{r2} + A_\Phi{}^{r2}\} \hat{R}_0 \quad (6)$$

It follows from (6) that the Poynting vector in the far zone is real and directed radially outward on each spherical surface of constant phase.

The complex electromagnetic energy-transfer function

$$T = T_r + jT_i$$

is real if $\mathbf{S}$ is real.

$$T = T_r = \int_{\Sigma_r} (\hat{n}, \mathbf{S}) d\sigma; \qquad T_i = 0 \quad (7)$$

Choosing Σ_r to be the surface of a great sphere,

$$T = R_0^2 \int_0^{2\pi} \int_0^\pi S \sin \Theta \, d\Theta \, d\Phi \quad (8)$$

With (6),

$$T = \frac{\omega^2 R_0^2}{2\zeta_0} \int_0^{2\pi} \int_0^\pi \{A_\Theta{}^{r2} + A_\Phi{}^{r2}\} \sin \Theta \, d\Theta \, d\Phi \quad (9)$$

Here T is a measure of the total power transferred over a time average from the moving charges within a sphere of radius R_0 that contribute to $A_\Theta{}^r$ and $A_\Phi{}^r$ to moving charges outside this sphere.

The function $(A_\Theta{}^{r2} + A_\Phi{}^{r2})$ contains factors that are not functions of Θ and Φ. They can be removed from under the sign of integration to leave a dimensionless function $K_p^2(\Theta, \Phi)$ in the integrand. This is the *space-radiation function* and is defined by

$$K_p^2(\Theta, \Phi) = \left(\frac{4\pi\omega R_0}{\zeta_0 I_p}\right)^2 (A_\Theta{}^{r2} + A_\Phi{}^{r2}) = \left(\frac{4\pi\omega R_0}{\zeta_0}\right)^2 \left(\frac{1}{I_p I_p{}^*}\right)$$
$$(A_\Theta{}^r A_\Theta{}^{r*} + A_\Phi{}^r A_\Phi{}^{r*}) \quad (10)$$

The subscript on K is the same as on I since its value depends on the choice of reference current. Using (10) in (9)

$$T = \frac{I_p^2 \zeta_0}{32\pi^2} \int_0^{2\pi} \int_0^\pi K_p^2(\Theta, \Phi) \sin \Theta \, d\Theta \, d\Phi \quad (11)$$

If rotational symmetry with respect to the coordinate Φ obtains, as in the case of a single antenna with its axis along the z axis

about which Φ is measured, $K_p^2(\Theta,\Phi)$ reduces to $K_p^2(\Theta)$. In this case,

$$T = \frac{I_p^2 \zeta_0}{16\pi} \int_0^\pi K_p^2(\Theta) \sin \Theta \, d\Theta \qquad (12)$$

The function T in (11) or (12) is a measure of the time-average rate of decrease of energy functions within the great sphere of radius R_0 and of the corresponding time-average rate of increase of energy functions in the rest of the universe. It is the total *transferred or radiated power* over a time average. It is independent of the radius of the sphere provided R_0 is large enough so that all points of the sphere are in the far zone of the currents contributing to the field.

The Poynting vector may be expressed in terms of the space radiation function.

$$\mathsf{S} = \frac{I_p^2 \zeta_0}{32\pi^2 R_0^2} K_p^2(\Theta,\Phi)\hat{\mathsf{R}}_0 \qquad (13)$$

The radiated power divided by the square of the r.m.s. reference current is dimensionally a resistance. It is given the name *external or radiation resistance referred to the current* I_p and assigned the symbol R_p^e. The superscript stands for external; the subscript is the same as that on I_p. Thus

$$\mathsf{R}_p^e \equiv \frac{2T}{I_p^2} = \frac{\zeta_0}{16\pi^2} \int_0^{2\pi} \int_0^\pi K_p^2(\Theta,\Phi) \sin \Theta \, d\Theta \, d\Phi \qquad (14)$$

If I_p is chosen to be the input current I_0, R_0^e is the input resistance, and this is a physically measurable quantity provided the currents entering and leaving the antenna are equal. If I_p is not the input current at the driving terminals but an arbitrary reference current, R_p^e is a fictitious resistance and not a physically meaningful, *i.e.*, measurable quantity. In some instances both I_p and R_p^e are fictitious. In general, R_p^e is merely that quantity which, when multiplied by the square of the reference current I_p, gives the total power lost from or radiated from a system of moving charges contained within an imaginary great sphere.

The evaluation of T using (9) or (11) presupposes a knowledge of the amplitude and the distribution of current in all the antennas contributing to $A_\Theta{}^r$ and $A_\Phi{}^r$. This is usually not available so that T and R_p^e cannot be calculated. Arbitrary assump-

tions regarding the distribution of current are often made, and the significance of the values of T computed by using such an assumed distribution depends directly upon the validity of the assumed current. This is discussed in detail in the analysis of the distribution of current in an antenna in Volume II. The radiation resistance of an antenna or array may be a convenient figure of merit for comparing the radiating properties of antennas or arrays. Comparable reference currents are chosen in the several arrays to be compared. The radiation resistance is of no significance in the analysis of an antenna as a circuit element *unless the reference current is the input current in an actual and not merely in an assumed distribution.*

The space-radiation function and the radiation resistance for a Hertzian dipole are readily evaluated using (12.10) in (10) and (14). They are

$$K(\Theta) = 2\beta_0 h \sin \Theta \tag{15}$$

$$R^s = 80\beta_0^2 h^2 = 3{,}158 \frac{h^2}{\lambda_0^2} \tag{16}$$

Because the transfer function T is real, the condition (III.18.30) is satisfied so that (III.18.32) and (III.18.33) are true. This means that in the far zone the time-average electric and magnetic energy functions are equal.

The Poynting vector and the transfer function can be expressed directly in terms of the electromagnetic vectors in several ways. As a result of the fact that electric and magnetic fields in the far zone are mutually perpendicular, in phase, and proportional, it follows with (11.17) that

$$S = \nu_0 E^r B^r = \nu_0 \nu_0 B^{r2} = \frac{\nu_0}{\nu_0} E^{r2} = \frac{1}{\zeta_0} E^{r2} \tag{17}$$

That is, the Poynting vector in the far zone is proportional to B^{r2} and to E^{r2}. Accordingly, the space-radiation function $K_p^2(\Theta,\Phi)$ is also proportional to B^{r2} and E^{r2}. With (11),

$$K_p^2(\Theta,\Phi) = \left(\frac{32\pi^2 R_0^2}{I_p^2 \zeta_0^2}\right) E^{r2} = \left(\frac{32\pi^2 R_0^2 \nu_0^2}{I_p^2}\right) B^{r2} \tag{18}$$

It is shown in the analysis of receiving antennas in Volume II that the power that can be transferred from a distant transmitter to the load is proportional to E^{r2} or to B^{r2} at the receiving antenna.

Accordingly, the space-radiation function $K_p^2(\Theta,\Phi)$, because it is a measure of the magnitude of the power that can be transferred to such a load *as a function of* Θ *and* Φ, actually completely characterizes the *directional* properties of a transmitting array as a source of power for a distant receiving station. The same is true of the Poynting vector because it, too, is proportional to E^{r2} and B^{r2}. It is shown in the study of receiving antennas that the power transferred to the load is not in any way related to the area, *i.e.*, length times diameter, of the receiving antenna. Accordingly, the assumption that the Poynting vector itself specifies a "spatially distributed energy" actually transferred across a given unit area per second is not verified in the case of a receiving antenna. This is discussed in greater detail in conjunction with receiving antennas.

The space-radiation function and the Poynting vector calculated for a transmitting antenna or array both characterize the relative power that is transferred to the load of a receiving antenna as this is moved along the surface of a great sphere around the transmitter with the receiving unit always oriented for maximum power to the load. Neither is here assumed to be a measure of the energy per second transferred across each unit of area of the enclosing sphere.

18. Directivity of a Radiating System.—Since the directional properties of an antenna or of an array as an energy-transferring device can be described in terms of the space-radiation function $K_p^2(\Theta,\Phi)$, it is desirable to establish a criterion or a figure of merit for comparing the directional properties of different antennas. The quantity defined for this purpose is called the *absolute directivity of the antenna or array* and is denoted by D. The absolute directivity of an antenna or array with respect to the direction Θ_m, Φ_m in which the space-radiation function $K_p^2(\Theta,\Phi)$ is a maximum is defined as the ratio of the total power that would be radiated if the antenna or array were *omni*directional with a space-radiation function independent of Θ and Φ and equal to $K_p^2(\Theta_m,\Phi_m)$, to the total power actually radiated. Thus,

$$D = \left\{ \frac{\text{total power radiated with } K_p^2(\Theta,\Phi) \text{ fixed at } K_p^2(\Theta_m,\Phi_m)}{\text{total power radiated with } K_p^2(\Theta,\Phi) \text{ the actual function of } \Theta \text{ and } \Phi} \right\}$$

$$\tag{1}$$

Symbolically,

$$D = \frac{K_p^2(\Theta_m, \Phi_m) \int_0^{2\pi} \int_0^{\pi} \sin \Theta \, d\Theta \, d\Phi}{\int_0^{2\pi} \int_0^{\pi} K_p^2(\Theta, \Phi) \sin \Theta \, d\Theta \, d\Phi} \qquad (2)$$

Using (17.14) and after carrying out the integration in the numerator,

$$D = \frac{K_p^2(\Theta_m, \Phi_m)}{4\pi} \frac{\zeta_0}{R_p^e} \qquad (3)$$

Numerically, with $\zeta_0 = 120\pi$ ohms,

$$D = \frac{30 K_p^2(\Theta_m, \Phi_m)}{R_p^e} \qquad (4)$$

The absolute directivity of the Hertzian dipole is

$$D = 1.5 \qquad (5)$$

If it is assumed that the Poynting vector is unique among the many possible and equally plausible "energy-flow" functions so that it specifies the actual flow of an "energy substance" across each unit area perpendicular to the Poynting vector, the space-radiation function is necessarily proportional to this flow. Then $K_p^2(\Theta, \Phi)$ is a measure of the energy traversing the unit area of the great sphere at (Θ, Φ). The conventional definition of directivity is based upon this assumption. It states that "directivity is the ratio of the power per unit solid angle flowing (in space) in the direction of maximum radiation to the average power per unit solid angle flowing (in space) in all directions from the radiating system."[1]   Since $K_p^2(\Theta_m, \Phi_m)$ is assumed to be proportional to the maximum energy crossing unit area, while

$$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} K_p^2(\Theta, \Phi) \sin \Theta \, d\Theta \, d\Phi$$

is the average energy crossing unit area, this definition leads exactly to (2) and (4). As was explained in Chapter III, the arbitrary assumption that the Poynting vector itself, rather than the integral of its normal component over a *closed* surface, is fundamental in the definition of electromagnetic energy functions is rejected here, together with the concept of spatially distributed

[1] CARTER, HANSELL, LINDENBLAD, *Proc. I.R.E.*, **19**, 1802 (October, 1931).

energy, as unnecessary and misleading. Accordingly, directivity is defined as in (1) and not in terms of a flow of spatially distributed energy.

19. Energy-transfer Function for the Surface of a Cylindrical Antenna.—The complex transfer function $T = T_r + jT_i$ is defined by

$$T = \int_{\Sigma_r} (\hat{n},\mathbf{S})d\sigma \tag{1}$$

where $d\sigma$ is an element of any *completely closed* surface Σ_r on which $\hat{n}$ is an external unit normal. $\mathbf{S}$ is the complex Poynting vector defined in space by

$$\mathbf{S} = \tfrac{1}{2}\nu_0[\mathbf{E},\mathbf{B}^*] \tag{2}$$

The value of the real part T_r of the integral (1) specifies the total transfer of power from within Σ_r so that T_r is necessarily independent of the shape and size of the envelope Σ_r provided the energy functions defined within Σ_r are unchanged. In particular, T_r has the same value if Σ_r is a great sphere enclosing an antenna completely in its far zone, or if Σ_r is a cylindrical envelope only an infinitesimal amount larger than the actual surface of the antenna. T_i, on the other hand, is not necessarily independent of the shape and location of Σ_r, so that it may not be concluded because T_i vanishes when Σ_r is a sphere in the far zone it will also vanish if Σ_r is the cylindrical surface of the antenna itself.

On the cylindrical envelope $r = a$, using cylindrical coordinates r, θ, z with the z axis along the antenna,

$$(\hat{n},\mathbf{S}) = S_r = \tfrac{1}{2}\nu_0[\mathbf{E},\mathbf{B}^*]_r = \tfrac{1}{2}\nu_0\{E_\theta B_z^* - E_z B_\theta^*\} \tag{3}$$

On the end surfaces at $z = \pm h$

$$(\hat{n},\mathbf{S}) = \pm S_z = \pm\tfrac{1}{2}\nu_0[\mathbf{E},\mathbf{B}^*]_z = \pm\tfrac{1}{2}\nu_0\{E_r B_\theta^* - E_\theta B_r^*\} \tag{4}$$

The upper sign is for the top of the antenna, the lower sign is for the bottom.

It follows directly from (11.8b) and (11.16) that

$$E_\theta = 0; \qquad B_r = 0 \tag{5}$$

so that

$$S_r = -\tfrac{1}{2}\nu_0 E_z B_\theta^* \tag{6}$$

$$S_z = \pm\tfrac{1}{2}\nu_0 E_r B_\theta^* \tag{7}$$

It is now possible to form (1) assuming rotational symmetry. Thus,

$$T = -\tfrac{1}{2}\nu_0 \int_{-h}^{h} (E_z B_\theta{}^*)_{r=a} 2\pi a \, dz + \tfrac{1}{2}\nu_0 \int_0^a (E_r B_\theta{}^*)_{z=h} 2\pi r \, dr$$
$$-\tfrac{1}{2}\nu_0 \int_0^a (E_r B_\theta{}^*)_{z=-h} 2\pi r \, dr \quad (8)$$

If the area of the end surfaces is sufficiently small compared with the cylindrical surface so that it is correct to write

$$2\pi a^2 << 2\pi a h \qquad (9)$$

or

$$a << h \qquad (10)$$

the contribution to T by the last two integrals in (8) is negligible. This follows because the very small radial currents on the ends cannot lead to an average tangential electromagnetic field at the ends that equals the average tangential field on the cylindrical sides due to the axial current, because the radial currents must certainly vanish at $r = 0$, $z = \pm h$. Accordingly, a good approximation subject to (10) is

$$T \doteq -\frac{1}{2}\,\nu_0 \int_{-h}^{h} (E_z B_\theta{}^*)_{r=a} 2\pi a \, dz \qquad (11)$$

It is possible to express B_θ in terms of the total current using the Maxwell-Ampère theorem of circuitation (integral form of the third field equation) (II.11.3). For a single surface and complex vectors this is

$$\nu_0 \oint (\mathbf{B,ds}) = \bar{I} + j\omega\epsilon_0 \int_{S(\text{cap})} (\hat{\mathbf{N}},\mathbf{E})dS \qquad (12)$$

with

$$\bar{I} = \int_{S(\text{cap})} (\hat{\mathbf{N}},\overline{\rho_m \mathbf{v}})dS + \int_s (\hat{\mathbf{N}},\overline{\eta_m \mathbf{v}})ds \qquad (13)$$

Let $S(\text{cap})$ be the cross section of the cylindrical conductor at any point along its length; s the circumference. In the conductor (not perfect)

$$(\hat{\mathbf{N}},\overline{\rho_m \mathbf{v}}) = i_{zf}; \qquad (\hat{\mathbf{N}},\overline{\eta_m \mathbf{v}}) = l_{zf} = 0; \qquad \bar{I} = I_z \qquad (14)$$

By definition of a good conductor,

$$i_{zf} = \sigma E_z >> \omega\epsilon_0 E_z \qquad (15)$$

Since the antenna is rotationally symmetrical, (12) becomes

$$\nu_0 \oint B_\theta a \, d\theta = \nu_0 2\pi a B_\theta = I_z \qquad (16)$$

With (16), (11) becomes

$$T \doteq -\frac{1}{2}\int_{-h}^{h}(E_z)_{r=a}I_z^* dz \qquad (17)$$

If the antenna is symmetrical and center driven so that

$$I(z) = I(-z) \qquad (18)$$

$$T = -\frac{1}{2}\int_0^h (E_z)_{r=a}I_z^*(z)dz + \frac{1}{2}\int_0^{-h}(E_z)_{r=a}I_z^*(-z)dz \quad (19)$$

By writing z for $-z$ throughout in the second integral, it becomes like the first integral.   Then

$$T = -\int_0^h (E_z)_{r=a}I_z^* dz \qquad (20)$$

The real part of (20) gives the total power transferred from a cylindrical region of length $2h$ and radius a carrying a total current I_z. $(E_z)_{r=a}$ is the tangential component of the electric field at the surface of the cylinder.   At all points along the part of the conducting surface that contains no charge-separating agency (A to h and B to $-h$ in Fig. 19.1), $(E_z)_{r=a}$ and I_z are in the same direction.   (The phase difference between $(E_{z\,\text{inst}})_{r=a}$ and $I_{z\,\text{inst}}$ is in no case greater than 45° as will be shown in Chapter V.)   At points along the conducting surface of the charge-separating region, $(E_z)_{r=a}$ and I_z are oppositely directed. It follows from (20) that integration over the length of the conductor not including the length AB contributes a negative value to T, while the integral over the cylindrical surface of AB gives a positive value.   It is interesting to note that that part of the integration which is carried out over the surface of the antenna proper, not including the surface of the generating region, yields only the rate of increase of heat

Fig. 19.1.— Electric field and Poynting vector on the cylindrical surface of a good but not perfect conductor with a generator between A and B.

energy in the conductor, not at all the rate at which energy functions outside the conductor increase. This is given entirely by that part of the integration which is carried out over the surface enclosing the charge-separating region. Since no particular significance can be attached to integration of the normal component of the Poynting vector over a *part* of a closed surface, it is not legitimate to attempt to localize the "flow of energy" on the basis of the meaningless parts of a meaningful integral.

An erroneous and peculiarly inconsistent analysis using (20) proceeds as follows. (1) The conductor is assumed to be perfect. (2) A sinusoidal distribution is *assumed* along the antenna. (This is an incorrect assumption except for infinitely thin, perfect conductors as is shown in Volume II.) (3) The electric field $(E_z)_{r=a}$ is calculated using (7.15a) and the assumed current. A *nonvanishing* value is obtained in direct contradiction to postulate 1 which requires a vanishing field. (4) The field so obtained is substituted in (20) to determine T and ultimately the input impedance of the antenna. It is obvious that the impedance of an antenna cannot be obtained by this sequence of errors and contradictions. This is considered in greater detail in Volume II.

The Poynting vector on the cylindrical surface of the conductor is given by (6). If the conductor is perfect, E_z is zero and hence S_r vanishes at all points along the antenna. If the conductor is not perfect, E_z has a small value and S_r is negative, *i.e.*, S points radially *into* the conductor. Between the circumferences at A and B of the antenna near $z = 0$ (Fig. 19.1), an impressed field E^e is maintained; it is in the direction of the current, while the electric field E (which does not include E^e) is in the opposite direction. It follows that S_r is positive between A and B so that S is outwardly directed. An interpretation based on the assumption that the Poynting vector specifies the magnitude and direction of flow of "spatially distributed energy" leads to the following conclusions. No "energy" leaves the antenna or enters it if the conductor is perfect. "Energy" flows radially into the antenna from the "stored energy" in space if the conductor is imperfect to account for the increase in thermal energy in the conductor. "Energy" is "radiated" outward into space from the charge-separating region AB (Fig. 19.1).

If the antenna is driven from a two-conductor line which is in turn driven from a generator or charge-separating region AB

(Fig. 19.2) and the Poynting vector is determined at the surface
of the conductors including those of the line, the following is
learned.   If the conductors are perfect, the tangential electric
field vanishes everywhere along all conductors so that there is no
outwardly directed component.   If the conductors are imperfect,
the Poynting vector has a small inward component as shown in
Fig. 19.2.   The Poynting vector has a large outward component
only between the terminals *AB*.

It might be concluded from this discussion that since no
"energy" leaves the antenna in either case the moving charges

Fig. 19.2.—Poynting vector at the surfaces of antenna and transmission line.
A generator is between *A* and *B*.

in the antenna can play no part whatsoever in maintaining
currents in neighboring or distant conductors.   This conclusion is
in direct contradiction with the postulate that all moving charges
exert forces and hence do work on each other.   Since no "energy"
leaves the antenna *if the Poynting vector correctly localizes its
flow,* the antenna is apparently quite unnecessary so long as the
same potential difference is maintained between the surfaces *A*
and *B*.   In fact, if the antenna in Fig. 19.2 is replaced by a
lumped impedance consisting of a suitable resistance and react-
ance that constitute an identical load at the end of the line,
the Poynting vector is unchanged, and directed outward between
A and *B*, indicating that the same "energy flow into space" is
maintained by the generating region in both cases.   On the other
hand, the long antenna is capable of "receiving" only a very

small inward "flow of energy from space," whereas the lumped impedance "receives from space" practically all the energy that is "transferred to space" from the generator. It is to be noted that the long transmission line seems to have nothing to do with the outward "flow of energy" from the charge-separating region AB nor with the inward "flow of energy from space" to either the antenna or the lumped impedance. According to the Poynting vector, energy does not travel along the line.

The above discussion further emphasizes that nothing of value is gained by attaching *arbitrary* physical significance to the Poynting vector or to electromagnetic or other energy functions.

GENERAL THEOREMS

20. Image Fields.—A straight conductor extends in space from C_a to D_a at an arbitrary angle above the mathematical

Fig. 20.1.—Antenna $C_a D_a$ with image $C_i D_i$ with reversed current.

plane $z = 0$ as shown in Fig. 20.1. It carries a periodically varying axial current distributed in an unspecified way. Periodically varying concentrations of charge exist in appropriate distribution as required to satisfy the equation of continuity for electric charge. Below the plane $z = 0$ is an identical second (or image) conductor extending in space from C_i to D_i and arranged to be the exact geometrical image of the first conductor *except in one respect*. All currents and charges, while the same in magnitude at image points, are opposite in direction and sign,

respectively. For example, at the point shown in Fig. 20.1
the current I_a is directed from C_a to D_a in the first conductor
(subscript a), whereas at the image point the current I_i in the
image conductor (subscript i) is the same in magnitude as I_a
but in the opposite direction with respect to image points, *i.e.*,
from D_i to C_i. It is to be noted that if the two conductors are
parallel to the plane $z = 0$ as in Fig. 20.2a the reversed current
in the image makes *currents at image points* in the two conductors
opposite in direction; if the two conductors are colinear and
perpendicular to the plane $z = 0$ as in Fig. 20.2b, the reversed

Fig. 20.2.—Horizontal and vertical antennas C_aD_a with images C_iD_i with
reversed currents.

current in the image makes *currents at image points* in the two
conductors the *same in direction.* The lower conductor in Fig.
20.2a is sometimes called an image of the upper conductor
with current in the opposite direction; the lower conductor in
Fig. 20.2b, an image of the upper conductor with current in the
same direction. This is an unfortunate terminology. Actually,
both are images with currents *reversed* with respect to correspond-
ing image points. It follows directly from the equation of
continuity that a reversal of current in the image involves a
change in sign of the charge in the image. Accordingly, if there
is a concentration of *positive* charge near a given point in the
upper conductor, there is an equal concentration of *negative*
charge near the image point in the lower conductor.

The complex amplitude of the resultant vector potential $\mathbf{A}$
at any point P (Fig. 20.1) is the vector sum of the complex

amplitudes of the potentials due to the currents in the individual conductors. Thus for the two conductors

$$\mathbf{A} = \mathbf{A}_a + \mathbf{A}_i = \frac{1}{4\pi\nu_0} \left\{ \mathbf{s}_a \int_{C_a}^{D_a} \frac{I'_a}{R_a} e^{-j\beta_0 R_a} ds'_a + \mathbf{s}_i \int_{C_i}^{D_i} \frac{I'_i}{R_i} e^{-j\beta_0 R_i} ds'_i \right\} \quad (1)$$

If the point P where $\mathbf{A}$ is calculated is on the plane $z = 0$, $R_a = R_i$ and the two integrals in (1) are the same. The directions of $\mathbf{A}_a$ and $\mathbf{A}_i$ are those of $\mathbf{s}_a$ and $-\mathbf{s}_i$ where $\mathbf{s}_a$ and $\mathbf{s}_i$ are, respectively, unit vectors in direction $C_a D_a$ and $C_i D_i$ as shown in Fig. 20.1. The resultant vector potential $\mathbf{A}$ is necessarily perpendicular to the plane $z = 0$ at all points.

The scalar potential at any point P is

$$\phi = \phi_a + \phi_i = \frac{1}{4\pi\epsilon_0} \left\{ \int_{C_a}^{D_a} \frac{q'_a}{R_a} e^{-j\beta_0 R_a} ds'_a + \int_{C_i}^{D_i} \frac{q'_i}{R_i} e^{-j\beta_0 R_i} ds'_i \right\} \quad (2)$$

Since $q'_a = -q'_i$, the two integrals are equal and opposite in sign at all points on the plane $z = 0$ where $R_a = R_i$. Therefore, the scalar potential·vanishes everywhere on this plane, and the plane itself is an equipotential surface characterized by $\phi = 0$. If ϕ is positive at a point (x_1, y_1, z_1) slightly above the plane, it has an equal negative value at a corresponding point $(x_1, y_1, -z_1)$ below the plane. The gradient of ϕ is perpendicular to the plane since this is an equipotential surface.

The electromagnetic field at any point P may be computed from

$$\mathbf{E} = -\text{grad } \phi - j\omega\mathbf{A} = -j\frac{\omega}{\beta_0^2} \{\text{grad div } \mathbf{A} + \beta_0^2\mathbf{A}\} \quad (3)$$

$$\mathbf{B} = \text{curl } \mathbf{A} \quad (4)$$

At any point on the plane $z = 0$, $\mathbf{E}$ and $\mathbf{B}$ may be resolved into components tangent to the plane (subscript s) and components perpendicular to the plane (subscript n for downward normal).

$$E_n = -\frac{\partial\phi}{\partial n} - j\omega A_n = -2\left\{\frac{\partial\phi_a}{\partial n} + j\omega A_{an}\right\} = 2E_{na} \quad (5)$$

$$E_s = -\frac{\partial\phi}{\partial s} - j\omega A_s = 0 \quad (6)$$

$$B_s = \text{curl}_s \mathbf{A} = 2 \text{ curl}_s \mathbf{A}_a = 2B_{sa} \quad (7)$$

$$B_n = \text{curl}_n \mathbf{A} = \lim_{\Delta\Sigma\to 0} \frac{\oint A_s ds}{\Delta\Sigma} = 0 \quad (8)$$

The vanishing of B_n is due to the fact that there is no component A_s of $\mathbf{A}$ tangent to the plane $z = 0$ in which the contour of integration and $\Delta\Sigma$ in (8) must lie. The electromagnetic field in the upper half-space, *i.e.*, with $z \geq 0$, is defined by (3) and (4) using (1) and (2). The electric field is perpendicular, the magnetic field is tangent to the plane $z = 0$.

Let the lower conductor in Fig. 20.1 be imagined removed and the entire half-space below $z = 0$ filled with a *perfectly* conducting medium as in Fig. 20.3. In this medium, and hence for $z \leq 0$, the electromagnetic field must be zero. Above the

Fig. 20.3.—Antenna $C_a D_a$ over a perfectly conducting half-space.

perfectly conducting plane at $z = 0$ the potential functions and the electromagnetic field must be determined from the currents and charges in the upper conductor (which is exactly as before) and from the surface densities of current and charge in the conducting plane. Thus

$$\mathbf{A} = \mathbf{A}_a + \mathbf{A}_{\text{plane}} = \frac{\hat{s}_a}{4\pi\nu_0} \int_{C_a}^{D_a} \frac{I_a'}{R_a} e^{-j\beta_0 R_a} ds_a' + \frac{1}{4\pi\nu_0} \int_{\text{plane}} \frac{l_f'}{R} e^{-j\beta_0 R} d\sigma' \quad (9)$$

$$\phi = \phi_a + \phi_{\text{plane}} = \frac{1}{4\pi\epsilon_0} \int_{C_a}^{D_a} \frac{q_a'}{R_a} e^{-j\beta_0 R_a} ds_a' + \frac{1}{4\pi\epsilon_0} \int_{\text{plane}} \frac{n_f'}{R} e^{-j\beta_0 R} d\sigma' \quad (10)$$

Here $\mathbf{A}_a$ and ϕ_a are the same as in (1) and (2) because I_a' and q_a' are the same. The boundary conditions at $z = 0$ for the electromagnetic field calculated from (9) and (10) using (3) and (4) are

$$(\mathbf{\hat{n}}, \mathbf{E}) = -\frac{n_f}{\epsilon_0} \tag{11}$$

$$[\mathbf{\hat{n}}, \mathbf{E}] = 0 \tag{12}$$

$$[\mathbf{\hat{n}}, \mathbf{B}] = -\frac{l_f}{\nu_0} \tag{13}$$

$$(\mathbf{\hat{n}}, \mathbf{B}) = 0 \tag{14}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields in space just above the conducting plane; A i : an external normal to the upper half-space; $\mathbf{n}_f$ and $\mathbf{l}_f$ are surface densities of free charge and of moving free charge in the conducting plane. In terms of the magnitudes of the normal and tangential components, (11) to (14) may be written as follows

$$E_n = - \frac{\mathbf{n}_f}{\epsilon_0} \tag{15}$$

$$E_s = 0 \tag{16}$$

$$B_s = - \frac{l_f}{\nu_0} \tag{17}$$

$$B_n = 0 \tag{18}$$

If the boundary conditions (15) to (18) at the perfectly conducting plane $z = 0$ are compared with the field (5) to (8) at the mathematical plane $z = 0$, it is seen that they can be made to coincide if it is required that the following conditions be true:

$$-\frac{\mathbf{n}_f}{\epsilon_0} = 2E_{na} = -2 \left\{ \frac{\partial \phi_a}{\partial n} + j\omega A_{an} \right\} \tag{19}$$

$$-\frac{l_f}{\nu_0} = 2B_{sa} = 2 \operatorname{curl}_s \mathbf{A}_a \tag{20}$$

Subject to (19) and (20) the electromagnetic field calculated from $\mathbf{A}$ and ϕ as given in (1) and (2) is the same for $z \geq 0$ as the electromagnetic field calculated from $\mathbf{A}$ and ϕ as given in (9) and (10) subject to (11) to (14). This follows from the fact that the two solutions satisfy the same differential equations (the Maxwell-Lorentz equations) in the entire upper half-space ($z \geq 0$) and the same conditions at the boundary ($z = 0$). Since any solution that satisfies the differential equations and the boundary conditions is the unique solution, no matter how obtained, it follows that the assumed conditions (19) and (20) that lead to the unique solution must be correct themselves.

The following conclusions constitute the theorem of images. The electromagnetic vectors at any point P for which $z \geq 0$ in Fig. 20.3 are the same as at the corresponding point in Fig. 20.1. It is assumed that the conductor $C_a D_a$ is identical physically and electrically in the two cases and that $C_i D_i$ in Fig. 20.1 is a geometrical image of $C_a D_a$ but with currents reversed in direction and with charges altered in sign. After $\mathbf{E}$ and $\mathbf{B}$ have been deter-

mined (as by calculation from ϕ and $\mathbf{A}$) for $z \geq 0$ in the single-conductor, infinite-conducting-plane problem of Fig. 20.3 by solving the conductor-with-image problem of Fig. 20.1, the surface densities $\mathbf{n}_f$ and $\mathbf{l}_f$ in the problem of Fig. 20.3 may be determined using (11) and (13). The field vectors $\mathbf{E}$ and $\mathbf{B}$ due to the surface distributions of current and charge on the perfectly conducting plane in Fig. 20.3 are the same at all points in the upper half-space ($z \geq 0$) as the fields due to currents and charges in the image conductor in Fig. 20.1. This is not true of the vector potential as is clear from the directions of $\mathbf{A}_i$ and $\mathbf{A}_{\text{plane}}$ in (1) and (9).

The theorem of images permits the substitution of a relatively simple problem involving only a conductor and its image in space for a rather difficult one involving a single conductor over a perfectly conducting infinite plane. This is possible only for points on and above this plane. The theorem is easily generalized to any combination of conductors over a perfectly conducting infinite plane. If the plane is a good conductor ($\sigma_o/\omega\epsilon_e >> 1$) rather than a perfect one ($\sigma_o = \infty$), the solutions obtained by assuming the conductor perfect are usually good approximations. If the conducting surface is not infinite or not plane, the boundary conditions are much more complicated and the method of images is strictly not applicable. In some instances, moderately good approximations are obtained by applying the method of images even when the conducting surface is finite and not plane. An example is the approximate solution of problems involving antennas over the surface of the earth at frequencies that are sufficiently low so that over moist earth or salt water ($\sigma_o/\omega\epsilon_e$) is very large. The general boundary-value problem of an antenna over an imperfectly conducting earth is analyzed in Volume II together with the associated problems of reflection and refraction. Great care must be exercised in using the method of images in practical problems involving conducting surfaces of *finite* extent. A finite plane, even if quite large, is usually not a satisfactory substitute for an infinite plane, and analyses involving the latter may be grossly in error when applied carelessly to the former.

Half a Hertzian dipole (Sec. 12) erected vertically on a perfectly conducting plane is called an Abraham dipole. In the half-space above the conducting plane, its radiation function

and directivity are the same as for the Hertzian dipole.   The radiation resistance is one half that of the Hertzian dipole (17.16) and given by the so-called Rüdenberg formula

$$R^e = 40\beta_0^2 h^2 = 1{,}579 \left(\frac{h}{\lambda_0}\right)^2 \tag{21}$$

21. Rayleigh-Carson Reciprocal Theorem.—A very general and fundamentally important theorem which has wide application in the general theory of electric circuits including especially those involving antennas may be derived subject only to the condition that the essential density of moving charge $\overline{\varrho_m v}$ is linearly related to the electric field E.   It was shown in Sec. III.15 that this is true in all simply polarizing, magnetizing, and conducting media for which it is possible to write as in (III.14.14) to (III.14.16)

$$P = (\varepsilon_r - 1)\epsilon_0 E; \qquad -M = (\nu_r - 1)\nu_0 B; \qquad i_f = \delta(E + E^e) \tag{1}$$

Subject to (1) it follows as in (III.15.40) that

$$\overline{\varrho_m v} = KE + \mathfrak{u}_r \delta E^e; \qquad K \equiv \mathfrak{u}_r \{\sigma_e + j\omega\epsilon_0(\epsilon_{er} - \nu_r)\} \tag{2}$$

Here $\mathfrak{u}_r = 1/\nu_r$; E^e is an impressed field independent of $\overline{\varrho_m v}$ and E in the regions under study.   It is clear that wherever (1) is true the density of moving charge $\overline{\varrho_m v}$ is linearly related to the electric field.

The electric field is expressed in terms of vector and scalar potentials by the fundamental relation

$$E = -\operatorname{grad} \phi - j\omega A \tag{3}$$

If (2) solved for E is substituted in (3), the following equation is easily obtained:

$$\frac{\overline{\varrho_m v}}{K} - \frac{\mathfrak{u}_r \delta}{K} E^e = -\operatorname{grad} \phi - j\omega A \tag{4}$$

Since only practically available and hence imperfect conductors are involved, surface densities of moving free charge are not required so that $I_f = 0$.   The vector potential can then be represented entirely in terms of the volume density $\overline{\varrho_m v}$.

$$A = \frac{1}{4\pi\nu} \int_\tau \frac{\overline{\varrho_m v}}{R} e^{-j\beta_0 R} d\tau \tag{5}$$

Integration is with respect to the variables locating $d\tau$. Primes are omitted in (5) in order to avoid confusion when primes and double primes are introduced below for another purpose.

Let a vector $\mathbf{G}$ be defined by

$$\mathbf{G} \equiv \frac{\mathfrak{y}_r\delta}{K} \mathbf{E}^\bullet - \text{grad } \phi \tag{6}$$

If the nonvanishing values of $\mathbf{E}^\bullet$ are maintained in a good conductor for which $K \doteq \mathfrak{y}_r\delta$ and in which $\delta \doteq \sigma$,

$$\mathbf{G} = \mathbf{E}^\bullet - \text{grad } \phi \tag{7}$$

With (4) and (5) used in (6) or (7)

$$\mathbf{G} = \overline{\frac{\varrho_m\mathbf{v}}{K}} + \frac{j\omega}{4\pi\mathfrak{v}} \int_\tau \overline{\frac{\varrho_m\mathbf{v}}{R}} e^{-j\beta_0 R} d\tau \tag{8}$$

Equation (8) is a vector integral equation in $\overline{\varrho_m\mathbf{v}}$. The kernel of the integral is $e^{-j\beta_0 R}/R$. It is symmetrical with respect to any two points in space such as (x_1,y_1,z_1) and (x_2,y_2,z_2) since the distance between the two points

$$R = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

is invariant to an interchange of subscripts. As a consequence of this symmetry, the theory of integral equations allows the following general theorem to be written.

THEOREM: Any two pairs of functions $\overline{\varrho_m\mathbf{v}}'$ and $\mathbf{G}'$, $\overline{\varrho_m\mathbf{v}}''$ and $\mathbf{G}''$, which are functions of the space coordinates, and which satisfy the integral equation (8), obey the following reciprocal relation:

$$\int_\tau (\overline{\varrho_m\mathbf{v}}',\mathbf{G}'') d\tau = \int_\tau (\overline{\varrho_m\mathbf{v}}'',\mathbf{G}') d\tau \tag{9}$$

Upon expanding $\mathbf{G}$ using (6) and noting that significant values of $\overline{\varrho_m\mathbf{v}}$ occur only in good conductors where $K = \sigma\mathfrak{y}_r$ so that (6) reduces to (7),

$$\int_\tau (\overline{\varrho_m\mathbf{v}}',\mathbf{E}^{\bullet\prime\prime}) d\tau - \int_\tau (\overline{\varrho_m\mathbf{v}}'',\mathbf{E}^{\bullet\prime}) d\tau = \int_\tau (\overline{\varrho_m\mathbf{v}}', \text{grad } \phi'') d\tau$$
$$- \int_\tau (\overline{\varrho_m\mathbf{v}}'', \text{grad } \phi') d\tau \tag{10}$$

The integrals on the right may be transformed using (III.2.11) in the form

$$\text{div } \mathbf{C}\psi = \psi \text{ div } \mathbf{C} + (\mathbf{C}, \text{grad } \psi) \tag{11}$$

and the divergence theorem (II.6.1) in the form

$$\int_\tau \text{div } \mathbf{C} \, d\tau = \int_\Sigma (\mathbf{n}, \mathbf{C}) d\sigma \tag{12}$$

For example, the first integral on the right in (10) becomes

$$\int_\tau (\overline{\varrho_m \mathbf{v}}', \text{grad } \phi'') d\tau = \int_\tau \text{div } (\phi'' \overline{\varrho_m \mathbf{v}}') d\tau - \int_\tau \phi'' \text{ div } \overline{\varrho_m \mathbf{v}}' d\tau$$
$$= \int_\Sigma \phi''(\mathbf{n}, \overline{\varrho_m \mathbf{v}}') d\sigma - \int_\tau \phi'' \text{ div } \overline{\varrho_m \mathbf{v}}' d\tau \tag{13}$$

The surface integral in (13) vanishes if the surface Σ is chosen to cross no regions in which $\overline{\varrho_m \mathbf{v}}'$ differs from zero. With the equation of continuity

$$\text{div } \overline{\varrho_m \mathbf{v}} + j\omega\mathfrak{d} = 0 \tag{14}$$

(13) becomes

$$\int_\tau (\overline{\varrho_m \mathbf{v}}', \text{grad } \phi'') d\tau = j\omega \int_\tau \phi'' \mathfrak{d}' d\tau \tag{15}$$

The scalar potential satisfies the general wave equation (1.1)

$$\nabla^2 \phi + \beta_0^2 \phi = -\frac{\mathfrak{d}}{\epsilon_0} \tag{16}$$

With (16) solved for $\mathfrak{d}$ substituted in (15), this becomes

$$\int_\tau (\overline{\varrho_m \mathbf{v}}', \text{grad } \phi'') d\tau = -j\omega\epsilon_0 \int_\tau \phi'' \{\nabla^2 \phi' + \beta_0^2 \phi'\} d\tau \tag{17}$$

Similarly

$$\int_\tau (\overline{\varrho_m \mathbf{v}}'', \text{grad } \phi') d\tau = -j\omega\epsilon_0 \int_\tau \phi' \{\nabla^2 \phi'' + \beta_0^2 \phi''\} d\tau \tag{18}$$

Using (17) and (18) in (10),

$$\int_\tau (\overline{\varrho_m \mathbf{v}}', \mathbf{E}^{o\prime\prime}) d\tau - \int_\tau (\overline{\varrho_m \mathbf{v}}'', \mathbf{E}^{o\prime}) d\tau$$
$$= j\omega\epsilon_0 \int_\tau \{\phi' \nabla^2 \phi'' - \phi'' \nabla^2 \phi'\} d\tau \tag{19}$$

The right side can be transformed using Green's symmetrical theorem (III.2.16). It becomes

$$j\omega\epsilon_0 \int_\Sigma \left\{ \phi' \frac{\partial \phi''}{\partial n} - \phi'' \frac{\partial \phi'}{\partial n} \right\} d\sigma \tag{20}$$

with n an external normal to the enclosing surface Σ. If this is chosen to be a spherical shell of radius R that is very large

compared with the greatest dimension between regions in which $\mathbf{E}^e$ differs from zero, then

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial R} \tag{21}$$

on this shell. Furthermore, the scalar potential ϕ on the surface of a sufficiently large shell in the far zone has the form

$$\phi = C \frac{e^{-j\beta_0 R}}{R} \tag{22}$$

where C is independent of R. With (21) and in the limit as $R \to \infty$

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial R} = -j\beta_0 \phi \tag{23}$$

If (23) is written for ϕ' and ϕ'' and substituted in (20) this vanishes. Therefore the right side of (19) is zero and

$$\int_\tau (\overline{\rho_m \mathbf{v}'}, \mathbf{E}^{e''}) d\tau = \int_\tau (\overline{\rho_m \mathbf{v}''}, \mathbf{E}^{e'}) d\tau \tag{24}$$

These integrals are to be evaluated over all regions where both $\mathbf{E}^e$ and $\overline{\rho_m \mathbf{v}}$ differ from zero.

In applying (24) to electric circuits including antennas, it is usually possible to assume that regions where $\mathbf{E}^e$ differs from zero are equivalent to short distances between a finite number of pairs of surfaces each of small area. Each of these pairs constitutes the terminals A and B of a generator or its equivalent across which there is a total current

$$I_s = \int_S \overline{\rho_m v_s} dS \tag{25}$$

due to the field $\mathbf{E}^e$ maintained between A and B. These terminals are sufficiently close together so that the current I_s has the same amplitude at A, at B, and in the generator between A and B. Let $\mathbf{E}^e$ be a mean value on each surface S between n pairs of terminals A and B.

$$\int_\tau (\overline{\rho_m \mathbf{v}'}, \mathbf{E}^{e''}) d\tau = \sum_{j=1}^{n} (\tfrac{1}{2} I_s', \int_A^B \mathbf{E}^{e''} ds)_j \tag{26}$$

Let the positive direction of current for each set of terminals be from B to A. Also let

$$V_{AB}^e = \int_A^B E^e ds \tag{27}$$

be the driving potential difference maintained across each pair of terminals by the n generators.   Then,

$$\int_\tau (\overline{\rho_m \mathbf{v}'}, \mathbf{E}^{e\prime\prime}) d\tau = \sum_{j=1}^n I'_j V^{e\prime\prime}_{ABj} \tag{28}$$

With (28) and an expression like it but with primes and double primes interchanged, (24) becomes

$$\sum_{j=1}^n I'_j V^{e\prime\prime}_{ABj} = \sum_{j=1}^n I''_j V'_{ABj} \tag{29}$$

This is the final expression.   The theorem it expresses may be summarized as follows.

STATEMENT I:[1] A set of driving potential differences V^e_1, $\cdots$ V^e_j, $\cdots$ V^e_n maintained across n pairs of terminals in a network of conductors or an array of antennas produces a set of currents I'_1, $\cdots$ I'_j, $\cdots$ I'_n at these terminals.

STATEMENT II: A different set of driving potential differences V''^e_1, $\cdots$ V''^e_j, $\cdots$ V''^e_n maintained across the *same* n terminals produces the set of currents I''_1, $\cdots$ I''_j, $\cdots$ I''_n.

THEOREM: The driving potential differences and the currents in the two cases are related by the following reciprocal relation:

$$\sum_{j=1}^n I'_j V^{e\prime\prime}_j = \sum_{j=1}^n I''_j V^{e\prime}_j$$

CONDITION: The total moving charge in the entire region is everywhere linearly related to the electric field so that

$$\overline{\rho_m \mathbf{v}} \sim \mathbf{E}$$

The proportionality constant may be complex.

The reciprocal theorem may be expressed nonmathematically as follows: If a generator with an e.m.f. or driving potential difference of complex amplitude V^e between its terminals maintains a current of complex amplitude I through a load connected between any other pair of terminals in the same or in a coupled network, the current in the load is unaltered if load and generator are interchanged provided the impedances connected between

[1] RAYLEIGH, "Theory of Sound," Vol. I, p. 155.

CARSON, *Bell System Tech. Jour.*, Vol. III, July, 1924.

each pair of terminals are the same in both cases and the generator maintains the same e.m.f.

In applying the reciprocal theorem, it is usually convenient to consider the currents due to only one driving potential difference at a time, since it is possible to add algebraically the individually determined currents due to several. This follows from the fact that the differential equations involved are linear with constant coefficients. Suppose that when V_j'' is applied at terminals j a current I_i' exists at terminals i, and when V_i'' is applied at terminals i a current I_j'' exists at terminals j. The reciprocal theorem reduces to the important form

$$I_j'' V_j' = I_i' V_i'' \tag{30}$$

which is used, for example, in Chapter VI in proving an important theorem regarding mutual impedances.

Further simplification in (30) results if the same potential difference is applied in the one case across the terminals j as in the other case across the terminals i. When

$$V_j' = V_i'' \tag{31}$$

it follows that the reciprocal theorem becomes simply

$$I_j'' = I_i' \tag{32}$$

This form of the theorem is applied in Volume II to prove that under specified conditions the directional properties of an antenna or an array of antennas are the same for transmission as for reception.

22. Electrodynamical Similitude and the Theory of Models. The experimental study of electromagnetic phenomena (as in the design of apparatus or the determination of characteristics of complicated circuits that are too intricate to permit of exact theoretical analysis) is often greatly simplified if experiments can be carried out using scale models of convenient size. Antennas used at broadcast frequencies are so large that an investigation of the properties of different shapes, for example, could be carried out much more readily using small scale models. Microwave equipment, on the other hand, may be so small that a study of changes in design is more easily performed with larger scale models. The question arises, under what conditions is it possible to use scale models?

In order to answer this question, let the electrical properties of a configuration of imperfect conductors and dielectrics in unbounded space be compared with the properties of a scale model that differs from the original only in having all dimensions changed by a factor n that may be larger or smaller than one. Each length L_n in the model is related to the corresponding length L in the original according to

$$L_n = nL \tag{1}$$

At all points in space the electromagnetic field can be determined from the equation

$$\nabla^2 \phi + \beta_0^2 \phi = 0 \tag{2}$$

and a similar equation for the vector potential. The corresponding equation

$$\nabla_n^2 \phi_n + \beta_{0n}^2 \phi_n = 0 \tag{3}$$

is true at all points in space about the scale model. The subscript n merely distinguishes a quantity in the model from the corresponding quantity without subscript in the original. The requirement that the electrical properties of the model shall be the same as in the original except for a change in scale requires at least that the scalar field ϕ_n in the space about the model shall differ from the scalar field ϕ in space about the original at most by a constant factor m which need not necessarily be n. Accordingly, (3) must be true if, after multiplying by m, ϕ is written for $m\phi_n$. Since all lengths are to be changed by a factor n in the model, the operator ∇_n^2, which is equivalent to a second derivative with respect to the space coordinates, may be replaced by

$$\nabla_n^2 = \frac{\nabla^2}{n^2} \tag{4}$$

Accordingly, (3) must be equivalent to

$$\nabla^2 \phi + n^2 \beta_{0n}^2 \phi = 0 \qquad . \tag{5}$$

if the model is to be electrically equivalent to the original. Since (5) and (2) must be identical, it follows that

$$\beta_{0n}^2 = \frac{\beta_0^2}{n^2} \tag{6}$$

Since $\beta_0 = \omega/v_0$, where v_0 is a universal constant, it follows that (6) is equivalent to

$$\omega_n = \frac{\omega}{n} \quad \text{or} \quad f_n = \frac{f}{n} \tag{7}$$

That is, the frequency used in the model must be that of the original divided by n if the dimensions are to be those of the original multiplied by n. The same conclusion is reached if the vector potential $\mathbf{A}$ is written for ϕ in (2), (3), and (5).

In order to have the potential functions in the model and the original differ at most by a constant factor at all corresponding points in space about the conductors and dielectrics in the configuration, the distributions of charge and current in these conductors and dielectrics must be the same at corresponding points except for a constant factor. This is true because the scalar and vector potentials at points in space are computed from the Helmholtz integrals (1.33) and (1.34) in terms of the distributions of charge and current. If all dielectrics and conductors are assumed to be homogeneous, simply polarizing, magnetizing, and conducting regions, the distributions of current and charge may be computed from solutions of the equations

$$\nabla^2\phi + \beta^2\phi = 0; \quad \nabla^2\mathbf{A} + \beta^2\mathbf{A} = 0 \tag{8}$$

subject to appropriate boundary conditions. Corresponding equations with subscripts n on all symbols must be true in the model. By the same reasoning as above, it is clear that if the current and charge are to be the same except for a constant factor at corresponding points in original and model, the following condition must be true in each dielectric and conductor:

$$\beta_n^2 = \frac{\beta^2}{n^2} \tag{9}$$

The complex propagation constant is defined by

$$\beta^2 = \frac{\omega^2\xi}{\nu} = \frac{\omega^2\epsilon_s}{\nu} - \frac{j\omega\sigma_s}{\nu} \tag{10}$$

For simplicity let it be assumed that the reluctivity of all materials is real so that ν may be written for $\mathbf{v}$. In this case (9) with (7) and (10) reduces to

$$\frac{\epsilon_{en}}{\nu_n} = \frac{\epsilon_e}{\nu} \tag{11}$$

$$\frac{\sigma_{en}}{\nu_n} = \frac{\sigma_e}{\nu n} \tag{12}$$

In general, the effective dielectric constant and conductivity are defined by

$$\epsilon_e = \epsilon' - \frac{\sigma''}{\omega}; \qquad \epsilon_{en} = \epsilon_n' - \frac{\sigma_n'' n}{\omega} \tag{13}$$

$$\sigma_e = \sigma' + \omega\epsilon''; \qquad \sigma_{en} = \sigma_n' + \frac{\omega\epsilon''}{n} \tag{14}$$

as described in Sec. III.14. If (13) and (14) are substituted in (11) using (7), expressions are obtained that are functions of the frequency and, hence, are not generally convenient. On the other hand, if the configurations are both restricted only to conductors and dielectrics with *negligible time lags* in conduction and polarization response so that

$$\epsilon_e \doteq \epsilon' = \epsilon \tag{15}$$
$$\sigma_e \doteq \sigma' = \sigma \tag{16}$$

the following relatively simple conditions are true:

$$\frac{\epsilon_n}{\nu_n} = \frac{\epsilon}{\nu} \tag{17}$$

$$\frac{\sigma_n}{\nu_n} = \frac{\sigma}{\nu n} \tag{18}$$

After canceling out the universal constants ν_0 and ϵ_0 in $\nu = \nu_0\nu_r$, $\epsilon = \epsilon_0\epsilon_r$, and writing $\mu_r = 1/\nu_r$,

$$\frac{\epsilon_{rn}}{\nu_{rn}} = \frac{\epsilon_r}{\nu_r} \qquad \text{or} \qquad \epsilon_{rn}\mu_{rn} = \epsilon_r\mu_r \tag{19}$$

$$\frac{\sigma_n}{\nu_{rn}} = \frac{\sigma}{\nu_r n} \qquad \text{or} \qquad \sigma_n\mu_{rn} = \frac{\sigma\mu_r}{n} \tag{20}$$

These conditions must be maintained in all conductors and dielectrics.

If a model antenna that is $n = 0.1$ times an original in size is to be used at $1/n = 10$ times the frequency, a possible arrangement is to maintain dielectric constant and permeability of all media including conductors and possibly the earth the same in the model as in the original, while the conductivity is made

$1/n = 10$ times as great in the model as in the original. If copper conductors are used in the original, this is physically impossible. However, it is shown in Volume II that the distribution of current and charge in a cylindrical antenna depends only in negligible degree on its conductivity provided this is high. It follows that copper could be used in both model and original antennas without significant error. On the other hand, if the antenna is erected over the earth, the conductivity of the soil around the model must be made $1/n = 10$ times that around the original if measurements on the model are to be comparable to those on the original.

CHAPTER V

SKIN EFFECT AND INTERNAL IMPEDANCE

Since the most significant components of the electromagnetic field in the *interior* of cylindrical conductors and of thin dielectric disks between conducting plates are affected only slightly by the outside electromagnetic field, it is possible to determine in approximate but, nevertheless, very general terms the circuit properties of unit length of a conductor or of a condenser insofar as this depends upon the internal distributions of current and charge. The results so obtained may then be used in the solution of problems involving closed and quasi-closed circuits, transmission lines, and antennas. The general problem of closed circuits is considered in Chapter VI; transmission lines in Volume III; antennas in Volume II.

This chapter is also designed to serve as an introduction to the study of boundary-value problems and the solution of the homogeneous wave equation in cylindrical coordinates · using Bessel functions.

SKIN EFFECT IN A SOLID CYLINDRICAL CONDUCTOR ·

1. The Wave Equation in Cylindrical Coordinates.—The general equation for the complex amplitude of the vector potential in a simple medium is (III.14.34*b*)

$$\nabla^2 \mathbf{A} + \beta^2 \mathbf{A} = 0 \tag{1}$$

Let this equation be expressed in cylindrical coordinates r, θ, z, subject to the following conditions for rotational symmetry:

$$\frac{\partial \mathbf{A}}{\partial \theta} = 0; \qquad A_\theta = 0 \tag{2}$$

By definition the operator ∇^2 operating on a vector has the meaning

$$\nabla^2 \mathbf{A} \equiv \text{grad div } \mathbf{A} - \text{curl curl } \mathbf{A} \tag{3}$$

so that it is necessary to expand the expression on the right in (3)

in cylindrical coordinates. The components of the curl are written below using (2).

$$\text{curl}_r \, \mathbf{A} = \frac{1}{r}\frac{\partial A_s}{\partial \theta} - \frac{\partial A_\theta}{\partial z} = 0 \tag{4}$$

$$\text{curl}_\theta \, \mathbf{A} = \frac{\partial A_r}{\partial z} - \frac{\partial A_s}{\partial r} \tag{5}$$

$$\text{curl}_s \, \mathbf{A} = \frac{1}{r}\left[\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_s}{\partial \theta}\right] = 0 \tag{6}$$

$$\text{curl}_r \, \text{curl} \, \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial \theta}(\text{curl}_s \, \mathbf{A}) - \frac{\partial}{\partial z}(\text{curl}_\theta \, \mathbf{A})$$
$$= -\frac{\partial}{\partial z}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_s}{\partial r}\right) \tag{7}$$

$$\text{curl}_\theta \, \text{curl} \, \mathbf{A} = \frac{\partial}{\partial z}(\text{curl}_r \, \mathbf{A}) - \frac{\partial}{\partial r}(\text{curl}_s \, \mathbf{A}) = 0 \tag{8}$$

$$\text{curl}_s \, \text{curl} \, \mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r \, \text{curl}_\theta \, \mathbf{A}) - \frac{\partial}{\partial \theta}(\text{curl}_s \, \mathbf{A})\right]$$
$$= \frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_s}{\partial r}\right) \tag{9}$$

Again using (2) and interchanging the order of differentiation where convenient,

$$\text{div} \, \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_s}{\partial z} \tag{10}$$

$$\text{grad div} \, \mathbf{A} = \hat{r}\frac{\partial}{\partial r}(\text{div} \, \mathbf{A}) + \hat{z}\frac{\partial}{\partial z}(\text{div} \, \mathbf{A}) \tag{11}$$

$$= \hat{r}\left[\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial^2 A_s}{\partial r \, \partial z}\right] + \hat{z}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial A_r}{\partial z} + \frac{\partial^2 A_s}{\partial z^2}\right] \tag{12}$$

Upon forming (1) using (3), (7) to (12), and separating the $\hat{r}$ and $\hat{z}$ components the following equations are obtained:

$$\frac{\partial^2 A_r}{\partial z^2} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \beta^2 A_r = 0 \tag{13}$$

$$\frac{\partial^2 A_s}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial A_s}{\partial r} + \beta^2 A_s = 0 \tag{14}$$

Following the procedure already used in Sec. III.16 solutions of (14) may be sought in the form

$$A_s(r,z) = f_s(z)F_s(r) \tag{15}$$

where $f_s(z)$ is a dimensionless function of z alone and $F_s(r)$ is a function of r alone. Substitution of (15) in (14) leads to

$$F_s \frac{\partial^2 f_s}{\partial z^2} + f_s \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial F_s}{\partial r} + \beta^2 f_s F_s = 0 \tag{16}$$

This may be written as follows:

$$\frac{1}{f_s} \frac{\partial^2 f_s}{\partial z^2} + \beta^2 = -\frac{1}{F_s} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial F_s}{\partial r} \tag{17}$$

The left side in (17) is a function of z alone; the right side a function of r alone. Hence they are mutually independent and can be equal to each other for all possible values of r and z *only* if both are equal to a constant that, however, may be multi-valued. Let it be k^2. Also let

$$\gamma^2 = -\gamma'^2 = k^2 - \beta^2$$

Then

$$\frac{\partial^2 f_s}{\partial z^2} + \gamma'^2 f_s = 0 \tag{18}$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial F_s}{\partial r} + k^2 F_s = 0 \tag{19}$$

Upon carrying out the differentiation, (19) becomes

$$\frac{\partial^2 F_s}{\partial r^2} + \frac{1}{r} \frac{\partial F_s}{\partial r} + k^2 F_s = 0 \tag{20}$$

This may be transformed by changing the independent variable. Let

$$x = kr \tag{21a}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r} \frac{dr}{dx} = \frac{1}{k} \frac{\partial}{\partial r} \tag{21b}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{1}{k} \frac{\partial}{\partial r} \right] = \frac{1}{k^2} \frac{\partial^2}{\partial r^2} \tag{21c}$$

Then

$$\frac{\partial^2 F_s}{\partial x^2} + \frac{1}{x} \frac{\partial F_s}{\partial x} + F_s = 0 \tag{22}$$

This is called the *Fourier equation*.

In the same way let solutions be sought for (13) of the type

$$A_r(r,z) = f_r(z) F_r(r) \tag{23}$$

By the same procedure as above,

$$\frac{\partial^2 f_r}{\partial z^2} + \gamma'^2 f_r = 0 \tag{24}$$

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + k^2 F_r = 0 \tag{25}$$

where k is not necessarily the same constant as above.   Carrying out the differentiation, (25) becomes

$$\frac{\partial^2 F_r}{\partial r^2} + \frac{1}{r}\frac{\partial F_r}{\partial r} + \left(k^2 - \frac{1}{r^2}\right)F_r = 0 \tag{26}$$

Using (21a,b,c)

$$\frac{\partial^2 F_r}{\partial x^2} + \frac{1}{x}\frac{\partial F_r}{\partial x} + \left(1 - \frac{1}{x^2}\right)F_r = 0 \tag{27}$$

Equations (27) and (22) are special cases of the general *Bessel equation*

$$\frac{\partial^2 F}{\partial x^2} + \frac{1}{x}\frac{\partial F}{\partial x} + \left(1 - \frac{n^2}{x^2}\right)F = 0 \tag{28}$$

Solutions for $\mathsf{A}(r,z)$ of the restricted type with components defined in (15) and (23) may thus be obtained in terms of solutions of the Bessel equation (28).   Detailed calculations are given in the following sections.   It can be shown that any particular solution of (1) having cylindrical symmetry can be obtained as a converging series of solutions of the restricted type, somewhat as a periodic function of one variable can be developed in a Fourier series.   Solutions of this general type are used, for example, in the analysis of wave guides in Volume III.

2. Electromagnetic Field and the Boundary Conditions for a Cylindrical Conductor.—The electromagnetic vectors E and B for a simple medium may be derived from the complex vector potential using (III.14.35a,b)

$$\mathsf{E} = -\frac{j\omega}{\beta^2}\{\text{grad div }\mathsf{A} + \beta^2\mathsf{A}\} \tag{1}$$

$$\mathsf{B} = \text{curl }\mathsf{A} \tag{2}$$

Using (1.1) and (1.3)

$$\mathsf{E} = -\frac{j\omega}{\beta^2}\text{ curl curl }\mathsf{A} \tag{3}$$

at points in a simple medium.   Hence from (1.7) to (1.9)

$$E_r = \frac{j\omega}{\beta^2}\left[\frac{\partial}{\partial z}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\right]; \qquad B_r = 0 \tag{4}$$

$$E_\theta = 0; \qquad\qquad B_\theta = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \tag{5}$$

$$E_z = \frac{-j\omega}{\beta^2}\left[\frac{1}{r}\frac{\partial}{\partial r}\,r\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\right]; \qquad B_z = 0 \tag{6}$$

It is shown in Sec. IV.3 that at points outside a cylindrical conductor carrying an axial current the component A_r is negligible compared with A_z.   It will be assumed, subject to later verification, that the following condition is a good approximation inside as well as outside the conductor:

$$\frac{\partial A_r}{\partial z} << \frac{\partial A_z}{\partial r} \tag{7}$$

Subject to (7), the electric field is

$$E_r = \frac{-j\omega}{\beta^2}\frac{\partial^2 A_z}{\partial z\,\partial r}; \qquad B_r = 0 \tag{8}$$

$$E_\theta = 0; \qquad B_\theta = -\frac{\partial A_z}{\partial r} \tag{9}$$

$$E_z = \frac{j\omega}{\beta^2}\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial A_z}{\partial r}; \qquad B_z = 0 \tag{10}$$

Using (1.19) multiplied by f_z,

$$\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial A_z}{\partial r} = -k^2 A_z \tag{11}$$

so that

$$E_z = \frac{-j\omega k^2}{\beta^2}\,A_z \tag{12}$$

Hence

$$E_r = \frac{-j\omega}{\beta^2}\frac{\partial}{\partial z}\frac{\partial A_z}{\partial r}; \qquad B_r = 0 \tag{13}$$

$$E_\theta = 0; \qquad B_\theta = -\frac{\partial A_z}{\partial r} \tag{14}$$

$$E_z = -j\omega\frac{k^2}{\beta^2}A_z; \qquad B_z = 0 \tag{15}$$

The general boundary conditions at the surface between two simple media include (III.14.31b,c)

$$[ň_1, E_1] + [ň_2, E_2] = 0 \tag{16}$$

$$\nu_1[ň_1, B_1] + \nu_2[ň_2, B_2] = 0 \tag{17}$$

In terms of magnitudes

$$(E_{z1})_{r=a} = (E_{z2})_{r=a} \tag{18}$$

$$\nu_1(B_{\theta 1})_{r=a} = \nu_2(B_{\theta 2})_{r=a} \tag{19}$$

Using (13) to (15)

$$\frac{k_1^2}{\beta_1^2}(A_{s1})_{r=a} = \frac{k_2^2}{\beta_2^2}(A_{s2})_{r=a} \tag{20}$$

$$\nu_1\left(\frac{\partial A_{s1}}{\partial r}\right)_{r=a} = \nu_2\left(\frac{\partial A_{s2}}{\partial r}\right)_{r=a} \tag{21}$$

These are the equations from which the constant k must be determined. Before this can be done, expressions for A_s in the conductor and outside it must be obtained.

If the cylinder is a good conductor (region 1) and the medium in which it is immersed (region 2) is space, the following statements are true, using Sec. III.15:

In space,

$$\beta_2 = \beta_0 = \frac{\omega}{v_0}; \qquad \nu_2 = \nu_0 \tag{22}$$

In a good conductor, assuming ν_1 real,

$$\beta_1 = \beta_s - j\alpha_s = (1 - j)\sqrt{\frac{\omega\sigma_1}{2\nu_1}} = \sqrt{\frac{-j\omega\sigma_1}{\nu_1}} \tag{23}$$

The magnitude of β_1 is $\beta_1 = \sqrt{\omega\sigma_1/\nu_1}$ so that

$$\beta_1 = \frac{1 - j}{\sqrt{2}}\beta_1 = j^{-\frac{1}{2}}\beta_1 \tag{24}$$

3. Solutions for a Conducting Cylinder of Infinite Length.—Although an infinitely long cylindrical conductor is not physically realizable, it is of great practical importance because the distribution of current in any cross section may be determined by a relatively simple analysis; moreover, this distribution is to a high degree of approximation the same as in any cylindrical conductor that is long compared with its radius. This is discussed later. For the present, let the distribution of the axial component A_s of the vector potential be determined in a cylindrical conductor of radius a extending to infinity along the positive z axis. Let it be so excited that complete rotational symmetry obtains. This requires a solution of (1.18) and (1.19) subject to the boundary conditions (2.20) and (2.21).

The solution of (1.18) is simple and well known. It will be outlined in somewhat greater detail than required in order to show the similarity in the several forms of its solution as com-

pared with the solution of (1.19) in the form (1.22). A general solution of (1.18) may be obtained by writing a power series for f_z, substituting this in the differential equation, and evaluating the coefficients in the series from the resulting recurrence relations.   The solution is

$$f_z = C_1 \cos u + C_2 \sin u \qquad (1)$$

where $u = \gamma'z$ and

$$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots \qquad (2)$$

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots \qquad (3)$$

and C_1 and C_2 are complex constants of integration.   The functions $\cos u$ and $\sin u$ are tabulated in standard trigonometric tables with u real.   The solutions may also be written in the form

$$f_z(z) = C_1' e^{ju} + C_2' e^{-ju} \qquad (4)$$

where

$$e^{ju} = \cos u + j \sin u \qquad (5)$$

$$e^{-ju} = \cos u - j \sin u \qquad (6)$$

and C_1' and C_2' are complex constants.   The form (1) is preferred for *real* arguments $u = \gamma'z$ and for conductors of finite length. The form (4) is preferred for complex arguments $u = \gamma'z$ and if the conductor is assumed to be infinite in length.   In the case at hand, the wire is assumed to be infinitely long and γ' is complex. In particular with

$$\gamma' = -j\gamma = -j(\gamma_r + j\gamma_i) \qquad (7)$$

the solution (4) is

$$f_z = C_1' e^{(\gamma_r + j\gamma_i)z} + C_2' e^{-(\gamma_r + j\gamma_i)z} \qquad (8)$$

Since f_z must remain finite at $z = \infty$, it is necessary that $C_1' = 0$, leaving the solution

$$f_z = C_2' e^{-\gamma_r z} e^{-j\gamma_i z} \qquad (9)$$

The Fourier equation (1.22) may also be solved by writing a power series for $F_z(r)$, substituting this in the differential equation, and evaluating the coefficients from recurrence relations so obtained.   A general solution of (1.22) corresponding to (1) is

$$F_z(r) = D_1 J_0(x) + D_2 N_0(x) \qquad (10)$$

where D_1 and D_2 are complex constants, $x = kr$, and

$$J_0(x) = 1 - (\tfrac{1}{2}x)^2 + \frac{(\tfrac{1}{2}x)^4}{(2!)^2} - \frac{(\tfrac{1}{2}x)^6}{(3!)^2} + \cdots$$

$$= \sum_{i=0}^{\infty} (-1)^i \frac{(\tfrac{1}{2}x)^{2i}}{(i!)^2} \quad (11)$$

$$N_0(x) = \frac{2}{\pi} \left\{ \ln\left(\frac{1}{2}x\right) + C \right\} J_0(x)$$

$$- \frac{2}{\pi} \sum_{i=1}^{\infty} (-1)^i \frac{(\tfrac{1}{2}x)^{2i}}{(i!)^2} \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{i} \right\} \quad (12)$$

C is Euler's constant given by $C = 0.5772$.

The series $J_0(x)$ is called a *Bessel function of the first kind* and order zero; the series $N_0(x)$ is called a *Bessel function of the second kind* (or *Neumann function*) of order zero. These functions are tabulated in Appendix III for real arguments. $J_0(x)$ and $N_0(x)$ are both real with real arguments $x = kr$; $J_0(x)$ and $N_0(x)$ are complex with complex argument $x = kr$. Limiting values for zero argument are

$$\lim_{x \to 0} J_0(x) = 1 \quad (13a)$$

$$\lim_{x \to 0} N_0(x) = \infty \quad (13b)$$

If the argument is real,

$$\lim_{x \to \infty} J_0(x) = 0 \quad (14a)$$

$$\lim_{x \to \infty} N_0(x) = 0 \quad (14b)$$

If the argument is complex,

$$\lim_{x \to \infty} J_0(x) = \infty \quad (15a)$$

$$\lim_{x \to \infty} N_0(x) = \infty \quad (15b)$$

For small values of the argument, higher powered terms in (11) and (12) are negligible and the leading terms alone are good approximations. They are

$$J_0(x) \doteq 1; \quad x^2 \ll 1 \quad (16a)$$

$$N_0(x) \doteq \frac{2}{\pi} \left\{ \ln\left(\frac{1}{2}x\right) + C \right\}; \quad x^2 \ll 1 \quad (16b)$$

For moderately large values of the argument, the following asymptotic formulas are good approximations:

$$J_0(x) \doteq \pm \sqrt{\frac{2}{\pi x}} \cos \left(x - \frac{\pi}{4} \right) \Big\} \qquad x \geq 10 \qquad (17a)$$

$$N_0(x) \doteq \pm \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{\pi}{4} \right) \Big\} \quad -\frac{\pi}{2} \leq \arg x \leq \frac{\pi}{2} \quad (17b)$$

The error is of the order of 1 per cent at $x = 10$ so that moderately accurate approximations are possible for even smaller values of x. Complete asymptotic series are given in standard tables.[1] The relation between $J_0(x)$ and $\cos x$, $N_0(x)$ and $\sin x$ is evident from (17a) and (17b).

An alternative form of the solution of (1.19), corresponding to the exponential solution (4) of (1.18), is

$$F_s(x) = D_1' H_0^{(1)}(x) + D_2' H_0^{(2)}(x) \tag{18}$$

where $x = kr$ and

$$H_0^{(1)}(x) = J_0(x) + jN_0(x) \tag{19}$$
$$H_0^{(2)}(x) = J_0(x) - jN_0(x) \tag{20}$$

and D_1' and D_2' are complex constants. The functions $H_0^{(1)}(x)$ and $H_0^{(2)}(x)$ are called *Bessel functions of the third kind* and order zero (or *Hankel functions*). $H_0(x)$ is complex with x real, but $jH_0^{(1)}(jx)$ and $-jH_0^{(2)}(-jx)$ are real with x real.

$$\lim_{x \to 0} H_0^{(1)}(x) = \infty ; \qquad \lim_{x \to 0} H_0^{(2)}(x) = \infty \tag{21}$$

$$\lim_{x \to \infty} H_0^{(1)}(x) = 0 \text{ if the imaginary part of } x \text{ is positive} \qquad (22a)$$

$$\lim_{x \to \infty} H_0^{(2)}(x) = 0 \text{ if the imaginary part of } x \text{ is negative} \qquad (22b)$$

One of the Hankel functions always vanishes with infinite *complex* argument, as shown in (22). No other Bessel function does. Approximate formulas for small arguments are obtained from (19) and (20) using (16).

$$H_0^{(1)}(x) \doteq 1 + j\frac{2}{\pi} \left\{ \ln \left(\frac{1}{2} x \right) + C \right\} \Big\} \quad x^2 << 1 \qquad (23)$$

$$H_0^{(2)}(x) \doteq 1 - j\frac{2}{\pi} \left\{ \ln \left(\frac{1}{2} x \right) + C \right\} \Big\}$$

[1] JAHNKE-EMDE, "Tables of Functions," pp. 137, 138, 1938 ed.

Asymptotic formulas for large arguments are obtained from (19) and (20) using (17).

$$\left.\begin{array}{l} H_0^{(1)}(x) \doteq \pm \sqrt{\dfrac{2}{\pi x}}\, e^{j(x - \pi/4)} \\[3mm] H_0^{(2)}(x) \doteq \pm \sqrt{\dfrac{2}{\pi x}}\, e^{-j(x - \pi/4)} \end{array}\right\} \; x \geq 10 \qquad (24)$$

The relationship between the Hankel functions and the exponential function is clear from (24). $H_0^{(1)}(x)$ resembles e^{jx}, $H_0^{(2)}(x)$ resembles e^{-jx}. The limits (22a,b) can be derived from (24) directly. For solutions of bounded regions, the functions $J_0(x)$ and $N_0(x)$ are usually convenient. For solutions in unbounded space with k in $x = kr$ complex, one of the functions $H_0^{(1)}(x)$, $H_0^{(2)}(x)$ is required.

For the case at hand, an infinitely long cylindrical conductor of finite radius a in space, a function that is finite at the origin, *i.e.*, $J_0(k_1 r)$, is required for the interior of the conductor. For free space outside the conductor, the Hankel function that vanishes at $r = \infty$ must be used. Thus,

$$r < a, \qquad F_s(r) = D J_0(k_1 r) \qquad (25)$$
$$r > a, \qquad F_s(r) = D H_0^{(1)}(k_2 r) \text{ or } D H_0^{(2)}(k_2 r) \qquad (26)$$

In (26) the Hankel function that vanishes at $r = \infty$ is chosen. Values of k appropriate to the medium must be used as indicated. The conductor is medium 1, space is medium 2.

The solution for A_s is given below.
In the conductor,

$$r \leq a; \qquad A_s = C_1 J_0(k_1 r) \exp(-j\gamma_1' z) \qquad (27)$$

In space,

$$r \geq a; \qquad A_s = C_2 H_0(k_2 r) \exp(-j\gamma_2' z) \qquad (28)$$

with appropriate superscript (1) or (2) on H_0, whichever vanishes at $r = \infty$. In general

$$\gamma' = \sqrt{\beta^2 - k^2} \qquad (29)$$

An important special case which may be obtained either by resolving (1.20) or from the solution (10) is that for which $k = 0$. In this case (1.19) reduces to

$$\frac{1}{r} \frac{d}{dr} r \frac{dF_s}{dr} = 0 \qquad (30)$$

with solution

$$F_z(r) = C_1 + C_2 \ln r \tag{31}$$

This is not a possible solution for any region that contains the axis $r = 0$ where the logarithm becomes infinite. Since the logarithm also becomes infinite at $r = \infty$, it appears that (31) is not a physically meaningful solution. This is true. However, in the physically impossible but nevertheless useful case of an infinitely long conductor, the vector potential cannot vanish at infinity because the conductor extends to infinity. Accordingly, in space a possible solution is

$$r \geq a; \qquad A_z = [C_1 + C_2 \ln r] e^{-j\beta_0 z} \tag{32}$$

provided $k = 0$ and the conductor is infinitely long.

4. Determination of Parameters.—The parameters k_1 and k_2 appearing in the solutions for A_z must be determined from the boundary conditions (2.20) and (2.21) using (3.27) and (3.28).

$$\frac{k_1^2}{\beta_1^2} C_1 J_0(k_1 a) \exp (-j\gamma_1' z) = \frac{k_2^2}{\beta_2^2} C_2 H_0(k_2 a) \exp (-j\gamma_2' z) \tag{1}$$

$$\nu_1 C_1 k_1 J_0'(k_1 a) \exp (-j\gamma_1' z) = \nu_2 C_2 k_2 H_0'(k_2 a) \exp (-j\gamma_2' z) \tag{2}$$

The notation

$$J_0'(x) = \frac{\partial}{\partial x} [J_0(x)] \tag{3a}$$

$$H_0'(x) = \frac{\partial}{\partial x} [H_0(x)] \tag{3b}$$

is used. The superscript 1 or 2 is omitted from H_0 until it is determined which of the two functions $H_0^{(1)}$ or $H_0^{(2)}$ vanishes at infinity. This requires knowledge of the sign of the imaginary part of k_2.

It follows from (1) or (2) that

$$\gamma_1' = \gamma_2' = \gamma' \tag{4}$$

if the equations are to be valid for all values of z at all times. Hence with (3.29)

$$\gamma' = \sqrt{\beta_1^2 - k_1^2} = \sqrt{\beta_2^2 - k_2^2} \tag{5}$$

β_1 and β_2 are given by (2.23) and (2.22), respectively. Dividing (1) by (2)

$$\frac{k_1}{\nu_1 \beta_1^2} \frac{J_0(k_1 a)}{J_0'(k_1 a)} = \frac{k_2}{\nu_2 \beta_2^2} \frac{H_0(k_2 a)}{H_0'(k_2 a)} \tag{6}$$

Equations (5) and (6) are theoretically sufficient to determine k_1 and k_2. Actually an explicit solution is not possible in general. An approximation that permits a determination of the order of magnitude for a good conductor in space is obtained by allowing the conductivity to become indefinitely large. In this case,

$$\sigma_1 \to \infty, \qquad d_s \to 0, \qquad \beta_1 \to \infty \tag{7}$$

where d_s is the skin depth. Since γ' may be assumed finite it follows using (5) that

$$k_1 = \sqrt{\beta_1^2 - \gamma'^2} \to \beta_1 \tag{8}$$

With $k_1 \to \infty$ it is possible to use (3.17).

$$J_0(k_1 r) \to \pm \sqrt{\frac{2}{\pi k_1 r}} \cos\left(k_1 r - \frac{\pi}{4}\right) \tag{9}$$

Differentiating and retaining only the leading term as $k_1 \to \infty$,

$$J_0'(k_1 r) \to \mp \sqrt{\frac{2}{\pi k_1 r}} \sin\left(k_1 r - \frac{\pi}{4}\right) \tag{10}$$

With

$$\beta_1 = \frac{(1 - j)\beta_1}{\sqrt{2}} = j^{-\frac{1}{2}}\beta_1 \tag{11}$$

it follows that as $\beta_1 \to \infty$,

$$\cos\left(\beta_1 r - \frac{\pi}{4}\right) = \frac{1}{2}\left[e^{j\left(\beta_1 r - \frac{\pi}{4}\right)} + e^{-j\left(\beta_1 r - \frac{\pi}{4}\right)}\right] \to \frac{1}{2} e^{j\left(\beta_1 r - \frac{\pi}{4}\right)} \tag{12}$$

$$\sin\left(\beta_1 r - \frac{\pi}{4}\right) = \frac{1}{2j}\left[e^{j\left(\beta_1 r - \frac{\pi}{4}\right)} - e^{-j\left(\beta_1 r - \frac{\pi}{4}\right)}\right] \to \frac{-j}{2} e^{j\left(\beta_1 r - \frac{\pi}{4}\right)} \tag{13}$$

Also

$$\sqrt{\frac{2}{\pi\beta_1 r}} = \sqrt{\frac{2}{\pi\beta_1 r}} j^{\frac{1}{4}} = \sqrt{\frac{2}{\pi\beta_1 r}} e^{(j\pi/8)} \tag{14}$$

The last step in (14) follows from the relation

$$j = \cos\frac{\pi}{2} + j \sin\frac{\pi}{2} = e^{j\frac{\pi}{2}} \tag{15}$$

and

$$j^z = e^{jz\pi/2} \tag{16}$$

With (11) to (14)

$$\lim_{\beta_1 r \to \infty} J_0(\beta_1 r j^{-\frac{1}{2}}) = \lim_{\beta_1 r \to \infty} \frac{e^{\beta_1 r/\sqrt{2}}}{\sqrt{2\pi\beta_1 r}} e^{j(\beta_1 r/\sqrt{2} - \pi/8)} \quad (17)$$

$$\lim_{\beta_1 r \to \infty} J_0'(\beta_1 r j^{-\frac{1}{2}}) = \lim_{\beta_1 r \to \infty} \frac{j e^{\beta_1 r/\sqrt{2}}}{\sqrt{2\pi\beta_1 r}} e^{j(\beta_1 r/\sqrt{2} - \pi/8)} \quad (18)$$

Accordingly,

$$\frac{J_0(k_1 a)}{J_0'(k_1 a)} \to \frac{J_0(\beta_1 a)}{J_0'(\beta_1 a)} = -j \quad (19)$$

With (19) and (8), (6) becomes

$$\frac{k_2}{\nu_2 \beta_2^2} \frac{H_0(k_2 a)}{H_0'(k_2 a)} = \lim_{\beta_1 \to \infty} \frac{-j}{\nu_1 \beta_1} = 0 \quad (20)$$

This equation is satisfied by

$$k_2 = 0 \quad (21)$$

if it can be shown that $H_0(k_2 a)/H_0'(k_2 a)$ remains finite as k_2 approaches zero. The behavior of the Hankel functions at the zero value of the argument is determined by the leading term in the series for

$$H_0(kr) = J_0(kr) \pm j N_0(kr) \quad (22)$$

as $k \to 0$. Thus with (3.13) and the leading term in (3.12)

$$\lim_{k \to 0} H_0(kr) = 1 \pm j \lim_{k \to 0} \frac{2}{\pi} \left\{ \ln\left(\frac{1}{2} kr\right) + C \right\} = \pm j \lim_{k \to 0} \ln(kr) \quad (23)$$

Differentiating with respect to the argument kr

$$\lim_{k \to 0} H_0'(kr) = \pm j \lim_{k \to 0} \left(\frac{1}{kr}\right) \quad (24)$$

Hence

$$\lim_{k \to 0} \frac{H_0(kr)}{H_0'(kr)} = \lim_{k \to 0} \frac{\ln kr}{\dfrac{1}{kr}} \quad (25)$$

This is indeterminate and may be evaluated by differentiating numerator and denominator separately. Then

$$\lim_{k \to 0} \frac{H_0(kr)}{H_0'(kr)} = \lim_{k \to 0} \frac{\dfrac{1}{kr}}{-\dfrac{1}{(kr)^2}} = 0 \quad (26)$$

Accordingly, (21) is a correct solution of (20).   Using (21) and (5) with (2.22) it is clear that

$$\gamma' = \beta_2 = \beta_0 \tag{27}$$

It follows that the complex phase constant of a *perfect* conductor reduces to the real phase constant β_0.

Upon substituting (27) in (5)

$$k_1 = \sqrt{\beta_1^2 - \beta_0^2} \tag{28}$$

Thus

$$k_1 \doteq \beta_1 \tag{29}$$

provided

$$\beta_1^2 >> \beta_0^2 \tag{30}$$

which is true by definition when the cylinder is a good conductor. It may be assumed, therefore, that (29) is a good approximation for all sufficiently good conductors.

As a result of (29), the transverse distribution of the vector potential at points inside the conductor is independent of the axial distribution.   It is reasonable to assume that the transverse distribution is also independent of the axial distribution in a conductor of finite length, so that it may be used directly to determine the distribution of current and to define the impedance per unit length of a cylindrical conductor of any length that is great compared with the radius.

5. Distribution of Current in the Interior of a Cylindrical Conductor.—The transverse distribution of the vector potential in the interior of a long cylindrical conductor is given by (3.10) multiplied by $f_z(z)$.   It is

$$A_z(r) = [D_1 J_0(k_1 r) + D_2 N_0(k_1 r)] f_z(z) \tag{1}$$

The subscript 1 on k_1 denotes region 1, the conductor.   The two constants of integration, D_1 and D_2, may be evaluated as follows. Since $A_z(r)$ must be finite at all points in the conductor, and since from (3.13) $N_0(kr)$ is infinite at $r = 0$, a physically meaningful solution requires that $D_2 = 0$.   The constant D_1 may be expressed in terms of the value of $A_z(r)$ at $r = a$.   Thus

$$D_1 = \frac{A_z(a)}{J_0(k_1 a)} \tag{2}$$

and

$$A_z(r) = A_z(a) \frac{J_0(k_1 r)}{J_0(k_1 a)} \tag{3}$$

If $E_s(a)$ is the z component of the electric field at the surface of the conductor, it follows from (2.15) that

$$E_s(r) = E_s(a) \frac{J_0(k_1 r)}{J_0(k_1 a)} \tag{4}$$

Since

$$i_s = \sigma E_s \tag{5}$$

the density of axial current at any radius r is

$$i_s(r) = i_s(a) \frac{J_0(k_1 r)}{J_0(k_1 a)} \tag{6}$$

with $i_s(a)$ the volume density of current extrapolated to the surface $r = a$. In (3), (4), and (6) using (4.29) and (4.11)

$$k_1 \doteq \beta_1 = j^{-\frac{1}{2}} \beta_1 = j^{-\frac{1}{2}} \sqrt{\frac{\omega \sigma_1}{\nu_1}} = j^{-\frac{1}{2}} \sqrt{\omega \mu_1 \sigma_1} \tag{7}$$

so that these solutions involve Bessel functions of the form $J_0(j^{-\frac{1}{2}} y)$ with $y = \beta_1 r$ real. Since (1) with a negative sign written in front of k_1 also satisfies the differential equation (1.20), alternative solutions of the form $J_0(-j^{-\frac{1}{2}} y) = J_0(j^{\frac{3}{2}} y)$ may be used. Since only even powers of j occur in the series (3.11), it follows that $J_0(j^{\frac{3}{2}} y) = J_0(j^{-\frac{1}{2}} y)$. Thus

$$J_0(j^{\frac{3}{2}} y) = J_0(j^{-\frac{1}{2}} y) = \left[1 - \frac{(\frac{1}{2} y)^4}{(2!)^2} + \frac{(\frac{1}{2} y)^8}{(4!)^2} - \cdots \right]$$
$$+ j \left[(\frac{1}{2} y)^2 - \frac{(\frac{1}{2} y)^6}{(3!)^2} + \cdots \right] \tag{8}$$

This may be written as follows:

$$J_0(j^{\frac{3}{2}} y) = J_0(j^{-\frac{1}{2}} y) = \text{ber } y + j \text{ bei } y = M_0(y) e^{j\theta_0(y)} \tag{9}$$

The notation

$$\text{ber } y = \text{real part } J_0(j^{-\frac{1}{2}} y) = 1 - \frac{(\frac{1}{2} y)^4}{(2!)^2} + \frac{(\frac{1}{2} y)^8}{(4!)^2} - \cdots \tag{10a}$$

$$\text{bei } y = \text{imaginary part } J_0(j^{-\frac{1}{2}} y) = (\frac{1}{2} y)^2 - \frac{(\frac{1}{2} y)^6}{(3!)^2} + \cdots \tag{10b}$$

is conventional. These functions are tabulated in standard tables.[1] Unfortunately they are oscillating functions and there-

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[1] H. B. Dwight, "Tables of Functions," pp. 214–220. JAHNKE-EMDE, "Tables of Functions," pp. 244–258, 1938 ed. The function $J_0(j^{\frac{1}{2}} y) = \text{ber } y - j \text{ bei } y$ is given. N. W. McLachlan, "Bessel Functions for Engineers."

fore not well adapted to interpolation by proportional parts. In this respect, the amplitude function $M_0(y)$ and the phase angle $\theta_0(y)$ are preferable. They are given by

$$M_0(y) = |J_0(j^{-\frac{1}{2}}y)| = \sqrt{\text{ber}^2\, y + \text{bei}^2\, y} \tag{11}$$

$$\theta_0(y) = \arg J_0(j^{-\frac{1}{2}}y) = \tan^{-1}\left(\frac{\text{bei}\, y}{\text{ber}\, y}\right) \tag{12}$$

Conversely,

$$\text{ber}\, y = M_0(y)\, \cos\, \theta_0(y) \tag{13}$$

$$\text{bei}\, y = M_0(y)\, \sin\, \theta_0(y) \tag{14}$$

Fig. 5.1.—The functions $M_0(y)$, $M_1(y)$, $\theta_0(y)$, $\theta_1(y)$.

The functions $M_0(y)$ and $\theta_0(y)$ are represented graphically in Fig. 5.1 and are tabulated in Appendix III and in standard books of tables.

Instead of making use of the phase angle $\theta_0(y)$, it is possible to write

$$j = e^{j\frac{\pi}{2}} \tag{15a}$$

$$j^{x_0} = e^{jx_0\frac{\pi}{2}} = e^{j\theta_0(y)} \tag{15b}$$

so that

$$x_0 = \frac{2}{\pi}\theta_0(y) \tag{15c}$$

Then

$$J_0(j^{-\frac{1}{2}}y) = M_0(y)j^{x_0} \tag{16}$$

The functions $M_0(y)$ and x_0 are tabulated and represented graphically in standard tables.[1]

If y is sufficiently small, it follows from (8)

$$M_0(y) \doteq 1; \qquad \theta_0(y) \doteq 0; \qquad y^2 << 4 \tag{17}$$

If the argument y in (8) is sufficiently large it is possible to obtain a simpler formula for $M_0(y)$ in (11) and $\theta_0(y)$ in (12). Thus, substituting $x = j^{-\frac{1}{2}}y$ in (3.17a)

$$J_0(j^{-\frac{1}{2}}y) \doteq \pm \sqrt{\frac{2j^{\frac{1}{2}}}{\pi y}}\cos\left[j^{-\frac{1}{2}}y - \frac{\pi}{4}\right] \tag{18}$$

for sufficiently large values of y.
Using (15b) with $x_0 = \frac{1}{4}$,

$$j^{\frac{1}{4}} = e^{\frac{\pi}{8}} \tag{19}$$

With

$$j^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}(1 - j) \tag{20}$$

$$\cos\left[j^{-\frac{1}{2}}y - \frac{\pi}{4}\right] = \frac{1}{2}[e^{y/\sqrt{2}}e^{j(y/\sqrt{2}-\pi/4)}$$

$$+ e^{-y/\sqrt{2}}e^{-j(y/\sqrt{2}-\pi/4)}] \tag{21}$$

Since y is necessarily large if (18) is to be a good approximation, the second term in the sum on the right in (21) is negligible compared with the first term. With (19) and (21)

$$J_0(j^{-\frac{1}{2}}y) \doteq \pm \frac{1}{\sqrt{2\pi y}}e^{y/\sqrt{2}}e^{j(y/\sqrt{2}-\pi/8)}; \qquad y \geq 10 \tag{22}$$

[1] JAHNKE-EMDE, "Tables of Functions," pp. 260, 262–265, 1938 ed. The function $J_0(j^{\frac{1}{2}}y) = M_0(y)e^{-j\theta_0(y)}$ is represented in the form $b_0j^{\beta_0}$. Hence

$$M_0(y) = b_0; \qquad \theta_0(y) = -\frac{\pi}{2}\beta_0.$$

Hence for large argument and choosing the positive sign,

$$M_0(y) \doteq \frac{1}{\sqrt{2\pi y}}\, e^{y/\sqrt{2}}; \qquad y \gtrsim 10 \tag{23a}$$

$$\theta_0(y) \doteq \left(\frac{y}{\sqrt{2}} - \frac{\pi}{8}\right); \qquad y \gtrsim 10 \tag{23b}$$

If the negative sign is chosen, $\theta_0(y)$ differs from (23b) by π. Using (7) and (9)

$$\frac{J_0(k_1 r)}{J_0(k_1 a)} = \frac{M_0(\beta_1 r)}{M_0(\beta_1 a)}\, e^{-j[\theta_0(\beta_1 a) - \theta_0(\beta_1 r)]} \tag{24}$$

If (24) is substituted in (3), (4), and (6), the following ratios may be written:

$$\frac{A_z(r)}{A_z(a)} = \frac{E_z(r)}{E_z(a)} = \frac{i_z(r)}{i_z(a)} = \frac{M_0(\beta_1 r)}{M_0(\beta_1 a)}\, e^{-j[\theta_0(\beta_1 a) - \theta_0(\beta_1 r)]} \tag{25}$$

Where

$$\beta_1 = \sqrt{\frac{\omega \sigma_1}{\nu_1}} = \sqrt{\omega \sigma_1 \mu_1}$$

This relation characterizes completely the distribution of the *axial* components of the vector potential, electric field, and density of current in a section of the conductor in terms of the value at the surface $r = a$. For sufficiently large values of $\beta_1 r$, the asymptotic formulas (23a,b) may be used to give

$$\frac{A_z(r)}{A_z(a)} = \frac{E_z(r)}{E_z(a)} = \frac{i_z(r)}{i_z(a)} = \sqrt{\frac{a}{r}}\, e^{-\beta_1(a-r)/\sqrt{2}} e^{-j\beta_1(a-r)/\sqrt{2}}$$
$$\beta_1 r \gtrsim 10 \quad (26)$$

Since $(a - r)$ measures the distance s radially in from the surface and

$$\frac{\beta_1}{\sqrt{2}} = \sqrt{\frac{\omega \sigma_1}{2\nu_1}} = \frac{1}{d_s} \tag{27}$$

where d_s is the skin depth defined in (III.15.31),

$$\frac{A_z(s)}{A_z(s = 0)} = \frac{E_z(s)}{E_z(s = 0)} = \frac{i_z(s)}{i_z(s = 0)} = \sqrt{\frac{a}{a - s}}\, e^{-s/d_s} e^{-js/d_s} \tag{28}$$

The skin depth d_s is seen to be the radial distance from the surface at which i_s is reduced to $1/e$ of its value at the surface, provided

this distance is negligibly small compared with the radius a. Formulas (26) and (28) are good approximations of the more complicated relation (25) whenever $\beta_1 r \geq 10$.

If the current density at the surface varies as

$$i_s(a)_{(\text{inst})} = i_s(a) \cos \omega t$$

the instantaneous volume density of current at a radius r is

$$i_s(r)_{(\text{inst})} = i_s(a) \, \frac{M_0(\beta_1 r)}{M_0(\beta_1 a)} \cos \left[\omega t - \theta_0(\beta_1 a) + \theta_0(\beta_1 r)\right] \quad (29)$$

Curves for the relative amplitude $M_0(\beta_1 r)/M_0(\beta_1 a)$ and for the phase angle $\theta_0(\beta_1 a) - \theta_0(\beta_1 r)$ are shown in Fig. 5.2 for

Fig. 5.2.—Relative amplitude and phase angle of the volume density of axial current in a cylindrical conductor for four values of $\beta_1 a$; d is the skin depth for $a = 1$ and $\beta_1 a = 25$.

several different but relatively small values of $\beta_1 a$. Since $\beta_1 = \sqrt{\omega \sigma_1 / \nu_1}$, these apply to any combination of ω, σ_1, ν_1, and

a, which yield the same assumed value of $\beta_1 a$. It is clear that for materials with lower reluctivity (higher permeability) the

Fig. 5.3a.—Direction and relative instantaneous values of the density of axial current in a cylindrical conductor for which $\beta_1 a = 25$ at the instant when the density at the surface is maximum downward.

Fig. 5.3b.—Like Fig. 5.3a but a quarter period later when the density at the surface is zero.

same distribution is obtained for smaller values of $\omega\sigma_1$. A study of the curves of Fig. 5.2 shows that at very low frequencies for which $\beta_1 a$ is small the amplitude of the current density is

almost uniform from the surface of the cylindrical conductor to the axis, and all values are essentially in the same phase. At even moderately high frequencies, on the other hand, for which $\beta_1 a$ is larger, the amplitude diminishes very rapidly from the surface inward, and there is a progressive and almost linear increase in the phase angle of lag. For sufficiently large values of $\beta_1 a$, this angle may exceed π or many multiples of π. This means that the direction of current alternates in layers from the surface inward. This is illustrated in Fig. 5.3*a,b* in which the instantaneous value of the current density is shown at two instants differing by a quarter period. Values of β_1 as high as 10^4 or 10^5 radians per meter are quite possible at superfrequencies. For such large values the skin depth is of order of magnitude 10^{-4} or 10^{-5} meters, so that the amplitude curves drop so rapidly that they cannot be represented in Fig. 5.2 even for small values of a.

6. Total Current in the Conductor.—The total current across a circle of radius r at the center of a cross section of the conductor of radius a is defined by

$$I_s(r) = \int_0^r i_s(r) 2\pi r \, dr \tag{1}$$

Using (5.6) and changing variables to $x = k_1 r$

$$I_s(r) = \frac{2\pi i_s(a)}{k_1^2 J_0(k_1 a)} \int_0^{k_1 r} x J_0(x) dx \tag{2}$$

The integral is of the form

$$\int_0^x x J_0(x) dx = x J_1(x) \tag{3}$$

where $J_1(x)$ is the Bessel function of the first kind and first order. It is a solution of (1.27). In series form it is

$$J_1(x) = \frac{x}{2} \left[1 - \frac{(\tfrac{1}{2}x)^2}{1 \cdot 2} + \frac{(\tfrac{1}{2}x)^4}{1 \cdot 2 \cdot 2 \cdot 3} - \frac{(\tfrac{1}{2}x)^6}{1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 4} + \cdots \right]$$
$$= -J_0'(x) \tag{4}$$

This is easily verified by integrating the convergent series for $x J_0(x)$ term by term. Formulas like (3) are also true for Bessel functions of the second and third kinds. Limiting values of $J_1(x)$ for complex arguments are

$$\lim_{x \to 0} J_1(x) = 0; \qquad \lim_{x \to \infty} J_1(x) = \infty \tag{5}$$

For real arguments

$$\lim_{x \to 0} J_1(x) = 0; \qquad \lim_{x \to \infty} J_1(x) = 0 \qquad \qquad (6)$$

Approximate formulas for small and large values of the argument are

$$J_1(x) \doteq \frac{x}{2}; \qquad x^2 << 1 \qquad (7)$$

$$J_1(x) \doteq \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{3\pi}{4}\right); \qquad \begin{cases} x \geq 10 \\ -\frac{\pi}{2} \leq \arg x \lesssim \frac{\pi}{2} \end{cases} \qquad (8)$$

Using (3) in (2) with $k_1 \doteq \beta_1 = j^{-\frac{1}{2}}\beta_1 = \beta_1 e^{-j\frac{\pi}{4}}$ according to (5.7) and (5.15b)

$$I_s(r) = \frac{2\pi r i_s(a)}{\beta_1} \frac{J_1(j^{-\frac{1}{2}}\beta_1 r)}{J_0(j^{-\frac{1}{2}}\beta_1 a)} e^{j\frac{\pi}{4}} \qquad (9)$$

The total current in the conductor is

$$I_s(a) = \frac{2\pi a i_s(a)}{\beta_1} \frac{J_1(j^{-\frac{1}{2}}\beta_1 a)}{J_0(j^{-\frac{1}{2}}\beta_1 a)} e^{j\frac{\pi}{4}} \qquad (10)$$

with $\beta_1 = \sqrt{\omega\sigma_1/\nu_1} = \sqrt{\omega\sigma_1\mu_1}$.

Alternatively,

$$i_s(a) = \frac{\beta_1 I_s(a)}{2\pi a} \frac{J_0(j^{-\frac{1}{2}}\beta_1 a)}{J_1(j^{-\frac{1}{2}}\beta_1 a)} e^{-j\frac{\pi}{4}} \qquad (11)$$

Upon substituting (11) in (9),

$$I_s(r) = I_s(a) \frac{r J_1(j^{-\frac{1}{2}}\beta_1 r)}{a J_1(j^{-\frac{1}{2}}\beta_1 a)} \qquad (12)$$

Similarly

$$i_s(r) = \frac{I_s(a)}{\pi a^2} \left(\frac{\beta_1 a}{2}\right) \frac{J_0(j^{-\frac{1}{2}}\beta_1 r)}{J_1(j^{-\frac{1}{2}}\beta_1 a)} e^{-j\frac{\pi}{4}} \qquad (13)$$

The axial components of electric field and vector potential are easily obtained as follows using (13):

$$E_s(r) = \frac{i_s(r)}{\sigma_1} \qquad (14)$$

With (2.12) and $k_1 \doteq \beta_1$,

$$A_s(r) = \frac{j\beta_1^2}{\omega k_1^2} E_s(r) = \frac{j}{\omega} E_s(r) = \frac{j}{\omega\sigma_1} i_s(r) \qquad (15)$$

All the above expressions involve the function

$$J_1(j^{-\frac{1}{2}}\beta_1 r) = J_1(j^{-\frac{1}{2}}y)$$

with $y = \beta_1 r$ Its real and imaginary parts may be separated, and it may be expressed in amplitude-phase-angle form by following the same methods used in the case of $J_0(j^{-\frac{1}{2}}y)$.   The relation between $J_1(j^{-\frac{1}{2}}y)$ and the more commonly used $J_1(j^{\frac{3}{2}}y)$ is shown. Using (4)

$$J_1(j^{-\frac{1}{2}}y) = -J_1(j^{\frac{3}{2}}y)$$
$$= \frac{j^{-\frac{1}{2}}y}{2}\left[\left(1 - \frac{(\frac{1}{2}y)^4}{2!\,3!} + \cdots\right) + j\left(\frac{(\frac{1}{2}y)^2}{1!\,2!} - \frac{(\frac{1}{2}y)^6}{3!\,4!} + \cdots\right)\right] \quad (16)$$

Real and imaginary parts are obtained using

$$j^{-\frac{1}{2}} = \frac{(1 - j)}{\sqrt{2}}$$

They are,

$$\text{R.P. } J_1(j^{-\frac{1}{2}}y) = \frac{y}{2\sqrt{2}}\left(1 + \frac{(\frac{1}{2}y)^2}{1!\,2!} - \frac{(\frac{1}{2}y)^4}{2!\,3!} - \frac{(\frac{1}{2}y)^6}{3!\,4!} + \cdots\right)$$
$$= -\text{ber}_1 y \quad (17)$$
$$\text{I.P. } J_1(j^{-\frac{1}{2}}y) = \frac{-y}{2\sqrt{2}}\left(1 - \frac{(\frac{1}{2}y)^2}{1!\,2!} - \frac{(\frac{1}{2}y)^4}{2!\,3!} + \frac{(\frac{1}{2}y)^6}{3!\,4!} + \cdots\right)$$
$$= -\text{bei}_1 y \quad (18)$$

Accordingly,

$$J_1(j^{\frac{3}{2}}y) = \text{ber}_1 y + j\,\text{bei}_1 y = M_1(y)e^{j\theta_1(y)} \quad (19a)$$
$$J_1(j^{-\frac{1}{2}}y) = -(\text{ber}_1 y + j\,\text{bei}_1 y) = M_1(y)e^{j(\pi + \theta_1(y))} \quad (19b)$$

The real functions $M_1(y)$ and $\theta_1(y)$ are defined by

$$M_1(y) = \sqrt{\text{ber}_1^2 y + \text{bei}_1^2 y} \quad (20)$$
$$\theta_1(y) = \tan^{-1}\left(\frac{\text{bei}_1 y}{\text{ber}_1 y}\right) \quad (21)$$

An alternative representation that is more common than the ber$_1$ and bei$_1$ functions makes use of the derivative functions

$$\text{ber}' y = \frac{d}{dy}(\text{ber } y) \quad (22a)$$

$$\text{bei}' y = \frac{d}{dy}(\text{bei } y) \quad (22b)$$

with the ber and bei functions defined in (5.10a,b). Since with (4)

$$\frac{d}{dy}[J_0(j^{-\frac{1}{2}}y)] = j^{-\frac{1}{2}}J_0'(j^{-\frac{1}{2}}y) = -j^{-\frac{1}{2}}J_1(j^{-\frac{1}{2}}y) = e^{j\frac{3\pi}{4}}J_1(j^{-\frac{1}{2}}y) \quad (23)$$

it follows that

$$J_1(j^{-\frac{1}{2}}y) = e^{-j\frac{3\pi}{4}}\frac{d}{dy}[J_0(j^{-\frac{1}{2}}y)] = e^{-j\frac{3\pi}{4}}(\text{ber}'\ y + j\ \text{bei}'\ y) \quad (24)$$

With (19b), (20), and (21)

$$\text{ber}'\ y + j\ \text{bei}'\ y = M_1(y)e^{j\left(\theta_1(y)-\frac{\pi}{4}\right)} \quad (25)$$

$$M_1(y) = \sqrt{\text{ber}'^2\ y + \text{bei}'^2\ y} \quad (26)$$

$$\theta_1(y) = \frac{\pi}{4} + \tan^{-1}\left(\frac{\text{bei}'\ y}{\text{ber}'\ y}\right) \quad (27)$$

$$\text{ber}'\ y = M_1(y)\cos\left[\theta_1(y) - \frac{\pi}{4}\right] \quad (28)$$

$$\text{bei}'\ y = M_1(y)\sin\left[\theta_1(y) - \frac{\pi}{4}\right] \quad (29)$$

The ber$'$ and bei$'$ functions are tabulated in standard tables together with the ber and bei functions. The functions $M_1(y)$ and $\theta_1(y)$ are represented graphically in Fig. 5.1 and are tabulated in Appendix III and usually in standard tables wherever $M_0(y)$ and $\theta_0(y)$ are given.

The function $J_1(j^{-\frac{1}{2}}y)$ may also be written in the form

$$J_1(j^{-\frac{1}{2}}y) = M_1(y)e^{j(\theta_1(y)-\pi)} = M_1(y)j^{(x_1-2)} \quad (30)$$

where, corresponding to (5.15c)

$$x_1 = \frac{2}{\pi}\ \theta_1(y) \quad (31)$$

The functions $M_1(y)$ and x_1 are tabulated and represented graphically in standard books of tables.[1]

For small values of y it follows from (16) that

$$J_1(j^{-\frac{1}{2}}y) \doteq \frac{y}{2\sqrt{2}}\ (1-j) = \frac{y}{2}\ e^{-j\frac{\pi}{4}}; \qquad y^2 << 8 \quad (32a)$$

$$M_1(y) \doteq \frac{y}{2}; \qquad \theta_1(y) \doteq \frac{3\pi}{4} \quad (32b)$$

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[1] JAHNKE-EMDE, "Tables of Functions," p. 261–264, 1938 ed. The function $J_1(j^{\frac{1}{2}}y) = M_1(y)e^{j(\pi-\theta_1(y))}$ is represented in the formed $b_1 j^{\beta_1}$. Hence

$$M_1(y) = b_1;\ \pi - \theta_1(y) = \frac{\pi}{2}\ \beta_1.$$

For sufficiently large values of y in (19b), simpler formulas for $M_1(y)$ and $\theta_1(y)$ may be obtained. Using (8) with $x = j^{-\frac{1}{2}}y$

$$J_1(j^{-\frac{1}{2}}y) = \pm \sqrt{\frac{2j^{\frac{1}{2}}}{\pi y}} \cos\left(j^{-\frac{1}{2}}y - \frac{3\pi}{4}\right) \tag{33}$$

Using the same method as in deriving (5.22) from (5.18)

$$J_1(j^{-\frac{1}{2}}y) = \pm \sqrt{\frac{1}{2\pi y}} e^{y/\sqrt{2}}e^{j(y/\sqrt{2}-5\pi/8)}; \qquad y \geq 10 \tag{34}$$

Accordingly

$$M_1(y) \doteq \frac{1}{\sqrt{2\pi y}} e^{y/\sqrt{2}}; \qquad y \geq 10 \tag{35a}$$

$$\theta_1(y) \doteq \frac{y}{\sqrt{2}} + \frac{3\pi}{8}; \qquad y \geq 10 \tag{35b}$$

Using (19b) in (12) and (13) with phase referred to $I_s(a)$

$$I_s(r) = I_s(a) \frac{rM_1(\beta_1 r)}{aM_1(\beta_1 a)} e^{-j[\theta_1(\beta_1 a)-\theta_1(\beta_1 r)]} \tag{36}$$

$$i_s(r) = \frac{\beta_1 I_s(a)}{2\pi a} \frac{M_0(\beta_1 r)}{M_1(\beta_1 a)} e^{-j\left[\theta_1(\beta_1 a)-\theta_0(\beta_1 r)-\frac{3\pi}{4}\right]} \tag{37}$$

For small values of $(\beta_1 a)$, as at very low frequencies, using (32b) and (5.17)

$$\left.\begin{array}{l} I_s(r) = I_s(a) \dfrac{r^2}{a^2} \\[2mm] i_s(r) = \dfrac{I_s(a)}{\pi a^2} \end{array}\right\} (\beta_1 a)^2 << 4 \tag{38}$$

For large values of $(\beta_1 a)$, as at very high frequencies, using (35a,b) and (5.23a,b)

$$\left.\begin{array}{l} I_s(r) = I_s(a) \sqrt{\dfrac{r}{a}} e^{-\frac{\beta_1}{\sqrt{2}}(a-r)} e^{-j\frac{\beta_1}{\sqrt{2}}(a-r)} \\[3mm] i_s(r) = \dfrac{\beta_1 I_s(a)}{2\pi a} \sqrt{\dfrac{a}{r}} e^{-\frac{\beta_1}{\sqrt{2}}(a-r)} e^{-j\left[\frac{\beta_1}{\sqrt{2}}(a-r)-\frac{\pi}{4}\right]} \end{array}\right\} \beta_1 a \geq 10 \tag{39}$$

The electric field and the vector potential are given by

$$E_s(r) = \frac{i_s(r)}{\sigma_1}; \qquad A_s(r) = \frac{j}{\omega\sigma_1} i_s(r) \tag{40}$$

The axial electric field at the surface of the conductor is

$$E_s(a) = \frac{I_s(a)}{\pi a^2 \sigma_1} \frac{\beta_1 a}{2} \frac{M_0(\beta_1 a)}{M_1(\beta_1 a)} e^{-j\left[\theta_1(\beta_1 a)-\theta_0(\beta_1 a)-\frac{3\pi}{4}\right]} \tag{41}$$

7. Internal Impedance per Unit Length of a Cylindrical Conductor.—The ratio of the axial electric field at the surface of a given cross section of the conductor to the total current across that cross section is a complex quantity called the *internal or surface impedance per unit length*. It is denoted by $z^i = r^i + jx^i$ with r^i the internal or ohmic resistance per unit length, x^i the internal reactance per unit length.

$$z^i \equiv \frac{E_s(a)}{I_s(a)} = \frac{1}{\pi a^2 \sigma_1} \frac{\beta_1 a}{2} \frac{M_0(\beta_1 a)}{M_1(\beta_1 a)} \, e^{-j\left[\, \theta_1(\beta_1 a) - \theta_0(\beta_1 a) - \frac{3\pi}{4}\right]} \quad (1)$$

$$r^i = \frac{1}{\pi a^2 \sigma_1} \frac{\beta_1 a}{2} \frac{M_0(\beta_1 a)}{M_1(\beta_1 a)} \cos\left[\frac{3\pi}{4} + \theta_0(\beta_1 a) - \theta_1(\beta_1 a)\right] \quad (2)$$

$$x^i = \frac{1}{\pi a^2 \sigma_1} \frac{\beta_1 a}{2} \frac{M_0(\beta_1 a)}{M_1(\beta_1 a)} \sin\left[\frac{3\pi}{4} + \theta_0(\beta_1 a) - \theta_1(\beta_1 a)\right] \quad (3)$$

Fig. 7.1.—The functions $F_1 = \dfrac{y}{2} \dfrac{M_0(y)}{M_1(y)}$ and $\theta = 135° + \theta_0(y) - \theta_1(y)$.

The requirement that $\beta_1 a$ be small is the equivalent in the conductor of the condition for the near zone. In this case

(5.17) and (6.32b) give

$$r^i \doteq r_0 \equiv \frac{1}{\pi a^2 \sigma}; \qquad x^i = 0; \qquad (\beta_1 a)^2 << 4 \qquad (4)$$

The value r_0 is the *d.c. resistance per unit length.* The complex frequency factor for the conductor is

$$F_1 = \frac{y}{2} \frac{M_0(y)}{M_1(y)} e^{j\left[\frac{3\pi}{4} + \theta_0(y) - \theta_1(y)\right]}; \qquad y \equiv \beta_1 a \qquad (5)$$

The magnitude $F_1 = \frac{y}{2} \frac{M_0(y)}{M_1(y)}$ and the angle

$$\theta = 135° + \theta_0(y) - \theta_1(y)$$

are plotted in Fig. 7.1.

For small values of $\beta_1 a$, but not so small as assumed in (4), it is convenient to proceed from (6.14) with (6.13) and

$$\beta_1 = j^{-\frac{1}{2}} \beta_1 = \beta_1 e^{-j\frac{\pi}{4}}.$$

That is, from

$$z^i = \frac{E_s(a)}{I_s(a)} = \frac{1}{\pi a^2 \sigma_1} \frac{\beta_1 a}{2} \frac{J_0(\beta_1 a)}{J_1(\beta_1 a)} \qquad (6)$$

The following series expansion for small values of $x_1 = \beta_1 a$ is true:

$$\frac{x_1}{2} \frac{J_0(x_1)}{J_1(x_1)} = 1 - \frac{(\frac{1}{2}x)^2}{2} - \frac{(\frac{1}{2}x)^4}{12} - \frac{(\frac{1}{2}x)^6}{48} - \cdots \qquad (7)$$

Hence

$$r^i = r_0 \left[1 + \frac{(\beta_1 a)^4}{192} + \cdots \right] \qquad (8)$$

$$x^i = \frac{1}{\pi a^2 \sigma_1} \frac{(\beta_1 a)^2}{8} \left[1 - \frac{(\beta_1 a)^4}{384} + \cdots \right] \qquad (9)$$

With $\beta_1^2 = \omega \sigma_1 / \nu_1 = \omega \sigma_1 \mu_1$

$$x^i = \frac{\omega}{8 \pi \nu_1} \left[1 - \frac{(\beta_1 a)^4}{384} + \cdots \right] \qquad (10)$$

Accordingly,

$$r^i \doteq r_0; \qquad x^i \doteq \frac{\omega}{8 \pi \nu_1}; \qquad (\beta_1 a)^4 << 192 \qquad (11)$$

It is now possible to define an internal inductance per unit length that is independent of the frequency.

$$l_0^i \equiv \frac{1}{8 \pi \nu_1} = \frac{1}{2 \nu_r} \times 10^{-7} \text{ henry/meter} \qquad (12)$$

If $\beta_1 a$ is sufficiently large, (5.23a,b) and (6.35a,b) may be used. Then (1) gives

$$z^i = \frac{\beta_1}{2\pi a \sigma_1} e^{j\frac{\pi}{4}} = \frac{\beta_1}{2\pi a \sigma_1} \frac{1+j}{\sqrt{2}} \tag{13}$$

Accordingly, with $\beta_1 = \sqrt{\omega \sigma_1 / \nu_1} = \sqrt{\omega \sigma_1 \mu_1}$

$$r^i = x^i = \frac{1}{2\pi a} \sqrt{\frac{\omega}{2\nu_1 \sigma_1}} = \frac{1}{2\pi a} \sqrt{\frac{\omega \mu_1}{2\sigma_1}} \tag{14}$$

This is called the *Rayleigh formula*. If the skin depth d_s defined by

$$\frac{1}{d_s} = \frac{\beta_1}{\sqrt{2}} = \sqrt{\frac{\omega \sigma_1}{2\nu_1}} \tag{15}$$

is introduced in (14),

$$r^i = x^i = \frac{1}{2\pi a d_s \sigma_1} \tag{16}$$

The internal resistance per unit length for large values of $\beta_1 a$ is seen to be inversely proportional to the area of a ring of circumference $2\pi a$ and very small width equal to the skin depth d_s, rather than inversely proportional to the area of the entire cross section. This is consistent with the fact that the current is principally in a thin skin near the surface when $\beta_1 a$ is large, whereas it is almost uniformly distributed throughout the interior when $\beta_1 a$ is small. In fact, the resistance per unit length given by (16) is the same as the d.c. resistance per unit length of a tube of thickness d.

8. Surface Impedance.—If the argument $\beta_1 a$ is sufficiently large to permit the use of asymptotic formulas for the Bessel functions, the total axial current $I_z(a)$ is distributed in a layer sufficiently near the circumference to allow the definition of a *quasi-surface* current l'_s. This is defined to be the fraction of the rotationally symmetrical total current $I_z(a)$ in each sector of the cross section formed by unit distance along the circumference. It is called a quasi-surface current because with $\beta_1 a \geq 10$ most of the current is confined to a layer that is thin compared with the radius of the conductor but that is still very thick compared with the atomic dimension δ of the surface cells used in defining a true surface current l_s. Such a true surface current does not exist in physically available conductors

at the highest electrically meaningful frequencies. It is significant only in the idealized case of perfect conductors. A quasi-surface current, on the other hand, is easily produced in all good conductors at frequencies measured in hundreds, thousands, and tens of thousands of megahertz.[1] In a cylindrical conductor, the quasi-surface current is defined by

$$l'_s = \frac{I_s(a)}{2\pi a} \tag{1}$$

The surface impedance is defined to be

$$Z^s = \frac{E_s(a)}{l'_s} = \frac{2\pi a E_s(a)}{I_s(a)} = 2\pi a z^i \tag{2}$$

It is measured in ohms. With (7.13)

$$Z^s = R^s + jX^s = \sqrt{\frac{\omega}{2\nu_1\sigma_1}} \, (1 + j) \tag{3}$$

In terms of the skin depth d_s as given in (7.15)

$$R^s = X^s = \sqrt{\frac{\omega}{2\nu_1\sigma_1}} = \sqrt{\frac{\omega\mu_1}{2\sigma_1}} = \frac{1}{d_s\sigma_1} \tag{4}$$

Since the surface impedance is independent of the radius of the conductor if $\beta_1 a$ is sufficiently large or the skin depth d_s sufficiently small, its value does not change significantly with the radius of curvature of the surface so long as this is large compared with the skin depth. Furthermore, since the interior of the cylindrical conductor carries practically no current if $\beta_1 a$ is large, no significant change in the quasi-surface current and, hence, in the surface impedance can occur if the interior of the conductor is removed so long as a thickness of wall is left in the resulting tube or pipe that is very large compared with the skin depth for the solid cylinder. It follows that the surface impedance (3) is a good approximation for metal sheets that are very thick compared with the skin depth $d_s = \sqrt{2\nu_1/\omega\sigma_1}$ and that have a radius of curvature that is also very large compared with d_s. Moreover, if the surface current is restricted to a very thin layer near the *inner* surface of a metal pipe, (3) also applies. For example, in the case of a coaxial line with a solid inner conductor of radius a and an outer conductor of inner radius b and

[1] One hertz (abbreviated hz.) = 1 cycle per second.

outer radius c as shown in Fig. 12.1, equal and opposite axial quasi-surface currents can be maintained on the adjacent surfaces of the line at sufficiently high frequencies to make $\beta_1 a \geq 10$ and, therefore, also $\beta_1 b \geq 10$.   The internal impedance per unit length of the inner conductor is

$$z_a^i = \frac{E_s(a)}{I_s(a)} = \frac{E_s(a)}{2\pi a I_s} = \frac{Z^s}{2\pi a} \tag{5}$$

The internal impedance of the outer conductor is

$$z_b^i = \frac{Z^s}{2\pi b} \tag{6}$$

The internal impedance per loop unit length of the line (both conductors) is

$$z^i = z_a^i + z_b^i = \frac{Z^s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right); \qquad (\beta_1 a \geq 10) \tag{7}$$

The same result is obtained if the solution (3.10) is evaluated subject to the boundary conditions (2.20) applied at both $r = a$ and $r = b$ and $\beta_1 a$ is allowed to become sufficiently great.   It is obtained in a different way in Sec. 12.   The solution in the general case in which $\beta_1 a$ and $\beta_1 b$ are unrestricted is intricate.

The surface impedance (2) is useful in the analysis of transmission circuits using hollow metal conductors at extremely high frequencies.   This is discussed in Volume III.

SKIN EFFECT IN A TUBULAR CONDUCTOR

9. Distribution of Axial Current in a Conducting Tube.—It has been shown that at sufficiently large values of the argument $\beta_1 a$, which in practice means at sufficiently high frequencies, practically the entire axially directed current I_s is in a thin layer near the surface of a cylindrical conductor.   Since the interior carries so little current, it appears obvious that a tubular conductor must have practically the same internal impedance per unit length as a solid conductor if the thickness of wall of the tube is large compared with the skin depth.   This can be shown analytically by applying boundary conditions appropriate to the tube to the solution (3.10) for the transverse distribution in a region with rotational symmetry.

$$A_s(r) = D_1 J_0(kr) + D_2 N_0(kr) \tag{1}$$

Here D_1 and D_2 are complex constants of integration. Let region 1 be the conducting tube of inner radius b and outer radius c as shown in Fig. 9.1. Region 2 is outside the tube; region 3 inside. Both are space. The outer boundary between the hollow conductor and space is no different from that between space and a solid conductor of the same radius if both are driven in the same way so that rotationally symmetrical currents are maintained on the outside of the tube just as on the surface of the solid cylinder. It may be assumed, therefore, that (4.21) and (4.29) are true so that

FIG. 9.1.—Cross section of conducting tube.

$$k_1 \doteq \beta_1 = j^{-\frac{1}{2}}\beta_1 = j^{-\frac{1}{2}} \sqrt{\frac{\omega\sigma_1}{\nu_1}} \tag{2}$$

and

$$A_z(r) = D_1 J_0(\beta_1 r) + D_2 N_0(\beta_1 r); \qquad b \leq r \leq c \tag{3}$$

In space inside the tube the only possible rotationally symmetrical solution is

$$A_z(r) = D_1 J_0(k_3 r); \qquad r \leq b \tag{4}$$

because $N_0(kr)$ is infinite at $r = 0$. Let it be assumed that b is sufficiently small that

$$(k_3 b)^2 << 1; \qquad J_0(k_3 r) \doteq 1; \qquad r \leq b \tag{5}$$

The significance of this restriction is considered in detail in Volume III where the general problem of the transmission of currents on the inner surface of a metal tube is considered. The tubes considered here are intended merely to serve the purposes of solid conductors, and the restriction (5) eliminates the possibility of large currents along the inner surface. Accordingly,

$$A_z(r) \doteq D_1 = \text{const.} \qquad r \leq b \tag{6a}$$

$$\frac{\partial A_z(r)}{\partial r} \doteq 0 \tag{6b}$$

With (2.12), *viz.*, $E_z = \dfrac{-j\omega k^2}{\beta^2} A_z$, it follows that

$$E_z(r) = D_3 J_0(\beta_1 r) + D_4 N_0(\beta_1 r); \qquad b \leq r \leq c \tag{7}$$

where D_3 and D_4 are complex constants. Differentiating (7) with respect to r and using (2.12) with (6) gives

$$\left[\frac{dE_s(r)}{dr}\right]_{r=b} = \beta_1[D_3 J_0'(\beta_1 b) + D_4 N_0'(\beta_1 b)] = 0 \qquad (8)$$

Accordingly, with $J_0'(x) = -J_1(x); N_0'(x) = -N_1(x)$,

$$D_4 = -D_3 \frac{J_0'(\beta_1 b)}{N_0'(\beta_1 b)} = -D_3 \frac{J_1(\beta_1 b)}{N_1(\beta_1 b)} \qquad (9)$$

and

$$E_s(r) = D[J_0(\beta_1 r)N_1(\beta_1 b) - N_0(\beta_1 r)J_1(\beta_1 b)] \qquad (10)$$

with $D = D_3/N_1(\beta_1 b)$ a complex constant. D may be expressed in terms of the electric field at the outer surface $r = c$ of the tube. Thus,

$$E_s(c) = D[J_0(\beta_1 c)N_1(\beta_1 b) - N_0(\beta_1 c)J_1(\beta_1 b)] \qquad (11)$$

so that solving for D and substituting in (10) the following ratio is obtained. The relation

$$i_s = \sigma_1 E_s \qquad (12)$$

is also used.

$$\frac{E_s(r)}{E_s(c)} = \frac{i_s(r)}{i_s(c)} = \left[\frac{J_0(\beta_1 r)N_1(\beta_1 b) - N_0(\beta_1 r)J_1(\beta_1 b)}{J_0(\beta_1 c)N_1(\beta_1 b) - N_0(\beta_1 c)J_1(\beta_1 b)}\right] \qquad (13)$$

Here $i_s(c)$ is the complex amplitude of the volume density of current extrapolated to the surface $r = c$. The total current in the tube is

$$I_s(c) = \int_b^c i_s(r) 2\pi r \, dr \qquad (14)$$

Using (6.3) and the corresponding relation for $N_0(x)$, *viz.*,

$$\int_0^x x N_0(x) dx = x N_1(x) \qquad (15)$$

$$I_s(c) = \frac{2\pi i_s(c)}{\beta_1^2}$$
$$\left\{\frac{[\beta_1 c J_1(\beta_1 c) - \beta_1 b J_1(\beta_1 b)]N_1(\beta_1 b) - [\beta_1 c N_1(\beta_1 c) - \beta_1 b N_1(\beta_1 b)]J_1(\beta_1 b)}{J_0(\beta_1 c)N_1(\beta_1 b) - N_0(\beta_1 c)J_1(\beta_1 b)}\right\} \qquad (16)$$

The second and fourth terms in the numerator cancel so that

$$i_s(c) = \frac{I_s(c)}{\pi c^2}\left(\frac{\beta_1 c}{2}\right)\left\{\frac{J_0(\beta_1 c)N_1(\beta_1 b) - N_0(\beta_1 c)J_1(\beta_1 b)}{J_1(\beta_1 c)N_1(\beta_1 b) - N_1(\beta_1 c)J_1(\beta_1 b)}\right\} \qquad (17)$$

Hence, substituting in (13)

$$i_s(r) = \frac{I_s(c)}{\pi c^2} \left(\frac{\beta_1 c}{2}\right) \left\{\frac{J_0(\beta_1 r)N_1(\beta_1 b) - N_0(\beta_1 r)J_1(\beta_1 b)}{J_1(\beta_1 c)N_1(\beta_1 b) - N_1(\beta_1 c)J_1(\beta_1 b)}\right\} \quad (18)$$

This expression for the density of axial current in the metal tube corresponds to (6.13) for the solid cylinder, if β_1 is expanded according to

$$\beta_1 = \beta_1 j^{-\frac{1}{2}} = \beta_1 e^{-j\frac{\pi}{4}} \quad (19)$$

The function $N_1(x)$ as defined in (15) may be expressed by an intricate series corresponding to (3.12) for $N_0(x)$. For small values of the argument, the leading term comparable to (3.16b) for $N_0(x)$ is

$$N_1(x) \doteq -\frac{2}{\pi x}; \quad x << 1 \quad (20)$$

For large values of the argument, the formula corresponding to (3.17b) for $N_0(x)$ is

$$N_1(x) \doteq \mp \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right); \quad \left\{\begin{array}{l} x \geq 10 \\ -\frac{\pi}{2} \leq \arg x \leq \frac{\pi}{2} \end{array}\right. \quad (21)$$

The functions $N_0(j^{-\frac{1}{2}}\beta_1 r)$ and $N_1(j^{-\frac{1}{2}}\beta_1 r)$ are not tabulated in standard tables, whereas $H_0^{(1)}(j^{-\frac{1}{2}}\beta_1 r)$ and $H_1^{(1)}(j^{-\frac{1}{2}}\beta_1 r)$ can be obtained from tabulated functions. It is, therefore, convenient to rearrange (18) by writing (see 3.19)

$$N_0(x) = j[J_0(x) - H_0^{(1)}(x)] \quad (22)$$

A relation corresponding to (3.19) with subscript 1 defines $H_1^{(1)}(x)$. It is

$$N_1(x) = j[J_1(x) - H_1^{(1)}(x)] \quad (23)$$

If (22) and (23) are used in (18), this becomes

$$i_s(r) = \frac{I_s(c)}{\pi c^2} \left(\frac{\beta_1 c}{2}\right) \left\{\frac{J_0(\beta_1 r)H_1^{(1)}(\beta_1 b) - H_0^{(1)}(\beta_1 r)J_1(\beta_1 b)}{J_1(\beta_1 c)H_1^{(1)}(\beta_1 b) - H_1^{(1)}(\beta_1 c)J_1(\beta_1 b)}\right\} \quad (24)$$

The functions

$$H_{0,1}^{(1)}(j^{-\frac{1}{2}}y) = H_{0,1}^{(1)}\left(\frac{1-j}{\sqrt{2}}y\right) \quad (25)$$

are the complex conjugates of

$$H_{0,1}^{(1)}(j^{\frac{1}{2}}y) = H_{0,1}^{(1)}\left(\frac{1+j}{\sqrt{2}}y\right) \quad (26)$$

Furthermore,

$$j^{\frac{1}{2}}H_1^{(1)}(j^{\frac{1}{2}}y) = -\frac{d}{dy}[H_0^{(1)}(j^{\frac{1}{2}}y)] \tag{27}$$

The real and imaginary parts of $H_0^{(1)}(j^{\frac{1}{2}}y)$ and of $j^{\frac{1}{2}}H_1^{(1)}(j^{\frac{1}{2}}y)$ are tabulated and represented graphically, the magnitudes and phase angles of $H_0^{(1)}(j^{\frac{1}{2}}y)$ and $H_1^{(1)}(j^{\frac{1}{2}}y)$ are shown graphically in standard books of tables[1] so that (24) may be evaluated directly in the general case.

For large values of $\beta_1 b$, (18) leads to an important expression. Using (3.17a,b), (6.8), and (21),

$$i_s(r) = \frac{I_s(c)}{\pi c^2}\left(\frac{\beta_1 c}{2}\right)\sqrt{\frac{c}{r}}$$

$$\left\{\frac{\cos\left(\beta_1 r - \frac{\pi}{4}\right)\cos\left(\beta_1 b - \frac{\pi}{4}\right) + \sin\left(\beta_1 r - \frac{\pi}{4}\right)\sin\left(\beta_1 b - \frac{\pi}{4}\right)}{\sin\left(\beta_1 c - \frac{\pi}{4}\right)\cos\left(\beta_1 b - \frac{\pi}{4}\right) - \cos\left(\beta_1 c - \frac{\pi}{4}\right)\sin\left(\beta_1 b - \frac{\pi}{4}\right)}\right\} \tag{28}$$

or

$$i_s(r) = \frac{I_s(c)}{\pi c^2}\left(\frac{\beta_1 c}{2}\right)\sqrt{\frac{c}{r}}\left[\frac{\cos\beta_1(r-b)}{\sin\beta_1(c-b)}\right]; \qquad \beta_1 b \geq 10 \tag{29}$$

Since

$$\beta_1 = j^{-\frac{1}{2}}\beta_1 = (1-j)\frac{\beta_1}{\sqrt{2}} = \frac{1-j}{d_s} \tag{30}$$

with d_s the skin depth,[2]

$$\cos\beta_1 u = \cos(1-j)\frac{u}{d_s} = \cos\frac{u}{d_s}\cosh\frac{u}{d_s} + j\sin\frac{u}{d_s}\sinh\frac{u}{d_s} \tag{31a}$$

$$\sin\beta_1 u = \sin(1-j)\frac{u}{d_s} = \sin\frac{u}{d_s}\cosh\frac{u}{d_s} - j\cos\frac{u}{d_s}\sinh\frac{u}{d_s} \tag{31b}$$

or, in polar form,

$$\cos\beta_1 u = \sqrt{\tfrac{1}{2}(\cosh 2u/d_s + \cos 2u/d_s)}\, e^{-j\psi_r} \tag{32a}$$

$$\sin\beta_1 u = \sqrt{\tfrac{1}{2}(\cosh 2u/d_s - \cos 2u/d_s)}\, e^{-j\psi_s} \tag{32b}$$

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[1] JAHNKE–EMDE, "Tables of Functions," pp. 250–259, 1938 ed.

[2] The hyperbolic functions appearing in (31) are defined by

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

in the general case of complex argument.   In (31) the argument u/d is real.

with

$$\tan \psi_r = - \tanh \frac{u}{d_s} \tan \frac{u}{d_s}; \qquad \tan \psi_c = \frac{\tanh \dfrac{u}{d_s}}{\tan \dfrac{u}{d_s}} \qquad (33)$$

Using (32) in (29) with phase referred to $I_s(c)$

$$i_s(r) = \frac{I_s(c)}{\pi c^2} \left(\frac{\beta_1 c}{2} \right) \sqrt{ \frac{c \left[\cosh \dfrac{2(r-b)}{d_s} + \cos \dfrac{2(r-b)}{d_s} \right]}{r \left[\cosh \dfrac{2(c-b)}{d_s} - \cos \dfrac{2(c-b)}{d_s} \right]} }$$
$$e^{-j\left(\psi_r - \psi_c + \frac{\pi}{4} \right)} \qquad (34)$$

with

$$\tan \psi_r = - \tanh \frac{(r-b)}{d_s} \tan \frac{(r-b)}{d_s} \qquad (35a)$$

$$\tan \psi_c = \frac{\tanh \dfrac{(c-b)}{d_s}}{\tan \dfrac{(c-b)}{d_s}} \qquad (35b)$$

If the thickness of the wall of the tube $c - b$ is sufficiently large compared with the skin depth d_s so that

$$\frac{(c-b)}{d_s} \geq 4 \qquad (36)$$

$$\sinh \frac{2(c-b)}{d_s} \doteq \frac{1}{2} e^{2(c-b)/d_s} \doteq \cosh \frac{2(c-b)}{d_s} >> 1 \qquad (37)$$

$$\tanh \frac{(c-b)}{d_s} \doteq 1 \qquad (38)$$

Then, since $\cos 2(c-b)/d_s \leq 1$,

$$i_s(r) = \frac{I_s(c)}{\pi c^2} \left(\frac{\beta_1 c}{2} \right) \sqrt{\frac{c}{r}} e^{-\frac{(c-b)}{d_s}} \sqrt{2 \left[\cosh \frac{2(r-b)}{d_s} + \cos \frac{2(r-b)}{d_s} \right]}$$
$$e^{-j\left(\psi_r - \psi_c + \frac{\pi}{4} \right)} \qquad (39)$$

with

$$\psi_c = \frac{\pi}{2} - \left(\frac{c-b}{d_s} \right) \qquad (40)$$

Because $(c-b)/d_s$ is large, tho magnitude $i_s(r)$ decreases very

rapidly as r decreases from $r = c$. The density at radial distances $s = c - r$ from the outer surface which are sufficiently small so that

$$\frac{(r - b)}{d_s} \geq 4 \tag{41}$$

. is

$$\frac{i_s(s)}{i_s(s = 0)} \doteq \sqrt{\frac{c}{c - s}}\, e^{-s/d_s} e^{-js/d_s} \tag{42}$$

This is exactly the same in form as (5.28) for a solid conductor of the same radius. Consequently, the radial distribution of axial current in a tube of sufficiently large radius ($\beta_1 b \geq 10$) with a wall that is thick compared with the skin depth is practically the same as that in a solid conductor of the same radius except near the inner surface of the tube.

10. Internal Impedance per Unit Length of a Conducting Tube.
The internal impedance per unit length of the tube is obtained using (9.18) or (9.24) and $i_s = \sigma_1 E_s$. Thus, with (9.18) and $r = c$,

$$z^i = \frac{E_s(c)}{I_s(c)} = \frac{1}{\pi c^2 \sigma_1}\left(\frac{\beta_1 c}{2}\right)\left\{\frac{J_0(\beta_1 c)N_1(\beta_1 b) - N_0(\beta_1 c)J_1(\beta_1 b)}{J_1(\beta_1 c)N_1(\beta_1 b) - N_1(\beta_1 c)J_1(\beta_1 b)}\right\} \tag{1}$$

The limiting value of z^i for small arguments, in particular the steady current value at zero frequency, may be obtained from (1) by allowing β_1 to approach zero. If the values for small arguments are inserted for the several Bessel functions and β_1 is made sufficiently small, the following formula is obtained:

$$(z^i)_{f=0} = r_0 = \frac{1}{\pi(c^2 - b^2)\sigma_1} \tag{2}$$

With (2), the general formula (1) may be written as follows:

$$z^i = r_0\left(1 - \frac{b^2}{c^2}\right)\left(\frac{\beta_1 c}{2}\right)\left\{\frac{J_0(\beta_1 c)N_1(\beta_1 b) - N_0(\beta_1 c)J_1(\beta_1 b)}{J_1(\beta_1 c)N_1(\beta_1 b) - N_1(\beta_1 c)J_1(\beta_1 b)}\right\} \tag{3}$$

For numerical evaluation using available tables it is more convenient to express (3) in terms of the Hankel functions, as explained in writing (9.18) in the form (9.24). For this purpose it is necessary merely to write $H^{(1)}$ for N in (3).

The internal impedance per unit length of the tube for large values of $\beta_1 b$ is obtained from (9.34) using $E_s = i_s/\sigma_1$ and writing $r = c$. Thus

$$z^i = \frac{E_z(c)}{I_z(c)} = \frac{1}{\pi c^2 \sigma_1}\frac{\beta_1 c}{2}\sqrt{\frac{\cosh\dfrac{2(c-b)}{d_s} + \cos\dfrac{2(c-b)}{d_s}}{\cosh\dfrac{2(c-b)}{d_s} - \cos\dfrac{2(c-b)}{d_s}}}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\epsilon^{-j\left(\psi_r - \psi_o + \frac{\pi}{4}\right)} \qquad (4)$$

$$\tan\psi_r = -\tanh\frac{(c-b)}{d_s}\,\tan\frac{(c-b)}{d_s} \qquad (5a)$$

$$\tan\psi_c = \frac{\tanh\dfrac{(c-b)}{d_s}}{\tan\dfrac{(c-b)}{d_s}} \qquad (5b)$$

provided that

$$\beta_1 b \geqq 10 \qquad (6)$$

For sufficiently thick walls so that

$$\frac{(c-b)}{d_s} \geqq 4 \qquad (7)$$

$$\cosh\frac{2(c-b)}{d_s} >> 1 \qquad (8a)$$

$$\tanh\frac{(c-b)}{d_s} \doteq 1 \qquad (8b)$$

Then,

$$\psi_r = -\frac{(c-b)}{d_s}; \qquad \psi_o = \frac{\pi}{2} - \left(\frac{c-b}{d_s}\right) \qquad (9)$$

and

$$z^i = \frac{\beta_1}{2\pi c \sigma_1}\cdot \epsilon^{j\frac{\pi}{4}} = \frac{\beta_1}{2\pi c \sigma}\frac{1+j}{\sqrt{2}} \qquad (10)$$

provided that (6) and (7) are both satisfied. This is identical with (7.13) for a solid conductor. It follows that the internal impedance per unit length of a tubular conductor is the same as that for a solid conductor if (6) and (7) are satisfied.

11. Electromagnetic Shielding.—The electric field in the space inside a tubular conductor that satisfies the condition (9.5) is constant according to (9.6a) using (2.12). It is, therefore, equal to the value at the inner surface of the metal tube because the tangential component of the electric field is continuous across a boundary. Accordingly, the electric field in the space inside a metal tube that satisfies the condition (10.6) or (9.36) is obtained

from (9.39) using $E_s = i_s/\sigma_1$ and writing $r = b$.   It is

$$E_{s1}(b) = E_{s3}(r \leq b) = \frac{I_s(c)}{\pi c^2 \sigma_1} \beta_1 c \sqrt{\frac{c}{b}}\, e^{-(c-b)/d_s}\, e^{j\left[\frac{\pi}{4} - (c-b)/d_s\right]} \quad (1)$$

since $\psi_r = 0$, $\psi_c = \frac{\pi}{2} - \left(\frac{c-b}{d_s}\right)$.   The field at the outer surface
is obtained with $r = c$.

$$E_{s1}(c) = E_{s2}(c) = \frac{I_s(c)}{\pi c^2 \sigma_1} \frac{\beta_1 c}{2}\, e^{j\frac{\pi}{4}} \quad (2)$$

since $\psi_r = -\dfrac{(c-b)}{d_s}$; $\psi_c = \dfrac{\pi}{2} - \left(\dfrac{c-b}{d_s}\right)$.

The ratio of the field inside the tube to the field in space
just outside the tube is

$$\frac{E_{s3}}{E_{s2}} = 2 \sqrt{\frac{c}{b}}\, e^{-(c-b)/d_s} e^{-j(c-b)/d_s} \quad (3)$$

The magnitude of this ratio is

$$\frac{E_{s3}}{E_{s2}} = 2 \sqrt{\frac{c}{b}}\, e^{-(c-b)/d_s} \quad (4)$$

By making the ratio $(c-b)/d_s$ sufficiently large, the field in the
space enclosed by the metal tube can be made as small as desired.
The skin depth d_s is given by

$$\frac{1}{d_s} = \sqrt{\frac{\omega \sigma_1}{2\nu_1}} = \sqrt{\frac{\omega \sigma_1 \mu_1}{2}} \quad (5)$$

The exponential reduces to 0.007 at $(c-b)/d_s = 5$.   Hence if

$$\frac{(c-b)}{d_s} \geq 5 \quad (6)$$

approximately

$$2e^{-(c-b)/d_s} \leq 0.014 \quad (7)$$

If both b and c are large compared with $(c-b)$, the factor $\sqrt{c/b}$
is practically unity.   Accordingly, a cylindrical metal tube that
is strictly infinitely long may be used as a shield.   Apparatus
placed inside the tube is exposed to an extremely small fraction
of a periodically varying electric and, hence, also magnetic field
which is maintained *outside* the tube.   In particular, the field

inside the shield is no greater than about 1 per cent of that out-side if (6) is satisfied. Although the rigorous analysis of metal shields of other shapes than infinitely long cylinders is analytically difficult, it is clear that a comparable situation exists for any completely enclosed metal box and the ratio (4) may be used as a criterion for any shape for all qualitative purposes to determine the effectiveness of completely closed metal shields.

At $\omega = 10^{10}$, for example, $d_s = 1.6 \times 10^{-6}$ meter for copper, so that (6) will be satisfied if the wall thickness $(c - b)$ is

$$(c - b) \geq 8 \text{ microns}$$

At $\omega = 377$, on the other hand, $d_s = 8.5 \times 10^{-3}$ meter for copper, so that (6) requires $(c - b) \geq 4.25$ centimeters. Thus an extremely thick copper shield is required for low-frequency dis-turbances. If an iron shield is used at $\omega = 377$, $d_s = 0.28 \times 10^{-3}$ meter, with relative reluctivity (reciprocal permeability)

$$\nu_r = \frac{1}{\mu_r} = \frac{1}{6,000} = 1.67 \times 10^{-4}$$

it is necessary that $(c - b) \geq 1.4$ millimeters. In spite of the lower conductivity of iron compared with copper, its very low reluctivity (high permeability) at 60 cycles makes the skin depth considerably less than for copper so that it is a better material for shielding. Alloys with lower reluctivities than iron are still better. At high frequen-cies copper, brass, or aluminum of such thickness as is required for physical strength makes an excel-lent shield. At low frequencies, very thick copper or a ferromag-netic alloy is required.

12. Internal Impedance per Unit Length of a Coaxial Line.— The coaxial line is one of the most important transmission cir-cuits. It is analyzed in detail in Volume III. The radial dis-

Fig. 12.1.—Cross section of coax-ial line.

tribution of current in a coaxial line with a cross section like that shown in Fig. 12.1 is an intricate boundary-value problem because four regions with three boundaries between them are

involved.   The general case will not be solved here.   On the other hand, if rotational symmetry exists and the inner radius b of the outer conductor is sufficiently large so that

$$\beta_1 b \geq 10 \tag{1}$$

most of the current in the outer conductor is confined to a very thin layer near the *inner* surface if the line is driven by a generator that maintains a rotationally symmetrical potential difference between the two conductors.   Since (9.42) is independent of the curvature of the surface, it may be assumed to approximate very closely the distribution of current whether the thin layer is on the inner or the outer surface of a metal tube that is sufficiently thick compared with the skin depth.   Hence, the axial current density in the outer conductor of the coaxial line may be assumed to be approximately

$$\frac{i_z(s)}{i_z(s = 0)} \doteq \sqrt{\frac{b}{b + s}}\; e^{-s/d_s} e^{-js/d_s} \tag{2}$$

provided (1) is satisfied and

$$\frac{(c - r)}{d_s} \geq 4 \tag{3}$$

Here $s = r - b$ is the radial distance from the surface $r = b$ *outward* into the conductor and d_s is the skin depth.   The radial distribution of current in the inner conductor is the same as in the absence of the outer conductor (Sec. 5).

Subject to (1) and

$$\frac{(c - b)}{d_s} \geq 4 \tag{4}$$

as in (10.7), the internal impedance per unit length of the outer conductor (region 3) is given by (10.10) with b written for c.

$$z^i = \frac{\beta_3}{2\pi b \sigma_3}\left(\frac{1 + j}{\sqrt{2}}\right) \tag{5}$$

or

$$r^i = x^i = \frac{1}{2\pi b}\sqrt{\frac{\omega}{2\nu_3 \sigma_3}} = \frac{1}{2\pi b}\sqrt{\frac{\omega \mu_3}{2\sigma_3}} \tag{6}$$

The internal impedance per unit length of the inner conductor is the same as with the outer conductor absent (Sec. 7).   In the

most important special case in which

$$\beta_1 a \geq 10, \qquad \frac{(c-b)}{d_s} \geq 4 \tag{7}$$

and the two conductors are made of the same metal and carry equal and opposite currents, the internal impedance per unit length of the two conductors in agreement with (8.7) is

$$z^i = z_a^i + z_b^i = \frac{\beta_1}{2\pi\sigma_1}\left(\frac{1+j}{\sqrt{2}}\right)\left(\frac{1}{a}+\frac{1}{b}\right) \tag{8}$$

$$r^i = x^i = \frac{1}{2\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\sqrt{\frac{\omega}{2\nu_1\sigma_1}} \tag{9}$$

SKIN EFFECT IN A CONDUCTING PLANE

13. Vertical Distribution of Radial Current in a Conducting Half-space below a Vertical Antenna; Surface Impedance; Dissipation.—A detailed analysis of an antenna over an *imperfectly* conducting half-space is given in Volume II.   An antenna

Conducting half-space: Region 1
Fig. 13.1.—Antenna over a conducting half-space.

over a *perfectly* conducting half-space was considered in Chapter IV.   The present discussion is concerned entirely with the vertical distribution of horizontally directed currents in an imperfectly conducting half-space.   This is a relatively simple problem if appropriate specializing conditions are imposed. The arrangement of conductors is shown in Fig. 13.1.   A vertical antenna of small radius a and length h carrying an axial current I_z is erected on an infinite, conducting half-space.   The region not occupied by the conductors, region 2, is space; the conducting

half-space is region 1.   The antenna is so driven that complete
rotational symmetry exists and, as in (1.2),

$$\frac{\partial \mathbf{A}}{\partial \theta} = 0; \qquad A_\theta = 0 \tag{1}$$

so that (1.1) may be separated into (1.13) and (1.14) and these,
in turn, into (1.18) and (1.20); (1.24) and (1.26).   The electro-
magnetic field is in this case also given by (2.4) to (2.6).   In the
conducting half-space, the principal current is directed radially,
not axially as in the cylindrical conductor analyzed in Sec. 2.
The condition corresponding to (2.7) is

$$\frac{\partial A_r}{\partial z} >> \frac{\partial A_s}{\partial r} \tag{2}$$

Let this be assumed satisfied in the conducting half-space ($z < 0$).
From (2.4) to (2.6) it follows that the electromagnetic field in the
conducting half-space is

$$E_r \doteq \frac{j\omega}{\beta^2} \frac{\partial^2 A_r}{\partial z^2}; \qquad\qquad B_r = 0 \tag{3}$$

$$E_\theta = 0; \qquad\qquad B_\theta \doteq \frac{\partial A_r}{\partial z} \tag{4}$$

$$E_s \doteq - \frac{j\omega}{\beta^2} \frac{\partial}{\partial z} \frac{1}{r} \frac{\partial}{\partial r} (rA_r); \qquad B_s = 0 \tag{5}$$

Using (1.24) multiplied by F_r in (3)

$$E_r = -j\omega \frac{\gamma'^2}{\beta^2} A_r \tag{6}$$

The general boundary conditions (2.16), (2.17) when written for
the surface of the conducting half-space become

$$\left.\begin{array}{l} (E_{r1})_{z=0} = (E_{r2})_{z=0} \\ \nu_1 (B_{\theta 1})_{z=0} = \nu_2 (B_{\theta 2})_{z=0} \end{array}\right\} r \geq a \tag{7}$$

With (6) and (4) and writing $\gamma^2 = -\gamma'^2$,

$$\left.\begin{array}{l} \dfrac{\gamma_1^2}{\beta_1^2} (A_{r1})_{z=0} = \dfrac{\gamma_2^2}{\beta_2^2} (A_{r2})_{z=0} \\[4mm] \nu_1 \left(\dfrac{\partial A_{r1}}{\partial z}\right)_{z=0} = \nu_2 \left(\dfrac{\partial A_{r2}}{\partial z}\right)_{z=0} \end{array}\right\} r \geq a \tag{8}$$

In space (region 2), (2.22) is true.

$$\beta_2 = \beta_0 = \frac{\omega}{v_0}; \qquad \nu_2 = \nu_0 \tag{9}$$

In the conductor (region 1), (2.24) applies.

$$\beta_1 = j^{-\frac{1}{2}}\beta_1 = j^{-\frac{1}{2}}\sqrt{\frac{\omega\sigma_1}{\nu_1}} \tag{10}$$

As in (1.18),

$$\gamma^2 = k^2 - \beta^2 \tag{11}$$

The boundary conditions (7) are not sufficient to determine A_r. The conditions (2.18), (2.19) leading to (2.20), (2.21) must be imposed at the cylindrical boundary between the antenna and space and an additional set at the upper end surface of the antenna unless this is negligibly small. Because interest in this section is focused exclusively on the vertical distribution of currents in the conducting half-space, in particular on the depth of penetration of a significant density of current below the boundary surface, the problem may be simplified greatly. The differential equation for A_r in the conducting half-space is (1.13)

$$\frac{\partial^2 A_r}{\partial z^2} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \beta_1^2 A_r = 0 \tag{12}$$

Since it has been shown that the amplitude of the component of the vector potential parallel to the surface of a good conductor decreases extremely rapidly as the conductor is penetrated, whereas the rate of change of this parallel component in directions along the boundary surface is *relatively* extremely slow, it may be assumed that the inequality

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rA_r) << \frac{\partial^2 A_r}{\partial z^2} \tag{13}$$

is a good approximation *in the conducting half-space*. Accordingly, (12) becomes

$$\frac{\partial^2 A_r}{\partial z^2} + \beta_1^2 A_r \doteq 0 \tag{14}$$

Although not demonstrated analytically, the approximations implied in writing (14) for (12) in the conducting half-space are essentially the same as those involved in writing $k_1 = \beta_1$ in (4.29) for the cylindrical conductor.

A general solution of (14) is

$$A(z) = Ce^{j\beta_1 z} + De^{-j\beta_1 z} \tag{15}$$

Since

$$\beta_1 = \frac{1-j}{\sqrt{2}}\beta_1 = \frac{1-j}{d_s} \tag{16}$$

where d is the skin depth,

$$d_s = \sqrt{\frac{2\nu_1}{\omega\sigma_1}} \tag{17}$$

it follows that (15) remains finite at $z = -\infty$ only if $D = 0$. Hence, with $A_r(0)$ the value at the surface,

$$A_r(z) = A_r(0)e^{z/d_s}e^{jz/d_s}; \qquad z \leq 0 \tag{18}$$

Using (6) and $i = \sigma E$,

$$\frac{A_r(z)}{A_r(0)} = \frac{E_r(z)}{E_r(0)} = \frac{i_r(z)}{i_r(0)} = e^{z/d}e^{jz/d_s}; \qquad z \leq 0 \tag{19}$$

This is the same as (5.28) if $-z$ is written for s and $a >> s$. This agreement with the solution for an infinite plane surface as obtained from (5.28) with $a = \infty$ justifies the assumptions implied in writing (14) for (12).

The surface impedance defined in Sec. 8 is, of course, obtainable directly from (19). In this case the density of the radial quasi-surface current l'_r is the total radial current traversing a unit circumference from $z = 0$ to $z = -\infty$. The complex amplitude of the total radial current is

$$I_r = 2\pi r \int_0^{-\infty} i_r(z)(-dz) = 2\pi r \frac{d_s}{1+j} i_r(0) \tag{20}$$

The complex amplitude of the total radial current per unit circumference is

$$l'_r = \frac{I_r}{2\pi r} = \frac{d_s}{1+j} i_r(0) \tag{21}$$

The ratio of the complex amplitude of the radial electric field at the surface to the complex amplitude of the density of the radial quasi-surface current is the surface impedance. With $i_r = \sigma_1 E_r$,

$$Z^s \equiv \frac{E_r(0)}{l'_r} = \frac{2\pi r E_r(0)}{I_r} = \frac{1+j}{d_s\sigma_1} \tag{22}$$

This is the same as (8.3).   If the skin depth is small, an integration from 0 to $-10d_s$ instead of from 0 to $-\infty$ differs by a negligible amount from (20).   Representative values of the skin depth d_s and the surface resistance and reactance, $R^s = X^s$, for

Fig. 13.2.— Skin depth d_s as a function of $\omega = 2\pi f$.   The curves are useful *only* for values of ω lying to the left of and below the broken line $d_s = 0.1\lambda$.   (In the diagram d is used for d_s.)

several metals, wet and dry earth, and salt water are given in Figs. 13.2 and 13.3 for a wide range of angular velocities, $\omega = 2\pi f$. These curves have a meaning only when the skin depth is a small fraction of a wave length.

The energy dissipated in heating the conducting half-space is readily calculated using the following relation obtained by equat-

ing the middle terms in (III.18.23) and (III.18.27):

$$A_q \overline{\frac{dQ}{dt}} = \frac{1}{2} \int_\tau \frac{ii^*}{\sigma} d\tau = \frac{1}{2} \int_\tau \frac{i^2}{\sigma} d\tau \qquad (23)$$

The fraction of the total energy dissipated per second in a square

Fig. 13.3.—Surface resistance and reactance as a function of $\omega = 2\pi f$. The curves are useful *only* for values of ω lying to the left of and below the broken line $d_s = 0.1\lambda$ in Fig. 13.2.

prism below a surface $\Delta x \, \Delta y$ extending from $z = 0$ to $z = -\infty$ is

$$\frac{A_q \overline{\frac{dQ}{dt}}}{\Delta x \, \Delta y} = \frac{1}{2\sigma_1} \int_0^{-\infty} i_r(z) i_r^*(z)(-dz) \qquad (24)$$

Using (19)

$$\frac{A_q \overline{\dfrac{dQ}{dt}}}{\Delta x \, \Delta y} = \frac{d_s}{4\sigma_1} \, i_r(0) i_r^*(0) \tag{25}$$

With (21)

$$\frac{A_q \overline{\dfrac{dQ}{dt}}}{\Delta x \, \Delta y} = \frac{d_s}{4\sigma_1} \frac{(1 + j)(1 - j)}{d_s^2} \, l_r' l_r'^* = \frac{1}{2\sigma_1 d_s} \, l_r' l_r'^* \tag{26}$$

Since (22) gives the surface resistance to be

$$R^s = \frac{1}{d_s \sigma_1} \tag{27}$$

it follows that the average energy dissipated in heat per second per unit area of surface is

$$\frac{A_q \overline{\dfrac{dQ}{dt}}}{\Delta x \, \Delta y} = \frac{1}{2} \, R^s l_r'^2 \tag{28}$$

This expression is equally correct for a unit surface of any sheet of metal or other conductor that is thick compared with the skin depth with a quasi-surface current in any direction. In this case, the subscript r in (28) is replaced by an appropriate subscript specifying the direction of the current in that particular area.

SKIN EFFECT IN A CIRCULAR CONDENSER

14. Electromagnetic Field in the Dielectric of a Circular Condenser.—A practically important and theoretically interesting problem is the behavior of a circular condenser as the frequency is increased. A complete formulation of this problem is possible only if the entire circuit containing the condenser is considered. This is a difficult problem in the general case and is considered in Chapter VI. The important characteristics of a circular condenser can be derived from a study of a somewhat specialized but thereby greatly simplified problem. Consider a condenser consisting of two parallel circular metal plates of thickness t separated by an imperfect dielectric of thickness w. The two plates are connected at the centers of their outer surfaces to long straight conductors of radius a as shown in Fig. 14.1. The z axis is the

axis of symmetry; its origin is at the center of the dielectric disk. Let it be assumed that the separation w of the plates is

Fig. 14.1.—Circular parallel-plate condenser in series with a long conductor.

extremely small both compared with the radius b of the plates and with the wave length. That is,

$$w <<< b; \qquad \jmath <<< \lambda \tag{1}$$

If (1) is well satisfied, edge effects may be neglected. Since the plates are highly conducting, the complex amplitude of the volume density of charge vanishes in the interior

$$\varrho = 0 \text{ in the plates} \tag{2}$$

Accordingly, all distributions of charge are confined to thin layers of atomic thickness δ. Let the external circuit be so designed and arranged with respect to the condenser that the total current I_{sL} *entering* the surface region of thickness δ on the left-hand plate is equal to the total current I_{sR} *leaving* the surface region of thickness δ on the right-hand plate.

$$I_{sL} = I_{sR} = I_{s1} \tag{3}$$

Complete rotational symmetry is assumed to prevail in and between the plates so that

$$I_{s1} = \int_0^b 2\pi r i_{s1}(r) dr \tag{4}$$

with $i_{s1}(r)$ the complex amplitude of the z component of the volume density of current at $z = \pm (\tfrac{1}{2}w + \delta)$ in the plate.

The equation of continuity for a boundary surface between two regions 1 (plates) and 2 (dielectric disk) is given by (I.24.13). Since perfect conductors are not involved, a surface density of current I is not required.   Furthermore, the dielectric is assumed to be so poorly conducting that a surface density of charge in the dielectric is insignificant compared with that along the same boundary but on the surface of the highly conducting plates. Hence $\eta_2 \doteq 0$.   There remains

$$\left. \begin{aligned} j\omega n_{1L} - i_{s1L} + i_{s2L} = 0 \\ j\omega n_{1R} + i_{s1R} - i_{s2R} = 0 \end{aligned} \right\} \tag{5}$$

Here i_{s1L} is the current per unit area entering the surface layer of the left-hand plate from the interior of the plate; i_{s2L} is the current per unit area leaving this surface layer to enter the imperfect dielectric; i_{s1R} is the current per unit area out from the surface layer of the right-hand plate into the plate; i_{s2R} is the current per unit area entering this surface layer from the imperfect dielectric.   The complex amplitude of the total charge on the plate is

$$q_{1L} = \int_0^b 2\pi r n_1 dr = \frac{-j}{\omega}(I_{s1L} - I_{s2L}); \qquad q_{1R} = -q_{1L} \tag{6}$$

I_{s2} is the total current of moving charges entering the *imperfect* dielectric (medium 2).

Since complete rotational symmetry exists in the dielectric disk,

$$\frac{\partial \mathsf{A}}{\partial \theta} = 0; \qquad A_\theta = 0 \tag{7}$$

just as in (1.2).   Accordingly, (1.13) and (1.14) apply, as do (2.4,5,6).   Upon differentiating (1.13) partially with respect to z and (1.14) partially with respect to r, interchanging the order of differentiation where required, and subtracting,

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \beta^2\right)\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) = 0 \tag{8}$$

With (2.5)

$$\frac{\partial^2 B_\theta}{\partial z^2} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) + \beta^2 B_\theta = 0 \tag{9}$$

If (1.14) is operated on by $\partial^2/\partial z\,\partial r$ and subtracted from (1.13)

operated on by $\partial^2/\partial z^2$, the result is

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r + \beta^2\right]\left[\frac{\partial^2 A_r}{\partial z^2} - \frac{\partial^2 A_z}{\partial z\,\partial r}\right] = 0 \qquad (10)$$

Multiplication by $j\omega/\beta^2$ and use of (2.4) gives

$$\frac{\partial^2 E_r}{\partial z^2} + \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \beta^2 E_r = 0 \qquad (11)$$

Similarly if (1.14) is operated on by $\dfrac{1}{r}\dfrac{\partial}{\partial r}r\dfrac{\partial}{\partial r}$ and subtracted from

(1.13) operated on by $\dfrac{1}{r}\dfrac{\partial}{\partial r}r\dfrac{\partial}{\partial z}$, the result is

$$\left[\frac{\partial^2}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \beta^2\right]\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial A_r}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial A_z}{\partial r}\right] = 0 \quad (12)$$

Upon multiplying by $-j\omega/\beta^2$ and using (2.6),

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial E_z}{\partial r} + \beta^2 E_z = 0 \qquad (13)$$

Equations (9), (11), and (13) could have been obtained directly from the telegraphist's equations (III.14.41a,b) with (III.14.42a,b) written in cylindrical coordinates with rotational symmetry and with E_θ, B_r, and B_z set equal to zero.

With identical plates and rotational symmetry, it follows from (3) that the complex amplitudes of the densities $i_{zL}(r)$ and $i_{zR}(r)$ must be equal in the two plates. It follows from (5) that

$$n_{1R} = -n_{1L} \qquad (14)$$

Using the boundary condition (III.14.20a)

$$\epsilon_0 E_{z1L} - \epsilon_2 E_{z2L} = -n_{1L} \qquad (15a)$$
$$-\epsilon_0 E_{z1R} + \epsilon_2 E_{z2R} = -n_{1R} \qquad (15b)$$

As a result of (3) it may be assumed with (14) and (15a,b) that $E_{z1R} = E_{z1L}$ so that

$$E_{z2L} = E_{z2R} \qquad (16)$$

Since the plates are very close together in terms of the wave length, (16) is possible only if E_{z2} is independent of z in the dielectric disk. Then

$$\frac{\partial^2 E_{z2}}{\partial z^2} = 0 \qquad (17)$$

With (17) the equation for E_{s2} in the imperfectly conducting dielectric is

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial E_{s2}}{\partial r} + \beta_2^2 E_{s2} = 0 \tag{18}$$

The equation for $B_{\theta 2}$ is obtained beginning with (2.5). It is

$$\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rB_{\theta 2}) + \beta_2^2 B_{\theta 2} = 0 \tag{19}$$

These equations are exactly the same in form, respectively, as (1.19) and (1.25). Their solutions for a region including $r = 0$ are

$$E_{s2} = CJ_0(\beta_2 r) \tag{20}$$
$$B_{\theta 2} = DJ_1(\beta_2 r) \tag{21}$$

with $\beta_2 = \beta_s - j\alpha_s$ defined by (III.15.1). Since the dielectric in the disk is a poorly conducting simple medium with effective parameters σ_c, ϵ_c such that the power factor h_c of the dielectric satisfies

$$h_c^2 = \left(\frac{\sigma_c}{\omega\epsilon_c}\right)^2 = \left(\frac{2\alpha_s}{\beta_s}\right)^2 << 1 \tag{22}$$

(III.15.20,21) may be used. These are

$$\beta_s \doteq \beta_c = \frac{\omega}{v_c} \tag{23}$$

$$\alpha_s \doteq \tfrac{1}{2}\sigma_c\zeta_c \tag{24}$$

Here v_c is the characteristic velocity, ζ_c the characteristic resistance of the dielectric medium as defined in (III.15.16a,b). The condition (III. 15.23)

$$\left(\frac{\alpha_s}{\beta_s}\right)^2 << 1 \tag{25}$$

is equivalent to (22). The constants C and D may be expressed in terms of the values of the components of the field at $r = b$ if edge effects are neglected. Thus

$$E_{s2}(r) = E_{s2}(b)\frac{J_0(\beta_2 r)}{J_0(\beta_2 b)} \tag{26}$$

$$B_{\theta 2}(r) = B_{\theta 2}(b)\frac{J_1(\beta_2 r)}{J_1(\beta_2 b)} \tag{27}$$

The amplitudes $E_s(b)$ and $B_\theta(b)$ may be evaluated in terms of the total current I_{s1} entering or leaving the surface layer.

This may be done in several ways.   An instructive one is to make use of the Maxwell-Ampère theorem for the circuitation of the **B** vector.   In space $\mathbf{H} = \nu_0\mathbf{B}$; in simple media $\mathbf{D} = \varepsilon\mathbf{E}$.   Hence (II.11.9) may be written

$$\nu_0 \oint_s (\mathbf{B},d\mathbf{s}) = I + j\omega\varepsilon \int_{S(\text{cap})} (\hat{\mathbf{N}},\mathbf{E})dS \tag{28}$$

Let the contour s be the circumference of the left-hand condenser plate at its inner edge.   Because of rotational symmetry, **B** is the same at all points on the circle.   It is $\mathbf{B} = \hat{\mathbf{\theta}}B_\theta(b)$.   Then

$$\nu_0 2\pi b B_\theta(b) = I + j\omega\varepsilon \int_{S(\text{cap})} (\hat{\mathbf{N}},\mathbf{E})dS \tag{29}$$

Here S is any cap surface bounded by the circle s.   As a first choice let S be a flat cap consisting of the area of the plate at a distance δ from its inner edge and the narrow rim of width δ. Integration over this narrow edge is negligible, and by definition of a good conductor $j\omega\varepsilon_1 E_s$ is negligible compared with $i_s = \delta_1 E_s$. Hence

$$\nu_0 2\pi b B_{\theta 2}(b) = I_{s1} \tag{30}$$

so that

$$B_{\theta 2}(b) = \frac{I_{s1}}{\nu_0 2\pi b} \tag{31}$$

If (31) is substituted in (27),

$$B_{\theta 2}(r) = \frac{I_{s1}}{\nu_0 2\pi b} \frac{J_1(\beta_2 r)}{J_1(\beta_2 b)} \tag{32}$$

As a second cap surface, choose the inner surface of the plate just in the dielectric.   Here

$$\nu_0 2\pi b B_{\theta 2}(b) = \int_0^b (\delta_2 + j\omega\varepsilon_2)E_{z2}(r)2\pi r\, dr \tag{33}$$

Hence, with (31), and using the complex dielectric factor ξ (III.14.28),

$$j\omega\xi_2 = (\delta_2 + j\omega\varepsilon_2) = j\omega\epsilon_{s2}(1 - jh_o) \tag{34}$$

$$I_{s1} = j\omega\xi_2 2\pi \int_0^b E_{z2}(r)r\, dr \tag{35}$$

Using (26)

$$I_{s1} = \frac{j\omega\xi_2 \cdot 2\pi E_{z2}(b)}{J_0(\beta_2 b)} \int_0^b J_0(\beta_2 r)r\, dr \tag{36}$$

With

$$\int_0^x J_0(x)x \, dx = xJ_1(x), \quad \text{and} \quad J_1(0) = 0$$

$$I_{z1} = \frac{j\omega\xi_2 \cdot 2\pi b E_{z2}(b)}{\beta_2 J_0(\beta_2 b)} J_1(\beta_2 b) \tag{37}$$

Solving (37) for $E_{z2}(b)$,

$$E_{z2}(b) = \frac{1}{j\omega\xi_2} \frac{I_{z1}}{\pi b^2} \left(\frac{\beta_2 b}{2}\right) \frac{J_0(\beta_2 b)}{J_1(\beta_2 b)} \tag{38}$$

This may be substituted in (26) to give

$$E_{z2}(r) = \frac{1}{j\omega\xi_2} \frac{I_{z1}}{\pi b^2} \left(\frac{\beta_2 b}{2}\right) \frac{J_0(\beta_2 r)}{J_1(\beta_2 b)} \tag{39}$$

The most important special case is that of a perfect dielectric,

$$\xi_2 = \epsilon_2; \qquad \beta_2 = \beta_o = \frac{\omega}{v_o}; \qquad v_o = \sqrt{\frac{v_2}{\epsilon_2}} \tag{40}$$

with small values of the argument

$$(\beta_2 b)^2 \ll 1 \tag{41}$$

Using the limiting values (3.16) and (6.7)

$$J_0(\beta_2 b) \doteq 1; \qquad J_1(\beta_2 b) \doteq \frac{\beta_2 b}{2} \tag{42}$$

(32) and (39) become

$$B_{\theta 2}(r) = \frac{I_{z1}}{v_0 2\pi b} \frac{r}{b}; \qquad (\beta_2 b)^2 \ll 1 \tag{43}$$

$$E_{z2}(r) = \frac{I_{z1}}{j\omega\epsilon_2 \cdot \pi b^2}; \qquad (\beta_2 b)^2 \ll 1 \tag{44}$$

It is to be noted that E_z is a constant independent of r as well as of z.

15. Distribution of Surface Charge in the Condenser Plates.— The determination of the distribution of charge on the inner surfaces of the condenser plates depends upon (14.15a,b). These boundary conditions can also be written in the form (III.14.31a) using the complex dielectric factor ξ (III.14.28). For both plates,

$$\xi_1 E_{z1} - \xi_2 E_{z2} = 0 \tag{1}$$

so that

$$\frac{E_{z1}}{E_{z2}} = \frac{\xi_2}{\xi_1} = \frac{\epsilon_{2o} - j\dfrac{\sigma_{2o}}{\omega}}{\epsilon_{1o} - j\dfrac{\sigma_{1o}}{\omega}} \tag{2}$$

From the definitions of a good conductor $(\sigma_{1o}/\omega\epsilon_0 >> 1)$ and of a slightly conducting dielectric $(\sigma_{2o}/\omega\epsilon_{2o})^2 << 1$, it follows that

$$\frac{\epsilon_0 E_{s1}}{\epsilon_{2o} E_{s2}} \doteq \frac{\omega\epsilon_0}{\sigma_{1o}} << 1 \tag{3}$$

For all electrically useful frequencies and available dielectrics, it is correct to assume

$$\epsilon_0 E_{s1} << \epsilon_{2o} E_{s2} \tag{4}$$

Accordingly (14.15a,b) reduce to

$$n_{1L} = -n_{1R} = \epsilon_2 E_s \tag{5}$$

With (14.39) this becomes

$$n_{1L}(r) = -n_{1R}(r) = \frac{\epsilon_2}{j\omega\xi_2}\left(\frac{I_{s1}}{\pi b^2}\right)\left(\frac{\beta_2 b}{2}\right)\frac{J_0(\beta_2 r)}{J_1(\beta_2 b)} \tag{6}$$

This relation can be separated into an amplitude factor that depends upon the radius b and a radial distribution factor that depends only upon r. The amplitude factor is the density of charge at the center $r = 0$.

$$n_{1L}(0) = \frac{\epsilon_2}{j\omega\xi_2}\left(\frac{I_{s1}}{\pi b^2}\right)\left(\frac{\beta_2 b}{2}\right)\frac{1}{J_1(\beta_2 b)} \tag{7a}$$

Using $\beta_2 = \omega/v_2$ with $v_2 \equiv \sqrt{v_2/\xi_2}$,

$$n_{1L}(0) = \frac{-j\epsilon_2}{v_2\xi_2}\frac{I_{s1}}{2\pi b}\frac{1}{J_1(\beta_2 b)} \tag{7b}$$

With (7a) or (7b), (6) becomes

$$n_{1L}(r) = n_{1L}(0)J_0(\beta_2 r) \tag{8}$$

If the dielectric is perfect so that $\beta_2 = \beta_e = \omega/v_e$ with

$$v_e = \sqrt{\frac{v_2}{\epsilon_{e2}}},$$

$$n_{1L}(0) = \frac{-j}{v_e}\frac{I_{s1}}{2\pi b}\frac{1}{J_1(\beta_e b)} \tag{9a}$$

$$n_{1L}(r) = n_{1L}(0)J_0(\beta_e r) \tag{9b}$$

In this case the charge density is in the same phase everywhere on the plate and in phase quadrature with the total current I_{s1} at the edge. The distribution (9b) is shown in Fig. 15.1. The total surface charge per unit distance along the radius, i.e., the charge

on a ring of radius r and unit width, is

$$q_{1L}(r) = 2\pi r n_{1L}(r) = \left(\frac{2\pi n_{1L}(0)}{\beta_e}\right)(\beta_e r) J_0(\beta_e r) \tag{9c}$$

FIG. 15.1.—Radial distribution of the surface density of charge $\eta(r) \sim J_0(\beta_e r)$ and of the quasi-surface density of current $l_r'(r) \sim J_1(\beta_e r)$.

FIG. 15.2.   Total surface charge on a ring of unit width and radius r, $q(r) = 2\pi r \eta(r) \sim \beta_e r J_0(\beta_e r)$; total radial current, $I_r(r) \sim \beta_e r J_1(\beta_e r)$.

The function $\beta_e r J_0(\beta_e r)$ is shown in Fig. 15.2 as a function of $\beta_e r$.

If the dielectric is slightly conducting so that (14.22) is satisfied

$$\beta_2 = \beta_s - j\alpha_s; \qquad \alpha_s^2 << \beta_s^2 \tag{10}$$

with β_s and α_s given by (14.23,24). As indicated, (14.25) is also true. Omitting the subscript s temporarily, and using (15.8)

$$\mathfrak{n}_{1L}(r) = \mathfrak{n}_{1L}(0)J_0\left(\beta r\left[1 - j\frac{\alpha}{\beta}\right]\right) \tag{11}$$

The Bessel function can be expanded in a Maclaurin series in powers of α/β and only the leading real and imaginary terms retained. Thus, with

$$f\left(\frac{\alpha}{\beta}\right) = f(0) + \frac{\alpha}{\beta}f'(0) \tag{12}$$

and

$$f\left(\frac{\alpha}{\beta}\right) = J_0\left(\beta r\left[1 - j\frac{\alpha}{\beta}\right]\right) \tag{13}$$

$$f'\left(\frac{\alpha}{\beta}\right) = J_0'\left(\beta r\left[1 - j\frac{\alpha}{\beta}\right]\right)(-j\beta r)$$

$$= j\beta r J_1\left(\beta r\left[1 - j\frac{\alpha}{\beta}\right]\right) \tag{14}$$

Accordingly, with distinguishing subscript s replaced,

$$\frac{\mathfrak{n}_{1L}(r)}{\mathfrak{n}_{1L}(0)} = J_0\left(\beta_s r\left[1 - j\frac{\alpha_s}{\beta_s}\right]\right) \doteq J_0(\beta_s r) + j\alpha_s r J_1(\beta_s r) \tag{15}$$

with β_s and α_s given by (14.23,24). In amplitude-phase-angle form,

$$\mathfrak{n}_{1L}(r) = \mathfrak{n}_{1L}(0)\sqrt{J_0^2(\beta_s r) + \alpha_s^2 r^2 J_1^2(\beta_s r)}\; e^{j\psi} \tag{16a}$$

$$\tan\psi = \left(\frac{2\alpha_s}{\beta}\right)\left[\frac{\beta r}{2}\frac{J_1(\beta_s r)}{J_0(\beta_s r)}\right] \tag{16b}$$

The surface charge on the plates of a condenser with a slightly conducting dielectric is not in phase everywhere on each plate. It differs from the density at the center by a phase angle that increases with distance from the center. The amplitude is essentially the same as with a perfect dielectric. The total charge on a ring of unit radial width is given by

$$q_1(r) = 2\pi r \mathfrak{n}_1(r) \tag{17}$$

If the condenser is sufficiently small,

$$\left.\begin{array}{c} \mathfrak{n}_{1L}(r) = \mathfrak{n}_{1L}(0) \\ \psi = 0 \end{array}\right\} \quad \beta_s^2 b^2 << 1 \tag{18}$$

That is, the density is practically uniform in amplitude and phase everywhere on each plate. In (18), $\beta_s \doteq \beta_e = \omega/v_e$. The condition $\beta_s^2 b^2 \ll 1$ is equivalent to the condition for the near zone. *At low frequencies, (18) is always well satisfied for most practically useful condensers* so that the total charge on the left plate is

$$Q_{1L} = n_{1L}(0)\pi b^2 \tag{19}$$

With (18) and (5), (19) becomes

$$Q_{1L} = E_{s2}\epsilon_2\pi b^2 \tag{20}$$

for a perfect dielectric. The ratio,

$$\frac{1}{c_0} = \frac{E_{s2}}{Q_{1L}} = \frac{1}{\epsilon_2\pi b^2} \tag{21}$$

is called the *static elastance per unit thickness* of the condenser. Since E_{s2} is constant between the plates and equal to $-\partial\phi_1/\partial z$ in the near zone, the *static elastance of the condenser* is

$$\frac{1}{C_0} = \frac{w}{c_0} = \frac{(\phi_L - \phi_R)}{Q_{1L}} = \frac{w}{\epsilon_2\pi b^2} \tag{22}$$

The *static capacitance C_0* of the condenser is

$$C_0 = \epsilon_2 \frac{\pi b^2}{w} \tag{23}$$

16. Distribution of Current in the Condenser Plates.—The analysis of the distribution of current in the metal condenser plates is excessively difficult in the general case. Since the plates are highly conducting and thin, their effect on the circuit as a whole is usually entirely negligible, so that the distribution of current in them is of no practical importance. For small values of $\beta_1 b$ for which it is possible to neglect $(\beta_1 b)^2$ compared with unity, the density of current in the cylindrical conductor of radius a is uniformly distributed in each cross section and it fans out in the condenser plate so that it is again uniformly distributed as it enters or leaves the surface layers of thickness δ. In this case, the surface density of charge is also uniformly distributed as explained above.

At frequencies which are so high that the thickness t of the plates is large compared with the skin depth, it may be assumed

that most of the current is in a very thin layer radially out-
ward (or inward) on the outer surfaces of the plates, around the
edges, and radially inward (or outward) along the inner surfaces.
If the conductor is not perfect, the layer of current is very thick
compared with the atomic thickness δ of the surface layer of
charge, even though very thin compared with the thickness t of
the plate.   In fact, a negligible fraction of the radial current
is in this surface layer so that a surface density of current I
need not be defined.   This means that all the current entering
the surface layer is directed perpendicular to the surface.   All
the charge entering or leaving across a particular unit area is
deposited in the surface layer bounded by this area—there is no
radial flow in the surface layer.   Accordingly, the distribution
of axial current at a depth δ below the inner surface of each
plate is obtained from the surface equation of continuity (I.24.13)
which simplifies to

$$i_{s1}(r) \;=\; j\omega n_1(r) \;=\; j\omega n_1(0)J_0(\beta_2 r) \tag{1}$$

The radial current is a quasi-surface current.   It exists princi-
pally near the surfaces of the plates at a depth between the
surface layer of thickness δ and several times skin depth d.   It
can be determined very simply using (14.28) with appropriate
choice of cap surface S and closed contour s.   Let the contour s
be a circle of radius r on the inner face of a plate; let the cap S
consist of a circular plane of radius r inside the condenser plate,
parallel to its sides at a distance several times the skin depth
from the inner face, and of a narrow cylinder of radius r as shown
in Fig. 16.1.   Since the entire cap surface is in the conductor,
the integral in (14.28) is negligible.   Furthermore, at several
times the skin depth from the inner surface the electric field
and the density of current are extremely small.   Accordingly,
the only significant contribution on the right side of (14.29)
is the radial quasi-surface current l'_r crossing the narrow cylin-
drical rim of the cap.   With rotational symmetry,

$$B_{\theta2}(r) \;\doteq\; \frac{I_{r1}(r)}{v_0 2\pi r} \;=\; \frac{l'_{r1}(r)}{v_0} \tag{2}$$

Hence, with (14.32)

$$l'_{r1}(r) \;=\; \frac{I_{s1}}{2\pi b}\,\frac{J_1(\beta_2 r)}{J_1(\beta_2 b)} \tag{3a}$$

Also

$$I_{r1}(r) = 2\pi r l'_{r1}(r) = I_{s1} \frac{r}{b} \frac{J_1(\beta_2 r)}{J_1(\beta_2 b)} \tag{3b}$$

Formula (3a) can be expressed in terms of an amplitude factor and a radial distribution factor. The amplitude factor is

$$\frac{I_{s1}}{2\pi b} \frac{1}{J_1(\beta_2 b)} = \mathfrak{n}_{1L}(0) \frac{j v_2 \xi_2}{\varepsilon_2} \tag{4}$$

where $\mathfrak{n}_{1L}(0)$ is the density of charge at the center of the left-hand plate. With (4)

$$l'_{r1}(r) = \frac{j v_2 \xi_2}{\varepsilon_2} \mathfrak{n}_{1L}(0) J_1(\beta_2 r) \tag{5}$$

Fig. 16.1.—Cross section of condenser plate very much exaggerated in thickness t.

If the dielectric is perfect so that $\xi_2 = \epsilon_2$, $\beta_2 = \beta_c$, $v_2 = v_c$,

$$l'_{r1}(r) = j v_c \mathfrak{n}_{1L}(0) J_1(\beta_c r) \tag{6a}$$

This formula is convenient for comparison with the distribution of surface charge in (15.9b); $\mathfrak{n}_{1L}(0)$ is the density of charge at the center of the plate. The density of the quasi-surface current is in the same or in opposite phase at all points and in phase quadrature with the density $\mathfrak{n}_{1L}(r)$ of the surface charge. The radial distribution of surface current (6a) is shown in Fig. 15.1 together with the distribution of surface charge. The total radial current

is

$$I_{r1}(r) = 2\pi r l'_{r1}(r) = \left(\frac{j2\pi v_e n_{1L}(0)}{\beta_e}\right)\beta_e r J_1(\beta_e r) \qquad (6b)$$

The function $\beta_e r J_1(\beta_e r)$ is shown in Fig. 15.2 with $\beta_e r J_0(\beta_e r)$.

If the dielectric is slightly conducting, (15.10) applies. In this case, temporarily omitting subscripts s on α_s and β_s,

$$f\left(\frac{\alpha}{\beta}\right) = J_1(\beta_2 r) = J_1\left(\beta r\left[1 - j\frac{\alpha}{\beta}\right]\right) \qquad (7)$$

$$f'\left(\frac{\alpha}{\beta}\right) = -j\beta r J_1'\left(\beta r\left[1 - j\frac{\alpha}{\beta}\right]\right) \qquad (8)$$

A standard formula for differentiating Bessel functions of any kind and order is[1]

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x) \qquad (9)$$

Hence

$$J_1'(x) = J_0(x) - \frac{1}{x}J_1(x) \qquad (10)$$

so that

$$f'\left(\frac{\alpha}{\beta}\right) = -j\beta r\left[J_0\left(\beta r\left[1 - j\frac{\alpha}{\beta}\right]\right) - \frac{J_1\left(\beta r\left[1 - j\frac{\alpha}{\beta}\right]\right)}{\beta r\left[1 - j\frac{\alpha}{\beta}\right]}\right] \qquad (11)$$

The Maclaurin series (15.12) neglecting terms in α^2 is

$$J_1(\beta_2 r) \doteq f(0) + \frac{\alpha}{\beta}f'(0)$$

$$J_1(\beta_2 r) \doteq J_1(\beta r) - j\frac{\alpha}{\beta}[\beta r J_0(\beta r) - J_1(\beta r)] \qquad (12)$$

Therefore, with the distinguishing subscript s replaced,

$$l'_{r1}(r) = j\frac{v_2\xi_2}{\epsilon_2}n_{1L}(0)\left\{J_1(\beta_s r) - j\frac{\alpha_s}{\beta_s}[\beta_s r J_0(\beta_s r) - J_1(\beta_s r)]\right\} \qquad (13)$$

The total radial current is

$$I_{r1}(r) = 2\pi r l'_{r1}(r) \qquad (14a)$$

[1] Jahnke-Emde, "Tables of Functions," p. 145, 1938 ed.

In terms of the total current, using 15.7b

$$I_{r1}(r) = I_{s1} \cdot \frac{r}{b} \left\{ \frac{J_1(\beta_s r) - j\dfrac{\alpha_s}{\beta_s}[\beta_s r J_0(\beta_s r) - J_1(\beta_s r)]}{J_1(\beta_s b) - j\dfrac{\alpha_s}{\beta_s}[\beta_s b J_0(\beta_s b) - J_1(\beta_s b)]} \right\} \quad (14b)$$

Since the term in α_s/β_s is important in the magnitude of the expression in braces only for values of r at which $J_1(\beta_s r)$ becomes extremely small, a satisfactory approximation is

$$\left\{ J_1(\beta_s r) - j\frac{\alpha_s}{\beta_s}[\beta_s r J_0(\beta_s r) - J_1(\beta_s r)] \right\}$$
$$\doteq \sqrt{J_1^2(\beta_s r) + \alpha_s^2 r^2 J_0^2(\beta_s r)} \; e^{j\psi_1} \quad (15a)$$

with

$$\tan \psi_1 = -\frac{\alpha_s}{\beta_s}\left[\beta_s r \frac{J_0(\beta_s r)}{J_1(\beta_s r)} - 1 \right] \quad (15b)$$

For very small values of $\beta_s r$, $J_1(\beta_s r) \doteq \frac{1}{2}\beta_s r$; $J_0(\beta_s r) \doteq 1$, so that

$$\left\{ J_1(\beta_s r) - j\frac{\alpha_s}{\beta_s}[\beta_s r J_0(\beta_s r) - J_1(\beta_s r)] \right\}$$
$$\doteq \left(1 - j\frac{\alpha_s}{\beta_s} \right) J_1(\beta_s r); \qquad (\beta_s r)^2 << 1 \quad (16)$$

It is important to note that the attenuation factor α_s includes only the effect of a slightly conducting dielectric, not that of imperfectly conducting metal plates. When a condenser is connected in a circuit, the attenuation due to the external circuit is usually very much greater than that due to both the imperfectly conducting condenser plates and the dielectric combined if this latter is good. If the dielectric is not so good that its attenuation is not negligible, it is usually also large compared with the attenuation due to the metal plates. A more complete and correspondingly much more complicated analysis therefore is usually not required. In effect, the present analysis is equivalent to one assuming perfectly conducting plates with a true surface current.

17. Internal Impedance of a Condenser; Equivalent Parallel Circuit.— The ratio of the electric field $E_{z2}(b)$ at the edge $r = b$ of a sufficiently thin dielectric disk between circular condenser plates to the total current I_{z1} charging or discharging the adjacent surfaces is defined to be the internal impedance per unit length.

Using (14.38)

$$z^i = \frac{E_{z2}(b)}{I_{z1}} = \frac{1}{j\omega\xi_2\pi b^2}\frac{\beta_2 b}{2}\frac{J_0(\beta_2 b)}{J_1(\beta_2 b)} \tag{1}$$

Although similar in form to the corresponding expression (7.6) for a conductor, z^i for the condenser differs greatly from z^i for the conductor because β_2 is predominantly real for a slightly conducting dielectric, whereas β_1 has equal real and imaginary parts in a good conductor. At sufficiently small values of $\beta_2 b$, it follows from (7.7) that

$$z^i = \frac{1}{j\omega\xi_2\pi b^2}; \qquad (\beta_2 b)^2 << 8 \tag{2}$$

The complex dielectric factor (III.14.25b,28) is

$$\xi_2 = \epsilon_{e2}(1 - jh_e) = \epsilon_{e2}\left(1 - j\frac{2\alpha_e}{\beta_e}\right) \tag{3}$$

with

$$\beta_2 = \beta_e - j\alpha_e \tag{4}$$

and β_e and α_e defined in (14.23,24). In a perfect dielectric

$$z^i = jx^i = \frac{1}{j\omega\epsilon_{e2}\pi b^2} \tag{5}$$

Using (15.21), let

$$x_0^i \equiv -\frac{1}{\omega\epsilon_{e2}\pi b^2} = -\frac{1}{\omega c_0} \tag{6}$$

Then, using (6) and (3) in (1)

$$z^i = jx_0^i\left[\frac{\beta_e b\left(1 - j\dfrac{\alpha_e}{\beta_e}\right)J_0\left(\beta_e b\left[1 - j\dfrac{\alpha_e}{\beta_e}\right]\right)}{2\left(1 - j\dfrac{2\alpha_e}{\beta_e}\right)J_1\left(\beta_e b\left[1 - j\dfrac{\alpha_e}{\beta_e}\right]\right)}\right] \tag{7}$$

For convenience in writing, let the following temporary notation be used:

$$B \equiv \beta_e b; \qquad p \equiv \frac{\alpha_e}{\beta_e} = \frac{\sigma_{e2}}{2\omega\epsilon_{e2}} \tag{8}$$

As in (15.15) and (16.12) the following relations are true:

$$J_0(B[1 - jp]) = J_0(B) + jpBJ_1(B) \tag{9}$$
$$J_1(B[1 - jp]) = J_1(B) - jp[BJ_0(B) - J_1(B)] \tag{10}$$

Using (8), (9), and (10) in (7) and omitting the argument $B = b\beta$

in writing the Bessel functions

$$z^i = jx_0^i \left[\frac{B(1-jp)[J_0 + jpBJ_1]}{2(1-j2p)[J_1 - jp(BJ_0 - J_1)]} \right] \tag{11}$$

Neglecting terms in $p^2 = \alpha_s^2/\beta_s^2$ and rearranging,

$$z^i = jx_0^i \frac{BJ_0}{2J_1} \left[\frac{1 + jp\left(\frac{BJ_1}{J_0} - 1\right)}{1 - jp\left(\frac{BJ_0}{J_1} + 1\right)} \right] \tag{12}$$

Since the axial electric field is constant in the dielectric, it follows that, neglecting edge effects, the internal impedance for a thickness of dielectric w is

$$Z^i = z^i w = jX_0^i \frac{BJ_0}{2J_1} \left[\frac{1 + jp\left(\frac{BJ_1}{J_0} - 1\right)}{1 - jp\left(\frac{BJ_0}{J_1} + 1\right)} \right] \tag{13}$$

where

$$X_0^i = x_0^i w = -\frac{w}{\omega\epsilon_{c2}\pi b^2} = -\frac{1}{\omega C_0} \tag{14}$$

Here C_0 is the static capacitance of the condenser as defined by (15.23).

$$C_0 = \epsilon_{c2}\frac{\pi b^2}{w} \tag{15}$$

The impedance (13) is the same as the impedance Z_{AB} of the circuit of Fig. 17.1. The impedance Z_{AB} is

Fig. 17.1.—Equivalent parallel circuit of condenser.

$$Z^i = Z_{AB} = \frac{jX_c(R + jX)}{R + j(X + X_c)} = jX_c \left[\frac{1 + j\frac{X}{R}}{1 + j\frac{(X + X_c)}{R}} \right] \tag{16}$$

This is equal to (13) if

$$X_c = X_0^i \frac{BJ_0}{2J_1} = -\frac{1}{\omega C_0} \left(\frac{\beta_s b}{2} \frac{J_0(\beta_s b)}{J_1(\beta_s b)} \right) \tag{17}$$

$$\frac{X}{R} = p\left(\frac{BJ_1}{J_0} - 1\right) \tag{18}$$

$$\frac{X + X_c}{R} = -p\left(\frac{BJ_0}{J_1} + 1\right) \tag{19}$$

Using (17), (18) and (19) may be solved for X and R. The results are

$$R = -\frac{X_0^i}{2p}\left[\frac{J_0^2}{J_0^2 + J_1^2}\right] = \frac{w}{\sigma_{s2}\pi b^2}\left[\frac{J_0^2}{J_0^2 + J_1^2}\right]$$

$$= R_0^i\left[\frac{J_0^2(\beta_s b)}{J_0^2(\beta_s b) + J_1^2(\beta_s b)}\right] \quad (20)$$

Here $R_0^i = wr_0$ is the d.c. resistance of the dielectric disk between the faces of the condenser plates.

$$X = \frac{X_0^i J_0}{2}\left[\frac{J_0 - BJ_1}{J_0^2 + J_1^2}\right] = -\frac{J_0(\beta_s b)}{2\omega C_0}\left[\frac{J_0(\beta_s b) - \beta_s b J_1(\beta_s b)}{J_0^2(\beta_s b) + J_1^2(\beta_s b)}\right] \quad (21)$$

Fig. 17.2.—The ratios X_c/X_0^i, R/R_0^i, X/X_0^i for a condenser. $X_0^i = -1/\omega C_0$; $R_0^i = w/\sigma\pi b^2$ with C_0 the static capacitance, R_0^i the d.c. resistance. (β is written for β_s.)

The ratios X_c/X_0^i from (17), R/R_0^i from (20), and X/X_0^i from (21) are plotted in Fig. 17.2; the ratio $|X_c/X_0^i|$ for a range of $\beta_s b$

near its first maximum is plotted in Fig. 17.3.   It follows that
the internal impedance of a circular condenser neglecting edge
effects is equivalent to the parallel circuit of Fig. 17.1 using the

Fig. 17.3.—The ratio $|X_c/X_0{}^i|$ in Fig. 17.2 near its first infinity.

values of X_c, R, and X given in (17), (20), and (21).   In most
cases involving a good dielectric, $p \equiv \alpha_s/\beta_s = \sigma_{e2}/2\omega\epsilon_{e2}$ is so
small that R as given in (20) is very large compared with X as
given in (21) except near values of $\beta_s b$ for which $J_0(\beta_s b)$ and Z^i
itself approach or equal zero.

$$|X| << R \qquad \text{if} \qquad \frac{\alpha_s}{\beta_s}\left|1 - \frac{\beta_s b J_1(\beta_s b)}{J_0(\beta_s b)}\right| << 1 \qquad (22)$$

Subject to (22) the equivalent circuit is shown in Fig. 17.4.

The most important special case is that which satisfies $\beta_s b < 1$. In this case, using (7.7) and $\beta_s^2 = \omega^2 \epsilon_{c2}/\nu_2 = \omega^2 \epsilon_{c2}\mu_2$

$$X_c = X_0^i \left[1 - \frac{\beta_s^2 b^2}{8} \right] = \left[-\frac{1}{\omega C_0} + \omega L_0^i \right]; \quad (\beta_s^4 b^4 << 192) \quad (23)$$

where

$$L_0^i = w l_0^i = w \frac{1}{8\pi \nu_2} = w \frac{\mu_2}{8\pi} \quad (24)$$

L_0^i is a series inductance that is significant whenever $\beta_s b$ approaches unity in value. In most dielectrics, the relative reluctivity ν_r is unity so that

$$L_0^i = \frac{w}{2} \times 10^{-7} \text{ henry;} \quad (w \text{ in meters}) \quad (25)$$

Similarly,

$$R = R_0^i \left(1 - \frac{\beta_s^2 b^2}{4} \right) = w r_0^i \left(1 - \frac{\beta_s^2 b^2}{4} \right) \quad (26)$$

With only slightly severer restrictions,

$$X_c \doteq X_0^i = -\frac{1}{\omega C_0}; \qquad R \doteq R_0^i = w r_0^i; \qquad \beta_s^2 b^2 << 4 \quad (27)$$

These are the conventional low-frequency formulas.

Since $\beta_s \doteq \beta_o = \omega/v_o$ for a slightly conducting or nonconducting dielectric, the restriction $\beta_s^2 b^2 << 4$ is not a very severe one for any but the very highest frequencies with condensers of rather large radius. For example, if a 1 per cent error is tolerated so that this condition is understood to mean

$$\beta_s^2 b^2 \leq 0.04, \qquad \beta_s b \leq 0.2 \quad (28)$$

an air condenser of radius $b = 2$ centimeters may be analyzed using the low-frequency formulas (27) provided

$$\omega \leq 3 \times 10^9 \qquad (\epsilon_{cr} = 1) \quad (29)$$

or

$$f \leq 480 \text{ megahertz} \quad (30)$$

On the other hand, with a 1 per cent error, the formulas (23) and (26) may be used if

$$\beta_s^4 b^4 \leq 1.92 \qquad \text{or} \qquad \beta_s b \leq 1.17 \quad (31)$$

This means

$$\omega \leq 1.76 \times 10^{10}, \qquad f \leq 2{,}800 \text{ megahertz} \tag{32}$$

Roughly, *if the radius does not exceed about* 0.03λ, *the low-frequency formulas* (27) *may be used; if the radius does not exceed* 0.2λ, *the formulas* (23) *and* (26) *are adequate.* It is interesting to note that, whereas in the case of the cylindrical conductor the low-frequency formulas were adequate only for low audio frequencies, the corresponding formulas may be used for the condenser at ultrahigh frequencies. The reason for this is found in the great difference in the magnitude of the phase constant β_s in a conductor and in a dielectric and the corresponding change in the distances involved in the condition for the near zone.

18. Internal Impedance of a Condenser; Equivalent Series Circuit.—The parallel circuit of Fig. 17.1 is readily transformed into an equivalent series circuit with the series impedance of the condenser given by

$$Z^i = R^i + jX^i \tag{1}$$

and

$$R^i = \frac{RX_c^2}{R^2 + (X + X_c)^2} = \frac{X_c^2}{R}\left[\frac{1}{1 + \left(\dfrac{X + X_c}{R^2}\right)^2}\right] \tag{2}$$

$$X^i = \frac{X_c[R^2 + X(X + X_c)]}{R^2 + (X + X_c)^2} = X_c\left[\frac{1 + \dfrac{X(X + X_c)}{R^2}}{1 + \left(\dfrac{X + X_c}{R}\right)^2}\right] \tag{3}$$

In a slightly conducting dielectric, the d.c. resistance R_0^i is so large that excluding narrow ranges where X_c becomes infinite or where R vanishes, discussed in the next section, it is correct to write

$$X_c^2 << R^2; \qquad X^2 << R^2 \tag{4}$$

Then

$$R^i \doteq \frac{X_c^2}{R} \tag{5}$$

$$X^i \doteq X_c \tag{6}$$

Using (17.17) and (17.20),

$$R^i = -2X_0^i p\left(\frac{\beta_s b}{2}\right)^2\left(1 + \frac{J_0^2(\beta_s b)}{J_1^2(\beta_s b)}\right) \tag{7}$$

Since $p \equiv \alpha_s/\beta_s = \sigma_{o2}/2\omega\epsilon_{o2}$ and $X_0^i = -w/\omega\epsilon_{o2}\pi b^2$, $R_0^i = w/\sigma_{o2}\pi b^2$,

$$R^i = R_0^i \left(\frac{X_0^i}{R_0^i}\right)^2 \left(\frac{\beta_s b}{2}\right)^2 \left(1 + \frac{J_0^2(\beta_s b)}{J_1^2(\beta_s b)}\right) \qquad (8a)$$

An alternative, simpler form using $\beta_s^2 \doteq \beta_{o2}^{2''} = \omega^2\epsilon_{o2}/\nu_2$ and

$$\zeta_{o2} \equiv \frac{1}{\sqrt{\nu_2\epsilon_{o2}}}$$

is

$$R^i = \frac{w\zeta_{o2}^2\sigma_{o2}}{4\pi}\left(1 + \frac{J_0^2(\beta_s b)}{J_1^2(\beta_s b)}\right) \qquad (8b)$$

Fig. 18.1.—Functions involved in the equivalent series impedance of a condenser.

The form $(8a)$ is convenient if it is desired to show the variation with frequency from the d.c. resistance R_0^i. The form $(8b)$ gives the relatively simple dependence on the parameter $\beta_s b$. Either $(8a)$ or $(8b)$ is the general formula for the equivalent

series resistance of a condenser except in ranges where (4) is not true.   The complete series impedance using (8a) is

$$Z^i = R^i + jX^i = R_0^i \left(\frac{1}{R_0^i \omega C_0}\right)^2 \left[\left(\frac{\beta_s b}{2}\right)^2 \left(1 + \frac{J_0^2(\beta_s b)}{J_1^2(\beta_s b)}\right)\right]$$
$$- \frac{j}{\omega C_0}\left[\left(\frac{\beta_s b}{2}\right)\frac{J_0(\beta_s b)}{J_1(\beta_s b)}\right] \quad (9)$$

Fig. 18.2. - Functions involved in the equivalent series impedance of a condenser.

The functions $\left(\frac{\beta_s b}{2}\right)^2 \left(1 + \frac{J_0^2(\beta_s b)}{J_1^2(\beta_s b)}\right)$ and $-\left(\frac{\beta_s b}{2}\right)\frac{J_0(\beta_s b)}{J_1(\beta_s b)}$, which characterize the change in impedance with frequency referred to the low-frequency form $Z_0^i = R_0^i - j/\omega C_0$, are plotted in Figs. 18.1 and 18.2 to different scales.   From these, R^i and X^i may be determined for any given values of C_0 and R_0^i except in the narrow ranges excluded by (4).   These occur only very near those values of $\beta_s b$ where X_a becomes infinite or R vanishes.   These are,

respectively, at the roots of $J_1(\beta_s b) = 0$, *viz.*, $\beta_s b = 3.83$, 7.02, etc., and at the roots of $J_0(\beta_s b) = 0$, *viz.*, $\beta_s b = 2.40$, 5.52, etc. The function $\left(1 + \dfrac{J_0^2(\beta_s b)}{J_1^2(\beta_s b)}\right)$, which represents the true variation of R^i with frequency, is shown in Figs. 18.2 and 18.3.

FIG. 18.3.—Function involved in the equivalent series impedance of a condenser.

The general formula (9) can be simplified greatly in the most important range. Using (7.7)

$$\left(\frac{\beta_s b}{2}\right)^2 \left(1 + \frac{J_0^2(\beta_s b)}{J_1^2(\beta_s b)}\right) \doteq 1 \qquad \text{when} \qquad (\beta_s b)^4 << 192 \quad (10)$$

Hence

$$R^i = \frac{X_0^{i2}}{R_0^i} = \frac{1}{R_0^i \omega^2 C_0^2}; \qquad (\beta_s b)^4 << 192 \quad (11)$$

Upon combining (11) and (17.23), the equivalent series imped-
ance of the condenser for $b \leqq 0.2\lambda$ is

$$Z^i = \frac{1}{R_0^i \omega^2 C_0^2} + j\left(\omega L_0^i - \frac{1}{\omega C_0}\right); \qquad (\beta_s b)^4 << 192 \quad (12)$$

where

$$R_0^i = w r_0^i = \frac{w}{\sigma_{c2}\pi b^2} \tag{13}$$

$$L_0^i = w l_0^i = \frac{w}{8\pi \nu_2} = \frac{w \mu_2}{8\pi} \tag{14}$$

$$\frac{1}{C_0} = \frac{w}{c_0} = \frac{w}{\epsilon_{c2}\pi b^2} \tag{15}$$

This formula is adequate for most ultrahigh-frequency purposes
using small condensers.   The internal impedance per unit thick-
ness of dielectric is

$$z^i = r_0^i\left(\frac{\sigma_{c2}}{\omega\epsilon_{c2}}\right)^2 + j\left(\omega l_0^i - \frac{1}{\omega c_0}\right); \qquad \beta_s^4 b^4 << 192 \tag{16}$$

**19. Critical Values of the Internal Impedance of a Condenser;
Resonance and Antiresonance.**—The general expressions (18.2)
and (18.3) for the series resistance and reactance

$$R^i = \frac{RX_c^2}{R^2 + (X + X_c)^2} \tag{1}$$

$$X^i = \frac{X_c[R^2 + X(X + X_c)]}{R^2 + (X + X_c)^2} \tag{2}$$

reduce to the simple forms (18.5) and (18.6) and their equivalent
in (18.9) except in ranges in which X_c approaches the parallel
resistance R of the condenser or R vanishes.   Since the entire
analysis has been limited to a condenser with R_0^i very great, it is
seen from Fig. 17.2 that since R is very large when X_c is large,
the simpler forms (18.5) and (18.6) are not adequate in narrow
ranges near values of $\beta_s b$ for which $J_1(\beta_s b)$ vanishes and X_c
becomes infinite.   Examination of (1) and (2) shows that when
X_c is made infinite with R finite, the series reactance X^i of the
condenser reduces to X, while R^i approaches the large value R.
Thus,

$$X_c = \infty; \qquad R^i = R; \qquad X^i = X \tag{3}$$

Since $X_c = \infty$ at $J_1(\beta_s b) = 0$, it follows from (17.20) and (17.21)

that

$$X_c = \infty; \qquad R^i = R = R_0^i; \qquad X^i = X = \frac{1}{2} X_0^i = \frac{-1}{2\omega C_0} \quad (4)$$

Clearly R^i has its maximum possible value, the d.c. value $R_0^i = w/\sigma_c 2\pi b^2$, at $X_c = \infty$. In a sufficiently narrow range near $X_c = \infty$, $|X|$ may be neglected in comparison with $|X_c|$. Then

$$R^i \doteq \frac{RX_c^2}{R^2 + X_c^2}; \qquad |X| << |X_c| \quad (5)$$

$$X^i \doteq \frac{X_c(R^2 + XX_c)}{R^2 + X_c^2}; \qquad |X| << |X_c| \quad (6)$$

It remains to determine R^i and X^i in the excluded ranges near zeros of R. These occur when $J_0(\beta_s b) = 0$. Since X and X_c both vanish at these same points, the corresponding values of R^i and X^i are best determined from the general formula (17.13) for $Z^i = R^i + jX^i$. With $J_0 = 0$ and neglecting terms $p^2 = \alpha_s^2/\beta_s^2$, (17.13) reduces exactly to (18.5) and (18.6) in the forms (18.7) and (17.17). Therefore, (5) and (6) together with (18.5) and (18.6) adequately describe the internal series impedance of the condenser for practically all values of $\beta_s b$.

Critical values of the impedance of the condenser are associated with vanishing reactance and extreme values of resistance and reactance. The equivalent circuit of Fig. 17.1 may exhibit characteristics of series resonance and parallel resonance. Series resonance, or simply resonance, occurs by definition when the reactance vanishes and the resistance is small. Parallel resonance or antiresonance occurs by definition when the reactance is zero and the resistance is very large—usually maximum. The condition for resonance is $X^i = 0$, so that with (2) and (17.17).

$$X^i = 0; \qquad X_c = 0; \qquad J_0(\beta_s b) = 0 \quad (7)$$

The roots are[1]

$$\beta_s b = 2.40, \ 5.52, \ 8.65, \ \text{etc.} \quad (8)$$

The resistance at resonance is given by (8b). It is

$$R_{res}^i = R_0^i \left(\frac{X_0^i}{R_0^i}\right)^2 \left(\frac{\beta_s b}{2}\right)^2 \quad (9)$$

[1] Additional values are tabulated in standard tables, *e.g.*, JAHNKE-EMDE, "Tables of Functions," p. 168, 1938 ed.

The condition for antiresonance is more intricate. From (2) it is

$$-X(X + X_c) = R^2 \tag{10}$$

Since R is large and X relatively small, X_c must be very large, so that it is correct to write

$$|X_c| >> X \tag{11}$$

Accordingly, a good approximation of (12) is

$$-XX_c = R^2 \tag{12}$$

Since the right side is positive, X and X_c must have opposite algebraic signs. Study of Fig. 17.2 discloses that X and X_c have opposite signs with large, including infinite, values of X_c in ranges extending to values of $\beta_s b$ for which $X_c = \infty$, *i.e.*, roots of $J_1(\beta_s b) = 0$, and beginning at values that are certainly much greater than those for which $X_c = 0$, *i.e.*, roots of $J_0(\beta_s b) = 0$. Two such ranges are shown in Fig. 17.2 extending from roughly $\beta_s b > 2.40$ to $\beta_s b = 3.83$ and from $\beta_s b > 5.52$ to $\beta_s b = 7.02$. The condition (12) can be expressed directly in terms of $\beta_s b$ using (17.20), (17.21), and (17.17). It is

$$\frac{\beta_s b}{2}\left[1 - \frac{J_0(\beta_s b)}{\beta_s b J_1(\beta_s b)}\right]^{\frac{1}{2}} = \frac{R_0^i}{|X_0^i|} = R_0^i \omega C_0 \tag{13}$$

If R_0^i is sufficiently great so that $R_0^i \omega C_0$ is greater than one, $J_0(\beta_s b)/\beta_s b J_1(\beta_s b)$ must be negative. Using Figs. 15.1 and 15.2, the possible ranges of values are seen to be from the roots of

$$J_1(\beta_s b) = 0 \tag{14}$$

to values slightly below these. For a perfect dielectric R_0^i is infinite, and antiresonance occurs exactly at values of $\beta_s b$ that satisfy (14). They are

$$\beta_s b = 3.83, \ 7.02, \ 10.17, \ \text{etc.} \tag{15}$$

For an imperfect dielectric the values of $\beta_s b$ producing antiresonance are slightly less than the values given in (15) depending upon the magnitude of $R_0^i \omega C_0$. The function on the left in (13) is represented graphically in Fig. 19.1 for the first range of values of $\beta_s b$. For values of $R_0^i \omega C_0$ larger than 20, no significant error is made by assuming antiresonance to occur at 3.83.

The extreme values of X^i and R^i occur near $X_c = \infty$. It is seen from Fig. 17.2 that near $X_c = \infty$ the rate of change of R is practically zero while that of X_c is great. Accordingly, (19.1) and (19.2) or the approximate formulas (5) and (6) may be

$$R_0^i \omega C = \frac{\beta_s b}{2}\left[1 - \frac{J_0(\beta_s b)}{\beta_s b\, J_1(\beta_s b)}\right]^{\!\frac{1}{2}}$$

FIG. 19.1.—Condition for antiresonance in a circular condenser with imperfect dielectric.

maximized and minimized by differentiating with respect to X_c assuming R constant and equating the derivatives to zero. Differentiating (19.1) in this way

$$\frac{dR^i}{dX_c} = \frac{[R^2 + (X + X_c)^2]2RX_c - RX_c^2[2(X + X_c)]}{[R^2 + (X + X_c)^2]^2} = 0 \quad (16)$$

The solution for this equation with R finite and X_c large is the condition for antiresonance (10) so that (12) to (18) apply.

The maximum value of R^i is practically that given in (4).

$$R^i_{max} = R_{max} = R^i_0 = \frac{w}{\sigma_{2e}\pi b^2} \quad \text{at} \quad \beta_e b \text{ defined in (13).} \quad (17)$$

Extreme values of X^i are derived from (2) with $|X| << |X_e|$.

$$\frac{dX^i}{dX_e} = 0 \quad \text{gives} \quad X_e^2 + 2XX_e - R^2 = 0 \quad (18)$$

The roots are

$$X_e = \pm \sqrt{R^2 + X^2} + X \quad (19)$$

Since X^i has the same sign as X_e, the maximum values of X^i occur at

$$X_e = \sqrt{R^2 + X^2} + X \doteq R + X \doteq R \quad (20)$$

The minimum values at

$$X_e = - \sqrt{R^2 + X^2} + X \doteq -R + X \doteq -R \quad (21)$$

Since X is small compared with R, it is possible to neglect X^2 compared with R^2 and even X compared with R depending upon the degree of approximation desired. For most purposes, it is adequate to write $X_e = \pm R$ at the extreme values of X^i. The corresponding magnitudes obtained from (2) with $|X| << |X_e|$ are

$$X^i_{extreme} \doteq \pm \frac{R}{2} \quad \text{at} \quad X_e = \pm R \quad (22)$$

It is possible to obtain the values of $\beta_e b$ giving extreme values of X^i from the equation $X_e \doteq \pm R$ using (17.17) and (17.20).

$$\frac{\beta_e b}{2} \frac{J_0(\beta_e b)}{J_1(\beta_e b)} \left(1 + \frac{J_1^2(\beta_e b)}{J_0^2(\beta_e b)}\right) = \mp R^i_0 \omega C_0 \quad (23)$$

or, more conveniently, from the intersections of the curves for R and $|X_e|$ which are obtained directly from Fig. 17.2 when X_e/X^i_0 and R/R^i_0 are multiplied, respectively, by values of X^i_0 and R_0 appropriate for a particular condenser.

In order to obtain a representative set of curves for the internal impedance of a typical condenser with variously conducting dielectrics, let the following specific case be computed. The frequency is assumed fixed and the impedance is determined as the radius of the condenser is increased from small values.

The conductivity is expressed in the form $\sigma_e = 10^{-n}$ so that a series of curves may be evaluated for conductivities ranging from small to large values.

Data:

$$f = 3.19 \times 10^9 \text{ hertz;}[1] \qquad \omega = 2 \times 10^{10} \text{ radians/second}$$

$$\epsilon_{er} = 2.25; \qquad \sigma_e = 10^{-n}; \qquad \nu_r = 1$$

$$\beta_e \doteq \beta_e = \omega\sqrt{\frac{\epsilon_e}{\nu}} = 10^2 \text{ radians/meter}$$

$$X_0^i = -\frac{1}{\omega C_0} = -\frac{w}{\omega\epsilon_e\pi b^2} = -\frac{1.6}{b^2} \times 10^{-3} \text{ ohm} = -\frac{16}{\beta_e^2 b^2}$$

$$R_0^i = \frac{w}{\sigma_e\pi b^2} = \frac{0.64}{b^2} \times 10^{n-3} \text{ ohm}$$

$$\left|\frac{R_0^i}{X_0^i}\right| = 0.4 \times 10^n; \qquad \left(\frac{X_0^i}{R_0^i}\right)^2 = \left(\frac{\sigma_e}{\omega\epsilon_e}\right)^2 = 6.25 \times 10^{-2n}$$

$$\frac{1}{R_0^i\omega^2 C_0^2} = \frac{4.0}{b^2} \times 10^{-n-3} = \frac{40}{\beta_e^2 b^2} \times 10^{-n}$$

The condition for a slightly conducting dielectric is

$$h_e^2 = \left(\frac{\sigma_e}{\omega\epsilon_e}\right)^2 << 1 \tag{24}$$

Hence n must exceed 1. Accordingly, using (18.9)

$$R^i \doteq 10^{1-n}\left(1 + \frac{J_0^2(\beta_e b)}{J_1^2(\beta_e b)}\right) \tag{25}$$

$$X^i \doteq -\frac{16}{\beta_e^2 b^2}\left(\frac{\beta_e b}{2}\frac{J_0(\beta_e b)}{J_1(\beta_e b)}\right) \tag{26}$$

except near $J_1(\beta_e b) = 0$, or $\beta_e b = 3.83, 7.02$, etc.

The antiresonant value is determined from $R_0^i\omega C_0 = 0.4 \times 10^n$ using Fig. 19.1. For values of n equal to 2 or greater, antiresonance may be assumed to occur with

$$\beta_e b = 3.83, 7.02, \text{ etc.} \tag{27a}$$
$$b = 0.0383 \text{ meter, } 0.0702 \text{ meter, etc.} \tag{27b}$$

The extreme values of R^i at the first two antiresonances are

$$(R_{max}^i)_1 = R_0^i = \frac{0.64}{(0.0383)^2} \times 10^{n-3} = 0.435 \times 10^n \text{ ohms} \tag{28a}$$

$$(R_{max}^i)_2 = R_0^i = \frac{0.64}{(0.0702)^2} \times 10^{n-3} = 0.13 \times 10^n \text{ ohms} \tag{28b}$$

---

[1] 1 hertz = 1 cycle per second, abbreviated hz.

The corresponding extreme values of X^i are

$$(X^i_{\text{ext}})_1 = \pm \tfrac{1}{2}(R^i_{\text{max}})_1 = \pm 0.218 \times 10^n \text{ ohms} \qquad (29a)$$

$$(X^i_{\text{ext}})_2 = \pm \tfrac{1}{2}(R^i_{\text{max}})_2 = \pm 0.065 \times 10^n \text{ ohms} \qquad (29b)$$

The location of the extreme values of X^i may be determined as follows. At and near antiresonance $R \doteq R^i_0$, so that the condition (22) $X_e = \pm R$ is approximately equivalent to $X_e = \pm R^i_0$. Hence $X_e/X^i_0 = \pm R^i_0/X^i_0 = \pm 0.4 \times 10^n$. If this ratio does not exceed 1,000, the values of $\beta_s b$ that maximize and minimize X^i may be read from Fig. 17.3. If the ratio exceeds 1,000, the extreme values occur at negligible distances below and above the value $\beta_s b = 3.8317$.

The internal impedance of a circular condenser with an imperfect dielectric may be studied either as a function of the frequency with the size of the plates fixed, or with the radius of the plates as variable, using a constant frequency. Although the dependence upon frequency and radius is the same in the Bessel function, it is quite different in other factors, particularly in $R^i_0 = w/\sigma_{e2}\pi b^2$ and $X^i_0 = -1/\omega C_0$ with $C_0 = \epsilon_{e2}\pi b^2/w$. Actually, the variation of impedance with radius of plates is very much simpler than the variation with frequency. This is due primarily to the fact that the real effective conductivity σ_{e2} and the real effective dielectric constant ϵ_{e2} of the dielectric medium 2 are, in general, not constants independent of frequency. For a simply polarizing and conducting dielectric with time lags in both polarizing and conducting response, it follows from (III.14.24a,c) that

$$\sigma_e = \sigma' + \omega\epsilon'' \qquad (30)$$

$$\epsilon_e = \epsilon' - \frac{\sigma''}{\omega} \qquad (31)$$

where σ', σ'' and ϵ', ϵ'' are the real and imaginary parts of complex conductivity and dielectric constant.

$$\sigma = \sigma' - j\sigma'' \qquad (32)$$

$$\epsilon = \epsilon' - j\epsilon'' \qquad (33)$$

In an electrolytic dielectric, both σ'' and ϵ'' may be significant; in a good or moderately good solid dielectric σ'' is probably negligible, and the term in ϵ'' may contribute more to σ_e than σ', especially in frequency bands in which molecular or atomic resonance phenomena with large time lags in polarization occur.

In general, it is not correct to assume that σ_e and ϵ_e are independent of frequency and the dependence upon frequency varies with different dielectrics.   It follows that it is not simple or convenient

Fig. 19.2.—Equivalent series resistance of a circular condenser with imperfect dielectric of thickness 2 mm. and with $\epsilon_{er} = 2.25$ as the radius b is varied.   The curve in broken line (scale on right) is the d.c. and low-frequency resistance.

to construct representative general curves of R^i and X^i as functions of frequency.   On the other hand, by assuming fixed values of σ_e and ϵ_e at a given frequency, and therefore a definite

power factor, $h_e = \sigma_e/\omega\epsilon_e$, curves of R^i and X^i are readily constructed as functions of the radius b of the condenser plates, or of $\beta_s b$ with β_s a constant.

FIG. 19.3.—Equivalent series reactance of a circular condenser with imperfect dielectric of thickness 2 mm. and with $\epsilon_{er} = 2.25$ as the radius b is varied at a fixed frequency. The curve in broken line is the reactance calculated, assuming it to be given by the low-frequency formula $-1/\omega C'_0$.

It follows from (18.8b) and (25) that at a fixed frequency the series resistance R^i is given in general by a constant $w\zeta_e^2\sigma_e/4\pi$ multiplied by the factor $\left(1 + \dfrac{J_0^2(\beta_a b)}{J_1^2(\beta_a b)}\right)$ except at and near antiresonance where R^i reaches a noninfinite maximum if the dielectric is not perfect. This maximum is given by (17) or, in the numerical case at hand, by (28). The resistance curve is, therefore, simply Fig. 18.3 with an appropriately adjusted scale determined by the constant $w\zeta_e^2\sigma_e/4\pi = 10^{1-n}$ and maxima given

by (28a,b). The curve is redrawn in Fig. 19.2 for $n = 3$ so that $\sigma_e = 10^{-3}$, a relatively high conductivity for a dielectric. With smaller values of σ_e, e.g., $\sigma_e = 10^{-12}$ with $n = 12$, the curve is unchanged except that it rises to very much greater maxima which may reach several hundred thousand megohms. The broken line shows the direct current resistance R_0^i for comparison.

The curve for the series reactance X^i is unaffected by the conductivity (if this is low) except near antiresonance where it has large extreme values determined by (22). Using $n = 3$ or $\sigma = 10^{-3}$ and the curve X_o/X_0^i in Fig. 17.2, Fig. 19.3 is readily determined. The broken line shows the low-frequency reactance $X_0^i = -1/\omega C_0$ for comparison.

20. Resonant and Antiresonant Amplitudes of Current and Charge in the Condenser.—The amplitudes of the radial distributions of surface charge and of quasi-surface current both depend upon the amplitude factor $n_{1L}(0)$ given in (15.7a). This may be rearranged in the form

$$n_{1L}(0) = -j\omega\left(\frac{\varepsilon_2}{v_2}\right)\left(\frac{I_{z1}}{2\pi}\right)\frac{1}{\beta_2 b J_1(\beta_2 b)} \tag{1}$$

For a given value of I_{z1}, $n_{1L}(0)$ can be varied by changing the radius b of the condenser plates. The dependence upon b is contained in the factor $1/\beta_2 b J_1(\beta_2 b)$. Using (16.12), with b written for r,

$$\beta_2 b J_1(\beta_2 b)$$
$$= \left(1 - j\frac{\alpha_s}{\beta_s}\right)\beta_s b\left\{J_1(\beta_s b) - j\frac{\alpha_s}{\beta_s}[\beta_s b J_0(\beta_s b) - J_1(\beta_s b)]\right\} \tag{2}$$

Neglecting terms in α_s^2

$$\beta_2 b J_1(\beta_2 b) = \beta_s b\{J_1(\beta_s b) - j\alpha_s b J_0(\beta_s b)\} \tag{3}$$

Hence,

$$\left|\frac{1}{\beta_2 b J_1(\beta_2 b)}\right| = \frac{1}{\beta_s b \sqrt{J_1^2(\beta_s b) + \alpha_s^2 b^2 J_0^2(\beta_s b)}} \tag{4}$$

Since α_s is small, the maxima of this amplitude factor occur practically at those values of b for which $J_1(\beta_s b)$ vanishes; i.e., at antiresonance.

The minima occur where $|J_1(\beta_s b)|$ is maximum. The values of $\beta_s b$ which maximize $|J_1(\beta_s b)|$ do not coincide with the roots of

$J_0(\beta_s b)$ which define resonance as can be seen from Fig. 15.1. For small values of $\beta_s b$, the difference is considerable; for large values, the difference is negligible and it may be assumed that the minima of (4) occur at resonance for $\beta_s b \geq 10$.

$$\eta_{1L}(0)_{\max} = \frac{\omega\epsilon_2}{\nu_2}\frac{I_{z1}}{2\pi}\cdot\frac{1}{\alpha_s\beta_s b^2|J_0(\beta_s b)|}; \qquad \begin{cases} \beta_s b \text{ a root of} \\ J_1(\beta_s b) = 0 \text{ or} \\ \beta_s b = 3.83,\, 7.02,\text{ etc.} \end{cases} \quad (5)$$

$$\eta_{1L}(0)_{\min} = \frac{\omega\epsilon_2}{\nu_2}\frac{I_{z1}}{2\pi}\cdot\frac{1}{\beta_s b|J_1(\beta_s b)|}; \qquad \begin{cases} \beta_s b \text{ to make } |J_1(\beta_s b)| \text{ a} \\ \text{maximum, or} \\ \beta_s b = 1.84,\, 5.34,\text{ etc.} \end{cases} \quad (6)$$

But

$$\beta_s \doteq \beta_o = \omega\sqrt{\frac{\epsilon_{e2}}{\nu_2}} \tag{7}$$

so that

$$\eta_{1L}(0)_{\max} = \frac{\epsilon_2}{\sqrt{\nu_2\epsilon_{e2}}}\frac{I_{z1}}{2\pi b}\cdot\frac{1}{\alpha_s b|J_0(\beta_s b)|}; \qquad \beta_s b = 3.83,\, 7.02,\text{ etc.} \quad (8)$$

$$\eta_{1L}(0)_{\min} = \frac{\epsilon_2}{\sqrt{\nu_2\epsilon_{e2}}}\frac{I_z}{2\pi b}\cdot\frac{1}{|J_1(\beta_s b)|}; \qquad \beta_s b = 1.84,\, 5.34,\text{ etc.} \quad (9)$$

If the effective dielectric constant depends primarily on a time lag in polarization and only to a minor extent on a time lag in conduction, $\epsilon_2 \doteq \epsilon_{e2}$. With $v_{e2} = \sqrt{\nu_2/\epsilon_{e2}}$,

$$\eta_{1L}(0)_{\max} = \frac{1}{v_{e2}}\frac{I_{z1}}{2\pi b}\frac{1}{\alpha_s b|J_0(\beta_s b)|}; \qquad \beta_s b = 3.83,\, 7.02,\text{ etc.} \quad (10)$$

$$\eta_{1L}(0)_{\min} = \frac{1}{v_{e2}}\frac{I_{z1}}{2\pi b}\frac{1}{|J_1(\beta_s b)|}; \qquad \beta_s b = 1.84,\, 5.34,\text{ etc.} \quad (11)$$

The values of $|1/J_0(\beta_s b)|$ corresponding to $\beta_s b = 3.83$ and 7.02 are 2.48 and 3.33. The values of $|1/J_1(\beta_s b)|$ for $\beta_s b = 1.84$ and 5.34 are 1.72 and 2.88.

For all values of $\beta_s b$, the actual amplitude of $\eta_{1L}(0)$ depends upon the total current I_{z1}. This is determined by the impedance of the entire circuit to which the condenser is connected. It includes the impedance of the outside surface of the plates of the condenser as well as its internal impedance. The important case in which the external circuit is a closed loop of conductors is considered in the next chapter.

CHAPTER VI

ELECTRIC CIRCUITS

One of the most important applications of electromagnetic theory in the field of electrical communication is to configurations of conductors of small and circular cross section. These include electric circuits in the conventional sense of closed loops formed of variously wound coils of copper or resistance wire, or of quasi-closed loops with gaps in the conducting path limited to condensers with parallel plates separated only very small distances. They also include open circuits particularly in the form of antennas. After formulating the problem of electric circuits in terms as general as possible for configurations involving only conductors of small circular cross section and condensers which have a dielectric that is so thin that edge effects are negligible, the analysis in this chapter will be concerned primarily with closed and quasi-closed circuits that are restricted in size so that all points are in the near zone with respect to each other. The study of open circuits and closed circuits that are not limited in size is the principal subject of Volume II on Antennas. Since this volume is an *introduction* to electromagnetic methods of analysis rather than a treatise on advanced mathematical techniques or a compendium of intricate formulas, attention is focused primarily on the simpler configurations of conductors.

ANALYTICAL FOUNDATIONS

1. General Formulation of Electric-circuit Theory.—The circuits to be analyzed in this chapter are constructed primarily of conductors of circular cross section with radius a that satisfies the inequality

$$\beta_0 a << 1 \tag{1}$$

They may also include parallel-plate condensers with circular plates of radius b separated a small distance w. The following

inequalities are assumed to be good approximations:

$$w <<< b; \qquad \beta_0 w <<< 1 \tag{2}$$

Since a.c. generators, and in particular high-frequency oscillators, are themselves intricate configurations of conductors that cannot be analyzed without first studying the properties of simple circuits, it is convenient to introduce highly simplified and idealized generators at the outset. After the analysis of simple circuits has been formulated, it is possible to apply the methods to a study of more intricate configurations of conductors including oscillators. The simplified generator to be used in the analysis consists of a section of conductor of negligible length between the ends of which a charge-separating field E^e is maintained by forces that are independent of the distribution of current and charge in the network. It will be called an idealized slice or point generator to distinguish it from more or less equivalent circuits of actual generators that may include coils, condensers, and resistors. At high frequencies, when currents are primarily in thin surface layers, the part of the cross section of the conductor in which E^e is active is primarily the outer ring of the section. In this case the generator is an idealized ring generator.

The conductors forming the circuits to be studied are permitted to have any moderately smooth and continuous configuration. It is reasonable to assume that, just as in the case of a straight conductor, the distribution of current and charge in any cross section is independent of the distribution in the axial direction except in extreme cases where two or more conductors are so close together that the cross-sectional distribution departs considerably from rotational symmetry. Even in such cases, it is usually a good approximation to assume that the axial distribution is independent of the cross-sectional distribution and determine the latter as a two-dimensional problem for each cross-sectional plane. If such planes are alike, as in a transmission line consisting of two or more parallel, very closely spaced conductors, a single determination of the transverse distribution in any typical cross-sectional plane suffices. Unless otherwise specified, it is assumed in the following that the internal impedance per unit length z^i as computed for an infinitely long conductor is a good approximation for each unit length of wires that are far apart compared with their radius according to

$a^2 << b^2$ where a is the radius of the wire and b the nearest distance between centers of conductors in a circuit.

The conductors in the circuits to be studied may turn and twist in any way. Let an arbitrarily chosen direction *along the central axis* of the conductor be designated by a unit vector $\mathfrak{s}$. This vector changes its direction from point to point along the conductor if this is not straight. The component of the electric field in the direction of $\mathfrak{s}$ at each point along the surface is

$$(\mathfrak{s}, E) = E_{\mathfrak{s}} \tag{3}$$

The total axial current at any cross section is

$$I_{\mathfrak{s}} = \int_{\mathfrak{s}} i_{\mathfrak{s}} ds \tag{4}$$

This current is related to the axial tangential electric field at the surface according to

$$E_{\mathfrak{s}} = z^i I_{\mathfrak{s}} \tag{5a}$$

where z^i is the internal impedance per unit length of the conductor (Chapter V). This relation is an integrated form of

$$E_{\mathfrak{s}} = \frac{i_{\mathfrak{s}}}{\mathfrak{d}} \tag{5b}$$

where $i_{\mathfrak{s}}$ is the density of conduction current and $\mathfrak{d}$ is the conductivity. In metallic conductors $\mathfrak{d} = \sigma$. In regions in which a charge-separating agency is active, in particular in the idealized slice generators, an axially tangential impressed field $E_{\mathfrak{s}}^{e}$ is maintained between the boundaries of the region at A and B. In this very short part of the circuit, and assuming the conductivity real,

$$i_{\mathfrak{s}} = \sigma(E_{\mathfrak{s}} + E_{\mathfrak{s}}^{e}) \tag{6a}$$

so that

$$E_{\mathfrak{s}} = \frac{i_{\mathfrak{s}}}{\sigma} - E_{\mathfrak{s}}^{e} \tag{6b}$$

This may be written in the form

$$E_{\mathfrak{s}} = z^i I_{\mathfrak{s}} - E_{\mathfrak{s}}^{e} \tag{7}$$

where z^i is the effective *internal* impedance per unit length of the charge-separating region, which is assumed to have the conducting properties of a simple conductor, and $I_{\mathfrak{s}}$ is the total current through it. If there is a condenser in the circuit which

satisfies (2), the axial electric field at its outer edge is given by an expression exactly like (5a) with I_s the total axial current entering or leaving the adjacent surface layers, and z^i the internal impedance per unit thickness of the dielectric in the condenser. The plates themselves are assumed to be perfectly conducting.

The electric field at all points in space outside the conductors and condensers is given by

$$\mathsf{E} = -\text{grad } \phi - j\omega \mathsf{A} = -\frac{j\omega}{\beta_0^2} (\text{grad div } \mathsf{A} + \beta_0^2 \mathsf{A}) \quad (8)$$

The component of this field axially tangent to the surface of the conductors or the surface of the condenser at its edge is

$$E_s = -\frac{\partial \phi}{\partial s} - j\omega A_s = \frac{-j\omega}{\beta_0^2} \left(\frac{\partial}{\partial s} \text{div } \mathsf{A} + \beta_0^2 A_s \right) \quad (9)$$

Since the tangential component of the electric field at a boundary is continuous, E_s as given by (5a) must be equal to E_s as given by (9) at the surface of a conductor or condenser. That is,

$$(E_s)_{\text{inner}} = (E_s)_{\text{outer}} \quad (10)$$

Hence, using (7) and (9) in (10),

$$-E_s^e + z^i I_s = -\frac{\partial \phi}{\partial s} - j\omega A_s = \frac{-j\omega}{\beta_0^2} \left(\frac{\partial}{\partial s} \text{div } \mathsf{A} + \beta_0^2 A_s \right) \quad (11)$$

Along those parts of the circuit between terminals A and B where the charge-separating agency is assumed to be localized, E_s^e differs from zero; elsewhere it is zero.

For simplicity let it be assumed that

$$\text{curl } \mathsf{E}^e = 0 \quad (12)$$

so that E^e may be derived from a scalar potential ϕ^e. Thus:

$$\mathsf{E}^e = \text{grad } \phi^e \quad (13)$$

$$E_s^e = \frac{\partial \phi^e}{\partial s} \quad (14)$$

The positive gradient is used since E^e points from lower to higher potentials. This is equivalent to assuming that a generator may be described mathematically by a discontinuity in scalar potential.

If (11) is integrated along the conductors of a circuit from one end at s_1 to the other end at s_2, the following expression is obtained. An idealized generator is assumed to exist between A

and B somewhere along the conductor.

$$\int_A^B d\phi^e = \int_{s_1}^{s_2} d\phi + j\omega \int_{s_1}^{s_2} A_s ds + \int_{s_1}^{s_2} z^i I_s ds \qquad (15)$$

Let the *driving potential difference* or e.m.f. maintained by the generator between the terminals A and B be defined by

$$V_{BA}^e \equiv \phi_B^e - \phi_A^e = \int_A^B d\phi^e \qquad (16)$$

Then

$$V_{BA}^e = \int_{s_1}^{s_2} d\phi + j\omega \int_{s_1}^{s_2} A_s ds + \int_{s_1}^{s_2} z^i I_s ds \qquad (17)$$

This is the fundamental integral equation for an *open* circuit containing a point generator. If the circuit is closed so that s_2 and s_1 coincide, the first integral in (17) vanishes and

$$V_{BA}^e = \oint z^i I_s ds + j\omega \oint (\mathsf{A}, ds) \qquad (18)$$

This is the fundamental integral equation for a *closed* circuit containing a generator between A and B.

2. General Equations for Two Coupled Circuits.—Let two coupled circuits be analyzed using the general equation (1.18) applied to the schematic diagram of Fig. 2.1. The primary

Fig. 2.1.—Diagram of two coupled circuits. A driving potential difference $V_{BA}^e = V_{10}^e$ is maintained between A and B in circuit (1); CD in circuit (2) is a load resistance.

circuit consists of a coil and a condenser; the secondary of a coil and a resistance. Only a few turns of the coils are shown. Since the vector potential in the primary circuit is due to the currents in the primary and the currents in the secondary, (1.18) with V_{10}^e written for V_{BA}^e becomes

$$V_{10}^e = \oint_{s_1} z^i I_{1s} ds_1 + j\omega \oint_{s_1} \{(\mathsf{A}_{11}, ds_1) + (\mathsf{A}_{12}, ds_1)\} \qquad (1)$$

Here A_{11} is the vector potential at the surface of the primary

circuit due to the current I_1; A_{12} is the vector potential at the surface of the primary circuit due to the current I_2. The corresponding equation for the secondary circuit is

$$0 = \oint_{s_2} z_2^i I_{2s} ds_2 + j\omega \oint_{s_2} \{(A_{22}, ds_2) + (A_{21}, ds_2)\} \qquad (2)$$

The contribution to the vector potential at any point in space due to the current I_s in an element ds' of a conductor is

$$dA = \frac{1}{4\pi\nu_0} \frac{I_s'}{R} e^{-j\beta_0 R} ds' \qquad (3)$$

The vector potential at the surface element ds_1 in the primary circuit (Fig. 2.1) due to the current in the primary is given approximately by

$$A_{11} = \frac{1}{4\pi\nu_0} \oint_{s_1} \frac{I_{1s}'}{R_{11}} e^{-j\beta_0 R_{11}} ds_1' \qquad (4)$$

The element ds_1' is at the axis of the circular conductor. The formula (4) was originally written (Sec. IV.3) for a straight conductor, but there is nothing in the analysis that restricts it in this way. It is a good approximation even on the surface except within distances of sharp bends comparable with the radius of the wire. Corresponding to (4),

$$A_{12} = \frac{1}{4\pi\nu_0} \oint_{s_2} \frac{I_{2s}'}{R_{12}} e^{-j\beta_0 R_{12}} ds_2' \qquad (5)$$

$$A_{22} = \frac{1}{4\pi\nu_0} \oint_{s_2} \frac{I_{3s}'}{R_{22}} e^{-j\beta_0 R_{22}} ds_2' \qquad (6)$$

$$A_{21} = \frac{1}{4\pi\nu_0} \oint_{s_1} \frac{I_{1s}'}{R_{21}} e^{-j\beta_0 R_{21}} ds_1' \qquad (7)$$

If (4) to (6) are substituted in (1) and (2), two simultaneous integral equations are obtained for determining the currents I_{1s} and I_{2s} as functions of the parameters and the configuration of the circuit. They are

$$V_{10}^e = \oint_{s_1} z_1^i I_{1s} ds_1 + \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \left(ds_1, \oint_{s_1} \frac{I_{1s}'}{R_{11}} e^{-j\beta_0 R_{11}} ds_1' \right)$$

$$+ \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \left(ds_1, \oint_{s_2} \frac{I_{2s}'}{R_{12}} e^{-j\beta_0 R_{12}} ds_2' \right) \qquad (8)$$

$$0 = \oint_{s_2} z_2^i I_{2s} ds_2 + \frac{j\omega}{4\pi\nu_0} \oint_{s_2} \left(ds_2, \oint_{s_2} \frac{I_{2s}'}{R_{22}} e^{-j\beta_0 R_{22}} ds_2' \right)$$

$$+ \frac{j\omega}{4\pi\nu_0} \oint_{s_2} \left(ds_2, \oint_{s_1} \frac{I_{1s}'}{R_{21}} e^{-j\beta_0 R_{21}} ds_1' \right) \qquad (9)$$

In general, it is not possible to solve these equations and so obtain explicit formulas for the distributions of current.

3. Self- and Mutual Impedance.—By introducing dimensionless distribution functions $f_1(s_1)$ and $f_2(s_2)$ for the currents in the general equations (2.8) and (2.9) for two coupled circuits, these can be reduced to standard form. In order to do this, it is first necessary to select convenient reference currents at points in each circuit to be designated by the zero value of the contour variable s. It is usually advantageous to choose these reference points at convenient points such as the center $s_1 = 0$ of the generator in the primary and at the center of the load $s_2 = 0$ in the secondary. The complex amplitudes of the currents at these points are respectively, I_{10} and I_{20}. The currents at other points in the circuit are given by

$$I_{1s} = I_{10}f_1(s_1) \tag{1}$$
$$I_{2s} = I_{20}f_2(s_2) \tag{2}$$

where $f_1(s_1)$ and $f_2(s_2)$ are dimensionless complex distribution functions.

Using (1) and (2) in (2.8) and (2.9), these may be written in the following standard form for two simultaneous equations:

$$V_{10}^s = I_{10}Z_{11} + I_{20}Z_{12} \tag{3}$$
$$0 = I_{10}Z_{21} + I_{20}Z_{22} \tag{4}$$

with

$$Z_{11} = Z_1^i + Z_1^e \tag{5}$$
$$Z_{22} = Z_2^i + Z_2^e \tag{6}$$

and

$$Z_1^i \equiv \oint_{s_1} z_1^i f_1(s_1) ds_1 \tag{7}$$

$$Z_2^i \equiv \oint_{s_2} z_2^i f_2(s_2) ds_2 \tag{8}$$

$$Z_1^e \equiv \frac{j\omega}{I_{10}} \oint_{s_1} (\mathbf{A}_{11}, ds_1) = \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \left(ds_1, \oint_{s_1} \frac{f_1(s_1')}{R_{11}} e^{-j\beta_0 R_{11}} ds_1' \right) \tag{9}$$

$$Z_2^e \equiv \frac{j\omega}{I_{20}} \oint_{s_2} (\mathbf{A}_{22}, ds_2) = \frac{j\omega}{4\pi\nu_0} \oint_{s_2} \left(ds_2, \oint_{s_2} \frac{f_2(s_2')}{R_{22}} e^{-j\beta_0 R_{22}} ds_2' \right) \tag{10}$$

$$Z_{12} \equiv \frac{j\omega}{I_{10}} \oint_{s_1} (\mathbf{A}_{12}, ds_1) = \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \left(ds_1, \oint_{s_2} \frac{f_2(s_2')}{R_2} e^{-j\beta_0 R_{12}} ds_2' \right) \tag{11}$$

$$Z_{21} \equiv \frac{j\omega}{I_{20}} \oint_{s_2} (\mathbf{A}_{21}, ds_2) = \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \left(ds_2, \oint_{s_1} \frac{f_1(s_1')}{R_{21}} e^{-j\beta_0 R_{21}} ds_1' \right) \tag{12}$$

The complex coefficient Z_{11} is called the *self-impedance* of the primary referred to I_{10}; Z_{22} is the self-impedance of the second-

ary referred to I_{20}; Z_{12} is the *mutual impedance* of the primary referred to I_{10}; Z_{21} is the mutual impedance of the secondary referred to I_{20}.   It is important to bear in mind that each impedance is changed if another reference current is selected, so that the impedances are functions *not only of the geometrical configuration of the circuit* (which determines the distribution of current) *but also of the arbitrarily selected reference current*.   The part Z^i of the self-impedance is called the *internal* impedance; the part Z^e of the self-impedance the *external* self-impedance. The former depends primarily on the internal impedance per unit length of the contour of the conductor, the latter entirely on the interaction of currents in various parts of what is recognized to be the same circuit.   Both self- and mutual impedances are complex.   The real part is the self- or mutual resistance, the imaginary part the self- or mutual reactance.   The notation

$$Z_{11} = R_{11} + jX_{11} \qquad (13)$$
$$Z_{12} = R_{12} + jX_{12} \qquad (14)$$

is used.[1]   All impedances are measured in ohms as can be seen directly from the fact that the vector potential is measured in volt-seconds per meter.   Multiplication by a frequency and a length leaves volts; and volts over amperes are ohms.   The internal resistance R_1^i is often called the *ohmic* resistance of the circuit; the external resistance R_1^e is also called the *self-radiation* resistance.

Since the distribution of current I_{1s} as determined from (2.8) and (2.9) is actually a function of both z_1^i and the configuration of the circuit, it is not possible to separate Z_1^i and Z_1^e except in special cases.   Therefore ohmic resistance and radiation resistance cannot be determined as two additive components in general.

4. Driving-point Impedance, Coefficient of Coupling, and Induced Voltage.—The simultaneous equations (3.3) and (3.4) are readily solved for I_{10} and I_{20}.   Thus,

$$I_{10} = \frac{V_{10}^i Z_{22}}{Z_{11}Z_{22} - Z_{12}Z_{21}} = \frac{V_{10}^i}{Z_{11} - \dfrac{Z_{12}Z_{21}}{Z_{22}}} \qquad (1)$$

$$I_{20} = -\frac{V_{10}^i Z_{21}}{Z_{11}Z_{22} - Z_{12}Z_{21}} \qquad (2)$$

[1] Note that resistances are in roman type, R_{11}, distances in italic type, R_{11}.

The driving-point impedance is defined to be the ratio of the driving potential difference V_{10}^2 across the terminals AB to the current entering one and leaving the other terminal. That is,

$$Z_{AB} = \frac{V_{10}^2}{I_A} = \frac{V_{10}^2}{I_B} \qquad (3)$$

This definition implies that

$$I_A = I_B \qquad (4)$$

and thereby imposes a condition on the part of the circuit between A and B. Assuming (4) true, (3) with (1) gives

$$Z_{AB} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_{22}} = Z_{11}\left(1 - \frac{Z_{12}Z_{21}}{Z_{11}Z_{22}}\right) \qquad (5)$$

For some purposes it is convenient to define the complex coefficient of coupling between the circuits to be

$$k_{12}^2 \equiv \frac{Z_{12}Z_{21}}{Z_{11}Z_{22}} \qquad (6)$$

Loose coupling exists when

$$k_{12}^2 << 1 \qquad (7)$$

so that

$$Z_{AB} \doteq Z_{11} \qquad (8)$$

It is well to note that Z_{11} in (8) and Z_{11} in (5) are not in general the same because $f_1(s_1)$ in (3.7) and $f_1(s_1')$ in (3.9) are altered if the relative orientation or the distance separating the two coupled circuits is changed. The self-impedance of the primary circuit is not only a function of the primary alone but also of the secondary insofar as this helps to determine the distribution of current in the primary.

In order to clarify this fundamentally important fact, consider the following two cases. First let the secondary be removed completely so that A_{12} and with it Z_{12} vanishes. Regardless of the choice of reference points, the second term on the right in (3.3), *i.e.*, in

$$V_{10}^2 = I_{10}Z_{11} + I_{20}Z_{12} \qquad (9)$$

vanishes leaving

$$V_{10}^2 = I_{10}Z_{11} \qquad (10)$$

with Z_{11} defined by (3.9) in which the distribution of function

$f_1(s_1')$ is determined entirely by interactions between currents in the several parts of the primary circuit alone. Now let the secondary be brought close to the primary, but let the circuit be broken at $s_2 = 0$ so that I_{20} vanishes. The second term in (9) is again zero, but this time because I_{20} vanishes and not Z_{12}. Since a zero current at $s_2 = 0$ does not in general mean that the current I_{2s} must be zero or even small anywhere else, it is clear that the distribution function $f_1(s_1')$ is determined not only by interactions between currents in different parts of the primary, but by interactions of currents in the primary with currents in the secondary. It cannot, therefore, be the same as with the secondary absent, so that Z_{11} is different in the two cases. It is clear, therefore, that *the self-impedance of a circuit as defined in* (3.9) *is not simply a constant of the geometry of a circuit, but a function of the proximity and distribution of currents in other circuits.*

The following term in (3.3), *viz.*,

$$V_{12}^i \equiv -I_{20}Z_{12} \qquad (11)$$

is called the voltage induced in the primary by the current in the secondary. The corresponding term in (3.4)

$$V_{21}^i \equiv -I_{10}Z_{21} \qquad (12)$$

is called the voltage induced in the secondary by the current in the primary. Using this notation, (3.3) and (3.4) become

$$V_{10}^i + V_{12}^i = I_{10}Z_{11} \qquad (13)$$
$$V_{21}^i = I_{20}Z_{22} \qquad (14)$$

In the important case of loose coupling as defined by (7),

$$V_{10}^i >> V_{12}^i \qquad (15)$$

so that

$$V_{10}^i = I_{10}Z_{11} \qquad (16)$$
$$V_{21}^i = I_{20}Z_{22} \qquad (17)$$

5. Generalization to n Circuits; Reciprocity in Mutual Imped-ance.—The analysis carried out in detail for two circuits with only one driven is readily extended to n circuits with any number driven. In this case, there are n simultaneous integral equations like (3.8) and (3.9) with n integrals in each in addition to one involving z^i. The left side is zero if a circuit includes

no generator. If distribution functions and reference currents are introduced for the currents, the equations corresponding to (3.3) and (3.4) are

$$
\left.
\begin{aligned}
V_{10}^e &= \sum_{j=1}^{n} I_{j0} Z_{1j} \\
V_{20}^e &= \sum_{j=1}^{n} I_{j0} Z_{2j} \\
&\cdots\cdots\cdots\cdots \\
V_{n0}^e &= \sum_{j=1}^{n} I_{j0} Z_{nj}
\end{aligned}
\right\}
\tag{1}
$$

The coefficients Z_{kj} in which $k = j$ are self-impedances defined by expressions like (3.5) with (3.7) and (3.9); the coefficients Z_{kj} with $k \neq j$ are mutual impedances defined by expressions like (3.11).

Despite the generality of the problem and the fact that explicit solutions for the distributions of current and the impedances cannot be determined, it is possible to prove that subject only to the conditions imposed in the formulation of the reciprocal theorem (Sec. IV.18)

$$
Z_{kj} = Z_{jk} \tag{2}
$$

for any configuration of circuits. This is accomplished as follows. If a set of driving potential differences $V_{10}'' \cdots V_{n0}''$ is maintained at n terminals, the circuit equations (1) are

$$
V_{10}'' = \sum_{j=1}^{n} I_{j0}' Z_{1j}; \qquad V_{20}'' = \sum_{j=1}^{n} I_{j0}' Z_{2j}; \qquad V_{i0}'' = \sum_{j=1}^{n} I_{j0}' Z_{ij};
$$

$$
V_{n0}'' = \sum_{j=1}^{n} I_{j0}' Z_{nj} \tag{3}
$$

If the driving potential differences are changed to $V_{10}'' \cdots V_{n0}''$, the equations are

$$
V_{10}'' = \sum_{j=1}^{n} I_{j0}'' Z_{1j}; \qquad V_{20}'' = \sum_{j=1}^{n} I_{j0}'' Z_{2j}; \qquad V_{i0}'' = \sum_{j=1}^{n} I_{j0}'' Z_{ij};
$$

$$
V_{n0}'' = \sum_{j=1}^{n} I_{j0}'' Z_{nj} \tag{4}
$$

The reciprocal theorem requires that

$$
\sum_{i=1}^{n} I_{i0}' V_{i0}'' = \sum_{i=1}^{n} I_{i0}'' V_{i0}' \tag{5}
$$

Upon substituting for V_{i0}' and V_{i0}'' from (3) and (4) in (5), this gives

$$\sum_{i=1}^{n} I_{i0}' \sum_{j=1}^{n} I_{j0}'' Z_{ij} = \sum_{i=1}^{n} I_{i0}'' \sum_{j=1}^{n} I_{j0}' Z_{ij} \tag{6}$$

Since the order of summation is immaterial, (6) is equivalent to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (I_{i0}' I_{j0}'' - I_{i0}'' I_{j0}') Z_{ij} = 0 \tag{7}$$

A change in notation does not alter the validity of (7). Hence, interchanging i and j,

$$\sum_{j=1}^{n} \sum_{i=1}^{n} (I_{j0}' I_{i0}'' - I_{j0}'' I_{i0}') Z_{ji} = 0 \tag{8}$$

Adding (7) and (8)

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \{(I_{i0}' I_{j0}'' - I_{i0}'' I_{j0}')(Z_{ij} - Z_{ji})\} = 0 \tag{9}$$

In general, this sum vanishes only if the expression in the braces is zero. This is always identically true if $i = j$. It is true when $i \neq j$ only if

$$Z_{ij} = Z_{ji} \tag{10}$$

It follows that irrespective of circuit configuration or choice of reference currents (provided these are unchanged once they have been chosen), the mutual impedance in circuit i due to circuit j is the same as the mutual impedance in circuit j due to circuit i. In particular, the distributions of current in the two circuits must be such that the general integrals (3.11) and (3.12) are equal. That is, for circuits i and j,

$$\oint_{s_i} \left(ds_i, \oint_{s_j} \frac{f_j(s_j')}{R_{ij}} e^{-j\beta_0 R_{ij}} ds_j' \right) = \oint_{s_j} \left(ds_j, \oint_{s_i} \frac{f_i(s_i')}{R_{ji}} e^{-j\beta_0 R_{ji}} ds_i' \right) \tag{11}$$

6. Circuit Elements in Unrestricted Circuits.—The self-impedances Z_{jj} and mutual impedances Z_{ij} defined in Sec. 3 for two coupled circuits or in Sec. 5 for n circuits are characteristics of a network as a whole. Since all currents in all parts of a network exert forces on each other, and the distribution of current is thus a function of the entire configuration, impedances

cannot be defined strictly and in general except for a complete network. Even apart from this fact, integrals for the self-impedance all have the form (3.9) which involves integration twice around the circuit, once along the axis using primed variables and once along the surface using unprimed variables. Similarly, integrals of the form (3.11) for mutual impedance require a complete integration around each of the two circuits involved. In special cases, the contributions to the contour integrals from different parts of a circuit may differ greatly in order of magnitude. In particular,[*] the contributions to Z_{11} due to integration along the wires connecting coils, condensers, etc., may be quite small compared, for example, with that part of Z_1^s which is evaluated from the double integration around the turns of a coil, or compared with the part Z_1^s obtained in the single integration along the edges of a condenser, or along a piece of resistance wire. In such cases, it is possible to neglect all contributions to the contour integral except those around coils, along the edges of condensers, or along resistance wires. The impedance Z_{11} of the circuit then reduces *approximately* to a sum of integrals between pairs of points around a circuit. It might then appear possible to call each of these individual integrals the impedance between the terminals of the particular part of the circuit, coil, condenser, or resistor along which the integral is evaluated. But this is not possible in any general sense because the distribution function $f_1(s_1')$ depends upon the configuration of the circuit as a whole, the location of the generator, and the reference point. In circuits that are unrestricted in size and shape, it is generally not possible to treat parts of a circuit as separate and independent elements with impedance characteristics determined entirely by their respective structures and individual configurations. For example, if the external self-impedance of a circuit element between points A and B were defined by

$$(Z_{11}^s)_{AB} = \frac{j\omega}{4\pi\nu_0} \int_A^B \left(ds_1, \int_A^B \frac{f_1(s_1')}{R_{11}} e^{-j\beta_0 R_{11}} ds_1' \right) \qquad (1)$$

the impedance so defined would be characteristic not only of the properties of the particular element but of the entire circuit. If the element in question is left unchanged in structure and location, but the rest of the circuit is modified so that the dis-

tribution function $f_1(s_1')$ is changed, the impedance $(Z_{11}^i)_{AB}$ is necessarily different. Similarly, if an element AB is interchanged in location with another, quite differently constructed element, CD, the distribution function for the circuit generally is not the same in the two cases, and hence the impedance of each element depends on its position in the circuit. Under these conditions, it is merely inviting confusion to attempt to define the impedance of a circuit element. A circuit can be analyzed only as a whole, not as a combination of independent parts. Special cases that are exceptions to this general rule are discussed in later sections.

An interesting and important application of the general expressions for the self-impedance of a circuit of any shape is found in the substitution method for the measurement of dielectric constants of liquids.[1] This method depends upon determining the resonant frequency of a coil or loop of wire either alone or in series with a small parallel-plate condenser, first with the circuit in air, and again when it is immersed in a sufficiently large volume of liquid. The coil in question serves as a loosely coupled secondary for a primary circuit that includes a generator of variable frequency. If resonance in the secondary is defined by

$$X_{22} = 0 \qquad (2)$$

with X_{22} referred to a convenient reference current, the following equations derived using (3.6), (3.8), and (3.10) to form (2) are true: With the coil in air and resonant at $\omega = 2\pi f$

$$\oint_{s_2} x_2^i f_2(s_2) ds_2$$
$$+ \frac{\omega}{4\pi\nu_0} \text{R.P.} \oint_{s_2} \left(ds_2, \oint_{s_2} \frac{f_2(\beta_0, s_2')}{R_{22}} e^{-j\beta_0 R_{22}} ds_2' \right) = 0 \quad (3)$$

With the coil in a dielectric and resonant at $\omega_d = 2\pi f_d$

$$\oint_{s_2} x_{2d}^i f_2(s_2) ds_2 + \frac{\omega_d}{4\pi\nu} \text{R.P.} \oint_{s_2} \left(ds_2, \oint_{s_2} \frac{f_2(\beta_d, s_2')}{R_{22}} \exp\left(-j\beta_d R_{22}\right) ds_2' \right)$$
$$= 0 \quad (4)$$

R.P. stands for the real part of the integral following it. These two equations do not lead to a simple relation between ω and ω_d in general. If the coil is a good conductor and the

[1] J. WYMAN, *J. Am. Chem. Soc.*, **53**, 3297 (1931).

frequency is high, x_2^i due to the wire of the coil and its connections is very small and the first integral in (3) and in (4) contributes significantly only if there is a condenser in the circuit. If a condenser is used, let it be sufficiently small so that the radius b of its plates satisfies the condition $b \leqq 0.2\lambda$ so that (V.18.12) applies. Let it be required further that the liquid in which the circuit including the condenser is immersed is a simple medium and such a good dielectric that the attenuation constant α_s (in $\beta_d = \beta_s - j\alpha_s$) and the equivalent series resistance R_0^i of the condenser are both negligible. In this case,

$$\beta_d \doteq \beta_s = \omega_d \sqrt{\frac{\epsilon_e}{\nu}} = \omega_d \sqrt{\frac{\epsilon_{er}\epsilon_0}{\nu_r\nu_0}} \tag{5}$$

$$Z^i = j\left(\omega L_0^i - \frac{1}{\omega C_0}\right) \text{ for the condenser.} \tag{6}$$

The two equations (3) and (4) reduce to the following:

$$\left(\omega L_0^i - \frac{1}{\omega C_0}\right) + \frac{\omega}{4\pi\nu_0} \text{ R.P. } \oint_{s_1}\left(ds_2, \oint_{s_1}\frac{f_2(\beta_0, s_2')}{R_{22}} e^{-j\beta_0 R_{22}}ds_2'\right) = 0 \tag{7}$$

$$\left(\omega_d L_{0d}^i - \frac{1}{\omega_d C_{0d}}\right) + \frac{\omega_d}{4\pi\nu_0\nu_r} \text{ R.P. } \oint_{s_1}\left(ds_2, \oint_{s_1}\frac{f_2(\beta_s, s_2')}{R_{22}} e^{-j\beta_s R_{22}}ds_2'\right) = 0 \tag{8}$$

Using (V.18.14) and (V.18.15),

$$L_{0d}^i = \frac{L_0^i}{\nu_r} = \mu_r L_0^i \tag{9}$$

$$C_{0d} = \epsilon_{er} C_0 \tag{10}$$

where ν_r ($= 1/\mu_r$) and ϵ_{er} are, respectively, the relative reluctivity (assumed to be real) and the effective relative dielectric constant of the liquid. Accordingly, (7) and (8) may be written as follows:
In air

$$L_0^i - \frac{1}{\omega^2 C_0} + \frac{1}{4\pi\nu_0} \text{ R.P. } \oint_{s_1}\left(ds_2, \oint_{s_1}\frac{f_2(\beta_0, s_2')}{R_{22}} e^{-j\beta_0 R_{22}}ds_2'\right) = 0 \tag{11}$$

In the liquid

$$L_0^i - \frac{\nu_r}{\omega_d^2\epsilon_{er} C_0} + \frac{1}{4\pi\nu_0} \text{ R.P. } \oint_{s_1}\left(ds_2, \oint_{s_1}\frac{f_2(\beta_s, s_2')}{R_{22}} e^{-j\beta_s R_{22}}ds_2'\right) = 0 \tag{12}$$

Clearly (12) reduces identically to (11) if

$$\beta_s = \beta_0 \tag{13}$$

or with (5) if

$$\omega_d \sqrt{\frac{\epsilon_{er}\epsilon_0}{\nu_r\nu_0}} = \omega \sqrt{\frac{\epsilon_0}{\nu_0}} \tag{14}$$

Therefore, if (11) is a true equation at $\omega = 2\pi f$, (12) must be a true equation at $\omega_d = 2\pi f_d$ which satisfies

$$\frac{\omega}{\omega_d} = \frac{f}{f_d} = \sqrt{\frac{\epsilon_{er}}{\nu_r}} = \sqrt{\epsilon_{er}\mu_r} \tag{15}$$

In most dielectrics $\nu_r = 1$, so that the square root of the relative effective dielectric constant is given by

$$N_s = \sqrt{\epsilon_{er}} = \frac{f}{f_d} \tag{16}$$

That is, the index of refraction $N_s = \sqrt{\epsilon_{er}}$ is given by the ratio of the resonant frequency with the circuit in air to the resonant frequency with the circuit completely immersed in a sufficiently large volume of a dielectric liquid. It is to be noted that the contribution to the contour integral in the integration along the outside surfaces of the condenser plates has been neglected. This is justified in Sec. 17. The assumption has been made that the volume of liquid is sufficiently great so that it is approximately equivalent to an infinite medium. The required volume is readily determined experimentally.

It is to be noted that (15) is also true if only the coil or if only the condenser is immersed in the liquid and contributions by connecting wires are neglected. In either case, (11) may be subtracted from (12) leaving the difference between the real parts of the two integrals if the coil alone is immersed, and the equation, $\left(\dfrac{1}{\omega^2} - \dfrac{\nu_r}{\omega_d^2\epsilon_{er}}\right) = 0$ if only the condenser is immersed. In both cases the resulting equation is satisfied if (13) or its equivalent (14) is true. Since the contribution to the contour integral from the outside surface of the plates of the condenser is negligible if these are sufficiently close together, it is sufficient to insert the dielectric only between the plates. In this case, it may be either a liquid or a solid disk.

NEAR-ZONE AND QUASI-NEAR-ZONE CIRCUITS

7. Conventional or Near-zone Electric Circuits.—The complexity of the general formulation of the problem of electric circuits is due primarily to the fact that the distribution of current is a function of the configuration. If the circuits are confined to the near zone and, in addition, are so restricted that all distribution functions such as $f_1(s_1)$ may be set equal to unity, great simplification is achieved. These conditions may be formulated as follows for two circuits:

$$f_1(s_1) \doteq f_2(s_2) \doteq 1 \tag{1}$$

$$\beta_0 R_{11} << 1; \qquad \beta_0 R_{22} << 1;$$

$$\beta_0 R_{12} << 1; \qquad \beta_0 R_{21} << 1 \tag{2}$$

If these conditions are satisfied, it is possible to write

$$f_i(s_i)e^{-j\beta_0 R_{ij}} \doteq 1; \qquad i = 1, 2; \qquad j = 1, 2, \tag{3}$$

in each integral in (3.7) to (3.12). These then become

$$Z_1^i = \oint_{s_1} z_1^i ds_1; \qquad Z_2^i = \oint_{s_2} z_2^i ds_2 \tag{4}$$

$$Z_1^e = \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \oint_{s_1} \frac{(ds_1, ds_1')}{R_{11}} = jX_1^e; \qquad R_1^e = 0 \tag{5}$$

$$Z_2^e = \frac{j\omega}{4\pi\nu_0} \oint_{s_2} \oint_{s_1} \frac{(ds_2, ds_2')}{R_{22}} = jX_2^e; \qquad R_2^e = 0 \tag{6}$$

$$Z_{12} = \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \oint_{s_2} \frac{(ds_1, ds_2')}{R_{12}} = jX_{12}; \qquad R_{12} = 0 \tag{7}$$

$$Z_{21} = \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \oint_{s_2} \frac{(ds_2, ds_1')}{R_{21}} = jX_{21}; \qquad R_{21} = 0 \tag{8}$$

All the integrals in (5) to (8) are real, so that for two coupled circuits with no parts common to both Z_i^e and Z_{ij} are pure imaginaries, as shown. Furthermore, the integrals themselves are independent of the frequency and are functions of the geometry of the circuit alone. It is possible to write

$$X_1^e = \omega L_1^e; \qquad L_1^e = \frac{1}{4\pi\nu_0} \oint_{s_1} \oint_{s_1} \frac{(ds_1, ds_1')}{R_{11}} = \frac{1}{I_1} \oint_{s_1} (A_{11}, ds_1)$$

$$\tag{9}$$

$$X_2^e = \omega L_2^e; \qquad L_2^e = \frac{1}{4\pi\nu_0} \oint_{s_2} \oint_{s_2} \frac{(ds_2, ds_2')}{R_{22}} = \frac{1}{I_2} \oint_{s_2} (A_{22}, ds_2)$$

$$\tag{10}$$

$$X_{12} = \omega L_{12}; \qquad L_{12} = \frac{1}{4\pi\nu_0} \oint_{s_1} \oint_{s_2} \frac{(\mathrm{ds}_1, \mathrm{ds}_2')}{R_{12}} = \frac{1}{I_2} \oint_{s_1} (\mathsf{A}_{12}, \mathrm{ds}_1)$$
(11)

$$X_{21} = \omega L_{21}; \qquad L_{21} = \frac{1}{4\pi\nu_0} \oint_{s_1} \oint_{s_2} \frac{(\mathrm{ds}_2, \mathrm{ds}_1')}{R_{21}} = \frac{1}{I_1} \oint_{s_2} (\mathsf{A}_{21}, \mathrm{ds}_2)$$
(12)

Here the primed elements ds' are along the axis, the unprimed elements ds along the surface of the circuit. The parameters L_1^e and L_2^e in (8) and (9) are, respectively, the external self-inductance of the entire primary circuit and of the entire secondary circuit. L_{12} and L_{21} in (11) and (12) are, respectively, the mutual inductance of the primary circuit as a whole due to the current in the entire secondary, and the mutual inductance of the secondary due to the current in the primary. Formulas (9) to (12) are powerful tools in calculating inductances in near-zone circuits. Examination of the formulas for L_{12} and L_{21} in (11) and (12) reveals at once that they are alike, so that the reciprocal theorem does not have to be invoked to prove that $L_{12} = L_{21}$ in conventional circuits. The integrals (11) and (12) for L_{12} and L_{21} are called *Neumann formulas*.

Since (9) depends only on the geometry of a circuit, it is possible to consider the double-contour integral in parts. Suppose, for example, that the contour s_1 is conveniently separated into the parts c extending from C to D, d extending from D to E, and e extending from E back to C (Fig. 7.1). The

Fig. 7.1. Closed circuit. An idealized point generator is assumed at E.

double integral (9) can then be written as follows:

$$L_1^e = L_{cc}^e + L_{cd}^e + L_{ce}^e + L_{dc}^e + L_{dd}^e + L_{de}^e + L_{ec}^e + L_{ed}^e + L_{ee}^e \quad (13)$$

Here L_{cc}^e, L_{dd}^e, L_{ee}^e involve integration twice along the *same part* of the circuit. For example,

$$L_{cc}^e = \frac{1}{4\pi\nu_0} \int_C^D \int_C^D \frac{(\mathrm{ds}_1, \mathrm{ds}_1')}{R_{11}} \quad (14)$$

On the other hand, $L_{cd}^e = L_{dc}^e$, $L_{cc}^e = L_{ec}^e$, $L_{de}^e = L_{ed}^e$ involve one integration along one *part* of the circuit and a second integration along another *part* of the same closed loop. For example,

$$L_{cd}^e = \frac{1}{4\pi\nu_0} \int_C^D \int_D^E \frac{(ds_1, ds_1')}{R_{11}} \tag{15}$$

If desired, external inductances of the type L_{cc}^e may be called external self-inductances of specified *parts* of a closed circuit. Similarly, external inductances of the type L_{cd}^e may be called mutual inductances of specified *parts* of a closed circuit. Thus L_{cc}^e may be called the self-inductance of the part of the circuit between points C and D; L_{cd}^e may be called the mutual inductance of the part of the circuit between C and D due to the current in the part DE. External inductances of *parts* of a closed circuit are defined conveniently only if (1) and (2) are satisfied. In general, the external inductance of a circuit depends on the distribution function $f_1(s_1)$ which is a characteristic of the circuit as a whole. If $f_1(s_1) \neq 1$, the self-inductance of a part of a circuit, if defined in a way analogous to (13), would have a variety of values depending upon the function $f_1(s_1)$.

Alternative and often used formulas for L_1^e and L_{12} are obtained by using Stokes's theorem (II.6.8) and the defining relation curl $\mathbf{A} = \mathbf{B}$ in the contour integrals as follows:

$$\oint_s (\mathbf{A}, \mathbf{ds}) = \int_{S(\text{cap})} (\hat{\mathbf{N}}, \text{curl } \mathbf{A})dS = \int_{S_1(\text{cap})} (\hat{\mathbf{N}}, \mathbf{B})dS \tag{16}$$

S is a cap or cup surface bounded by the closed contour s; $\hat{\mathbf{N}}$ is a unit vector everywhere perpendicular to the surface S and in a direction to satisfy the right-hand-screw convention referred to the direction of integration around the contour s. With (16), (9) and (11) may be written as follows

$$L_1^e = \frac{1}{I_1} \int_{S_1(\text{cap})} (\hat{\mathbf{N}}_1, \mathbf{B}_{11})dS_1 \tag{17}$$

$$L_{12} = \frac{1}{I_2} \int_{S_1(\text{cap})} (\hat{\mathbf{N}}_1, \mathbf{B}_{12})dS_1 \tag{18}$$

It is possible to evaluate the inductance of a circuit either by integrating the tangential component of the vector potential around its contour *or* by integrating the normal component of the magnetic $\mathbf{B}$ vector over the area enclosed by the contour. Although the method using the vector potential is more powerful, that using the $\mathbf{B}$ vector is often convenient.

The simplification achieved in using (4) to (8) instead of (3.7) to (3.12) is very much greater than may be apparent. This is

due primarily to the fact that the integrals in (4) to (8) are functions only of the geometry and of the internal impedance per unit length of the circuit. They are independent of distribution functions and reference points for current. This has important consequences in circuits in which the contributions to the integrals in L^e and L_{12} are significant only along the contours of coils, while the integration around the remainder of the circuit is negligible. In such cases it is evidently possible to define the impedance of coils, condensers, and resistive elements in a manner that is independent of the configuration of the circuit connecting them. This is considered again in later sections where specific cases are studied.

8. Self-impedance of Quasi-conventional Circuits.—The special case of conventional or near-zone electric circuits as defined by (7.1) and (7.2) is a good approximation for a great variety of electric circuits. In fact, all electric-circuit theory is based on the assumption that the conditions (7.1) and (7.2) are very well satisfied, so well, indeed, that it is seldom considered necessary even to mention the existence of such restrictions on the generality of the theory. While it is possible at power frequencies and the lower radio frequencies to ignore the fact that the ordinary theory of electric circuits is a special case of a more general theory, this is not true at ultrahigh and higher frequencies. In order to investigate the nature of the differences between the general theory and the much simpler special theory, it is desirable to investigate the border line between them by expanding both the distribution function $f_1(s_1)$ and the phase-delay factor $e^{-j\beta_0 R_{11}}$ in power series so that at least one term beyond the leading one (which is unity in both cases) may be retained. The exponential function is easily expanded into the complex series written below.

$$e^{-j\beta_0 R_{11}} = \left(1 - \frac{\beta_0^2 R_{11}^2}{2!} + \frac{\beta_0^4 R_{11}^4}{4!} - \cdots \right) \\ \qquad -j\beta_0 R_{11}\left(1 - \frac{\beta_0^2 R_{11}^2}{3!} + \frac{\beta_0^4 R_{11}^4}{5!} - \cdots \right) \quad (1)$$

In order to determine the exact form of the distribution function $f_1(s_1)$, it is necessary to solve the integral equation (3.8). Such a solution would necessarily yield a different function for each configuration. If such a function were evaluated in any particular case, Z_1^e would be determined completely and in gen-

eral, so that there would be no point in obtaining an approximate solution for the quasi-stationary state. Since it is not generally possible to solve the complicated integral equation (3.8), an approximate method will be pursued. This depends upon the fact that the principal contribution to significant higher order terms in the factor $f_1(s_1)e^{-j\beta_0 R_{11}}$ is from the exponential factor rather than from $f_1(s_1)$. That is, the approximation

$$f_1(s_1) \doteq 1 \tag{2}$$

may be retained in evaluating the leading terms in the real and imaginary parts of $Z_1^e = R_1^e + jX_1^e$, while the exponential factor is represented by (1). In order to show that this is true, it is necessary to make use of a solution derived in Volume III for a long and narrow rectangle of wire driven at the center of one of the shorter sides. This is a section of two-wire transmission line which has a distribution of current that is very well described by

$$I_{s1} = I_0 \frac{\cos \beta_0(\tfrac{1}{2}s_t - s_1)}{\cos \tfrac{1}{2}\beta_0 s_t}$$

$$= I_0 \left\{ \cos \beta_0 s_1 + \sin \beta_0 s_1 \tan \frac{1}{2}\beta_0 s_t \right\} \tag{3}$$

I_{s1} is the amplitude of the current at a distance s_1 in the positive direction around the rectangle measured from the generator at $s_1 = 0$. The perimeter of the rectangle is s_t. The currents at s_1 and $s_t - s_1$, *i.e.*, at opposite points in the long sides, are equal in magnitude and in the same direction *around* the rectangle. This may be verified using (3).

The present discussion is limited to quasi-near-zone circuits so that the following inequality must be satisfied

$$\tfrac{1}{2}\beta_0 s_t << 1 \tag{4}$$

With (4) the distribution function $f_1(s_1) = I_{s1}/I_0$ in (3) reduces to

$$f_1(s_1) \doteq 1 + \tfrac{1}{2}\beta_0 s_1(s_t - s_1) \cdots \tag{5}$$

This is the approximate distribution function for a rectangle that is long compared with its width and that satisfies (4). It may be assumed to approximate reasonably well the distribution of current around the same length of wire s_t if arranged in a loop of any other shape that does not involve folded-back or very closely coiled sections.

If (1) and (5) are substituted in (3.9), the inner integral becomes

$$\oint_{s_1} \frac{f_1(s_1)}{R_{11}} e^{-j\beta_0 R_{11}} ds_1$$

$$= \oint_{s_1} \frac{1}{R_{11}} \left\{ 1 + \frac{1}{2} \beta_0^2 s_1(s_t - s_1) \right\} \left\{ \left(1 - \frac{\beta_0^2 R_{11}^2}{2} \cdots \right) \right.$$

$$\left. - j\beta_0 R_{11} \left(1 - \frac{\beta_0^2 R_{11}^2}{6} \cdots \right) \right\} ds_1$$

$$= \oint_{s_1} \frac{1}{R_{11}} \left\{ 1 + \frac{1}{2} \beta_0^2 (s_1 s_t - s_1^2 - R_{11}^2) \right\} ds_1$$

$$- j\beta_0 \oint_{s_1} \left\{ 1 + \frac{1}{2} \beta_0^2 \left(s_1 s_t - s_1^2 - \frac{R_{11}^2}{3} \right) \right\} ds_1 \quad (6)$$

The leading term in the integrand of the first integral is simply $1/R_{11}$. The second integral includes the following:

$$\oint_{s_1} ds_1 = 0; \qquad \oint_{s_1} s_1 ds_1 = 0; \qquad \oint_{s_1} s_1^2 ds_1 = 0 \quad (7)$$

Accordingly, retaining only leading real and imaginary terms,

$$\oint_{s_1} \frac{f_1(s_1)}{R_{11}} e^{-j\beta_0 R_{11}} ds_1 \doteq \oint_{s_1} \frac{ds_1}{R_{11}} + \frac{j\beta_0^3}{6} \oint_{s_1} R_{11}^2 ds_1 \quad (8)$$

The same two integrals are obtained if instead of (5) the simpler assumption

$$f_1(s_1) \doteq 1 \quad (9)$$

is made. Let it be assumed that (8) and (9) are good approximations whenever the following inequality is true:

$$\beta_0^2 R_{11}^2 << 1 \quad (10)$$

This condition, with the understanding that it means that (8) and (9) may be used, is called the *condition for the quasi-near-zone or for quasi-conventional circuits*. It is less severe than the condition $\beta_0 R_{11} << 1$ which defines the near-zone and strictly conventional circuits.

In other words, in the evaluation of Z_1^e (9) is a reasonable approximation for all closed loops of any shape that satisfy the condition of the quasi-near zone $(\beta_0 R_{11})^2 << 1$ for all values of R_{11}. The approximation (5) is not required either for the rectangle or for a loop of wire of any other shape. If a circuit

is driven in phase at several symmetrically arranged points around its circumference, the condition (9) may be a good approximation for much larger values of $\beta_0 R_{11}$ than permitted by (10).

With (8) substituted in (3.9), real and imaginary terms are obtained in the form $Z_1^e = R_1^e + jX_1^e$. The imaginary term is simply (9).

$$X_1^e = \frac{\omega}{4\pi\nu_0} \oint_{s_1} \oint_{s_1} \frac{(\mathrm{d}s_1,\mathrm{d}s_1')}{R_{11}} \tag{11}$$

With $\omega/\nu_0 = \beta_0\zeta_0$, the real term is

$$R_1^e = -\frac{\zeta_0}{4\pi} \frac{\beta_0^4}{6} \oint_{s_1} \oint_{s_1} R_{11}^2(\mathrm{d}s_1,\mathrm{d}s_1') \tag{12}$$

The numerical value $\zeta_0 = 120\pi$ ohms is often convenient. With it

$$R_1^e = -5\beta_0^4 \oint_{s_1} \oint_{s_1} R_{11}^2(\mathrm{d}s_1,\mathrm{d}s_1') \text{ ohms} \tag{13}$$

Formula (12) is readily evaluated for a plane loop of any shape. In rectangular coordinates,

$$R_{11}^2 = (x - x')^2 + (y - y')^2 = (x^2 + x'^2 + y^2 + y'^2)$$
$$- 2(xx' + yy') \tag{14a}$$
$$(\mathrm{d}s_1,\mathrm{d}s_1') = dx\,dx' + dy\,dy' \tag{14b}$$

If (14a) and (14b) are substituted in (13), a number of different integrals are obtained. Since all integrations are around a closed path, integrals of the types $\oint dx$, $\oint x\,dx$, $\oint x^2 dx$ vanish so that

$$\oint \oint (x^2 + x'^2 + y^2 + y'^2)(dx\,dx' + dy\,dy') = 0 \tag{15}$$

For the same reason,

$$\oint \oint (xx'dx\,dx' + yy'dy\,dy') = 0 \tag{16}$$

The remaining integrals are

$$\oint \oint (xx'dy\,dy' + yy'dx\,dx') = \oint x\,dy \oint x'dy'$$
$$+ \oint y\,dx \oint y'dx' \tag{17}$$

The integrals involving primed coordinates give

$$S_1' = \oint x'dy' = \oint y'dx' \tag{18}$$

where S_1' is the area enclosed by a contour along the axis of the

conductor. Similarly

$$S_1 = \oint x\, dy = \oint y\, dx \tag{19}$$

where S_1 is the area enclosed by a contour along the surface of the conductor. Since wires of small radius have been assumed, a satisfactory approximation is

$$S_1 \doteq S_1' \tag{20}$$

so that

$$\oint \oint R_{11}^2 (ds_1, ds_1') = -4S_1^2 \tag{21}$$

With (21), (13) becomes

$$R_1^e = 20\beta_0^4 S_1^2 \text{ ohms} \tag{22}$$

This formula may be used as an approximation for the external or radiation resistance of any quasi-conventional circuit in which the over-all length of wire s_t satisfies the condition (4). If the wire is coiled, the enclosed area should be measured inside a contour through the axis of the coil. If a condenser is included in the circuit, the contour may be taken through its center.

It is clear from (22) that every a.c. circuit that encloses a finite area has a nonvanishing external or radiation resistance. At low frequencies, $\beta_0 = 2\pi f / v_0$ is so small that R_1^e is insignificant. At high frequencies, β_0 may be so large that the condition for the quasi-near zone $(\beta_0^2 R_{11}^2 << 1)$ is not satisfied so that (22) is not a good approximation. Important simple circuits that generally satisfy $\beta_0^2 R_{11}^2 << 1$ and in which the external or radiation resistance is important are loop or frame antennas. These are usually rectangular, square, or circular in shape. For a rectangle of length s and width b,

$$R_1^e = 20\beta_0^4 s^2 b^2 \tag{23}$$

For a square of side s,

$$R_1^e = 20(\beta_0 s)^4 \tag{24}$$

For a circle of mean radius b,

$$R_1^e = 20\pi^2 (\beta_0 b)^4 \tag{25}$$

In all formulas, $\beta_0 = 2\pi / \lambda_0$ may be substituted if convenient. All resistances are in ohms.

9. Mutual Impedance in Quasi-conventional Circuits.—If each of two coupled circuits is quasi-conventional so that it is

possible to write (8.8) and (8.9) for each as follows:

$$\oint_{s_1} \frac{f_1(s_1)}{R_{11}} e^{-j\beta_0 R_{11}} ds_1 \doteq \oint_{s_1} \frac{ds_1}{R_{11}} + \frac{j\beta_0^3}{6} \oint_{s_1} R_{11}^2 ds_1; \qquad f_1(s_1) \doteq 1 \quad (1)$$

$$\oint_{s_2} \frac{f_2(s_2)}{R_{22}} e^{-j\beta_0 R_{22}} ds_1 \doteq \oint_{s_2} \frac{ds_2}{R_{22}} + \frac{j\beta_0^3}{6} \oint_{s_2} R_{22}^2 ds_2; \qquad f_2(s_2) \doteq 1 \quad (2)$$

the self-impedance of each circuit is to the required approximation unaffected by the proximity of the other circuit. Its evaluation is discussed in Sec. 8. The mutual impedances are obtained from the general integrals (3.11) and (3.12) by writing $f(s) \doteq 1$ in each.

$$Z_{12} = \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \left(ds_1, \oint_{s_2} \frac{e^{-j\beta_0 R_{12}}}{R_{12}} ds_2' \right) \tag{3}$$

$$Z_{21} = \frac{j\omega}{4\pi\nu_0} \oint_{s_2} \left(ds_2, \oint_{s_1} \frac{e^{-j\beta_0 R_{21}}}{R_{21}} ds_1' \right) \tag{4}$$

The real and imaginary parts are easily separated to give explicit expressions for the mutual resistances and reactances. With $\omega/\nu_0 = \beta_0 \zeta_0$,

$$R_{12} = \frac{\zeta_0 \beta_0}{4\pi} \oint_{s_1} \left(ds_1, \oint_{s_2} \frac{\sin \beta_0 R_{12}}{R_{12}} ds_2' \right) \tag{5}$$

$$X_{12} = \frac{j\omega}{4\pi\nu_0} \oint_{s_1} \left(ds_1, \oint_{s_2} \frac{\cos \beta_0 R_{12}}{R_{12}} ds_2' \right) \tag{6}$$

The factor $\zeta_0/4\pi = 30$ ohms. Expressions for R_{21} and X_{21} are like these with subscripts 1 and 2 interchanged. It is to be noted that whereas the size of each circuit is restricted to the near zone, the distance between the circuits is completely unrestricted. If the circuits are close together *so that each is in the near zone with respect to the other i.e.,*

$$\beta_0 R_{12} << 1; \qquad \beta_0 R_{21} << 1 \tag{7}$$

then

$$\sin \beta_0 R_{12} \doteq \beta_0 R_{12} - \frac{(\beta_0 R_{12})^3}{3!} + \cdots \tag{8}$$

$$\cos \beta_0 R_{12} \doteq 1 - \frac{(\beta_0 R_{12})^2}{2!} + \cdots \tag{9}$$

Since $\oint_{s_2} ds_2' = 0$, the leading term in (5) is obtained from the second term in (8). The leading terms are

$$R_{12} \doteq -\frac{\zeta_0 \beta_0^4}{24\pi} \oint_{s_1} \left(ds_1, \oint_{s_2} R_{12}^3 ds_2' \right) = -5\beta_0^4 \oint_{s_1} \oint_{s_2} R_{12}^3 (ds_1, ds_2') \quad (10)$$

$$X_{12} \doteq \omega L_{12} = \frac{\omega}{4\pi \nu_0} \oint_{s_1} \oint_{s_2} \frac{(ds_1, ds_2')}{R_{12}} \quad (11)$$

The mutual resistance term is extremely small at low frequencies. At ultrahigh frequencies it is usually necessary to use (5) and (6) instead of (10) and (11).

This and the preceding sections contained the theoretical background for the analysis of conventional loop transmitting and receiving antennas. They are investigated in detail in Volume II. Special formulas for rectangular and circular loops are obtained in later sections.

10. Self-impedance of a Rectangle of Wire.—The simplest closed loop of wire from the analytical point of view is a rectangle of length s and width b constructed of wire of radius a and internal impedance per unit length z'. Let the length of the rectangle be such that the conditions for the quasi-near zone are satisfied so that (8.11) and (8.12) are applicable.

The external self-reactance of the circuit of Fig. 10.1 from the point of view of a generator at the center of one of the shorter sides is

$$X_1^e = \omega L_1^e; \qquad L_1^e = \frac{1}{4\pi \nu_0} \oint_{s_1} \oint_{s_1} \frac{(ds_1, ds_1')}{R_{11}} \quad (1)$$

Fig. 10.1.—(a) Rectangle of wire; (b) enlarged wire at right end.

This double-contour integral consists of eight separate integrals along the several sides which can be combined into four pairs of integrals because opposite sides are of the same length. Thus, referring to Fig. 10.1,

$$L_1^e = 2(L_{cc} + L_{ca} + L_{dd} + L_{df}) \quad (2)$$

since

$$L_{ca} = L_{cc}; \qquad L_{ca} = L_{cc}; \qquad L_{ff} = L_{dd}; \qquad L_{fd} = L_{df} \quad (3)$$

The integrals are

$$L_{co} = \frac{1}{4\pi\nu_0} \int_C^D dx \int_C^D \frac{dx'}{\sqrt{a^2 + (x - x')^2}}$$

$$= \frac{1}{4\pi\nu_0} \int_0^b \int_0^b \frac{dx'}{\sqrt{a^2 + (x - x')^2}} dx \quad (4)$$

$$L_{dd} = \frac{1}{4\pi\nu_0} \int_D^E dz \int_D^E \frac{dz_1'}{\sqrt{a^2 + (z - z')^2}}$$

$$= \frac{1}{4\pi\nu_0} \int_0^s \int_0^s \frac{dz'}{\sqrt{a^2 + (z - z')^2}} dz \quad (5)$$

$$L_{ce} = -\frac{1}{4\pi\nu_0} \int_C^D dx \int_E^F \frac{dx'}{\sqrt{s^2 + (x - x')^2}}$$

$$= -\frac{1}{4\pi\nu_0} \int_0^b \int_0^b \frac{dx'}{\sqrt{s^2 + (x - x')^2}} dx \quad (6)$$

$$L_{df} = -\frac{1}{4\pi\nu_0} \int_D^E dz \int_F^C \frac{dz'}{\sqrt{b^2 + (z - z')^2}}$$

$$= -\frac{1}{4\pi\nu_0} \int_0^s \int_0^s \frac{dz'}{\sqrt{b^2 + (z - z')^2}} dz \quad (7)$$

There are no integrals involving one integration along a short side and the second integration along a long side because the scalar product (ds_1, ds_1') vanishes if ds and ds' are perpendicular. The integrals (4) to (7) are all of the same form and readily integrable. Thus

$$L_{co} = \frac{b}{2\pi\nu_0} \left[\sinh^{-1}\left(\frac{b}{a}\right) + \frac{a}{b} - \sqrt{1 + \frac{a^2}{b^2}} \right] \quad (8)$$

$$L_{dd} = \frac{s}{2\pi\nu_0} \left[\sinh^{-1}\left(\frac{s}{a}\right) + \frac{a}{s} - \sqrt{1 + \frac{a^2}{s^2}} \right] \quad (9)$$

$$L_{ce} = -\frac{b}{2\pi\nu_0} \left[\sinh^{-1}\left(\frac{b}{s}\right) + \frac{s}{b} - \sqrt{1 + \frac{s^2}{b^2}} \right] \quad (10)$$

$$L_{df} = -\frac{s}{2\pi\nu_0} \left[\sinh^{-1}\left(\frac{s}{b}\right) + \frac{b}{s} - \sqrt{1 + \frac{b^2}{s^2}} \right] \quad (11)$$

If desired, the inverse hyperbolic sine may be replaced by the natural logarithm using

$$\sinh^{-1}\left(\frac{b}{a}\right) = \ln\left(\frac{b + \sqrt{a^2 + b^2}}{a}\right) \quad (12)$$

Since it has been assumed throughout that the radius of the wire

is small compared with its length, it is possible to neglect a^2/b^2 and a^2/s^2 compared with unity.   This will be done in the following.   If the symbol D is introduced as a shorthand for the diagonal of the rectangle

$$D = \sqrt{s^2 + b^2} \tag{13}$$

and (12) is used, the external self-inductance (2) becomes

$$L_1^e = \frac{1}{\pi \nu_0} \left\{ b \ln \frac{2bs}{a(b+D)} + s \ln \frac{2bs}{a(s+D)} + 2(a+D-b-s) \right\} \tag{14a}$$

The reactance is

$$X_1^e = \omega L_1^e \tag{14b}$$

The internal reactance per unit length is x_1^i.   For uniform wire of total length $2(b+s)$, the total internal reactance is

$$X_1^i = 2(b+s)x_1^i \tag{15}$$

At low frequencies when

$$x_1^i = \omega l_1^i \tag{16}$$

it is possible to write

$$X_1^i = \omega L_1^i; \qquad L_1^i = 2(b+s)l_1^i \tag{17}$$

The total self-reactance of the rectangle is

$$X_{11} = X_1^i + X_1^e \tag{18}$$

The external or radiation resistance of the rectangle has been determined.   From 8.23 it is

$$R_1^e = 20\beta_0^4 b^2 s^2 \tag{19}$$

The internal resistance is simply

$$R_1^i = 2(b+s)r_1^i \tag{20}$$

Hence the total self-resistance is

$$R_{11} = R_1^i + R_1^e \tag{21}$$

The impedance of the rectangle is

$$Z_{11} = R_{11} + jX_{11} = (R_1^i + R_1^e) + j(X_1^i + X_1^e) \tag{22}$$

using (14), (15), (19), and (20).

The impedance of a square of side s made of wire with radius a is obtained from formulas for a rectangle by setting $b = s$ and noting that the diagonal $D = s\sqrt{2}$.   It is given by (22) with

$$X_1^e = \omega L_1^e; \qquad L_1^e = \frac{2s}{\pi \nu_0}\left\{ \ln \frac{s}{a} - \frac{a}{s} - 0.774 \right\} \tag{23}$$

$$X_1^i = 4sx_1^i \tag{24}$$

$$R_1^e = 20(\beta_0 s)^4 \tag{25}$$

$$R_1^i = 4sr_1^i \tag{26}$$

In most cases the term a/s is negligible so that

$$L_1^e = \frac{2s}{\pi \nu_0}\left\{ \ln \frac{s}{a} - 0.77 \right\} \tag{27}$$

If this is expressed in terms of the total length of wire $s_t = 4s$,

$$L_1^e = \frac{s_t}{2\pi \nu_0}\left\{ \ln \frac{s_t}{a} - 2.16 \right\} \tag{28}$$

As a numerical example, let the total self-impedance be calculated for a square of copper wire of total length $s_t = 0.245$ meter and radius $a = 4 \times 10^{-4}$ meter at a frequency of 159×10^6 hertz. This satisfies the condition for the near zone, $\beta_0 R_{11} < < 1$, since $\beta_0 R_{11} = 0.03$ with R_{11} the diagonal. The term in a/s is negligible for present purposes, and (28) is adequate for determining L_1^e. Using (23) to (28),

$$L_1^e = \frac{s_t}{2\pi \nu_0}\left\{ \ln \frac{s_t}{a} - 2.16 \right\} = \frac{0.245 \times 4\pi}{2\pi \times 10^7}\{\ln 612.5 - 2.16\}$$
$$= 0.209 \times 10^{-6} \text{ henry}$$

$$X_1^e = \omega L_1^e = 209 \text{ ohms}$$

$$R_1^e = 20(\beta_0 s)^4 = 20\left(\frac{0.061}{0.3}\right)^4 = 0.034 \text{ ohm}$$

At the high frequency here involved, and with $\sigma = 5.8 \times 10^7$ mhos/meter, $\nu = \nu_0 = 10^7/4\pi$,

$$r_1^i = x_1^i = \frac{1}{2\pi a}\sqrt{\frac{\omega}{2\sigma\nu}} = \frac{10^4}{8\pi}\sqrt{\frac{10^9 \times 4\pi \times 10^{-7}}{2 \times 5.8 \times 10^7}}$$
$$= 1.31 \text{ ohms/meter}$$

$$R_1^i = X_1^i = 0.245 \times 1.31 = 0.321 \text{ ohm}$$

Hence,

$$Z_{11} = (R_1^i + R_1^e) + j(X_1^i + X_1^e) = (0.321 + 0.034)$$
$$+ j(0.321 + 209)$$

or

$$Z_{11} = 0.35 + j209$$

---

[1] Cycles/second.

It is seen that X_1^c is negligible compared with X_1^s and that R_1^c is small compared with R_1^s.

11. Terminal Impedance and the Two-wire Line with Uniform Current.—The two longer sides of the rectangle of wire analyzed in Sec. 10 constitute a two-wire line that is sufficiently short to satisfy the condition

$$\beta_0 s << 1 \tag{1}$$

Examination of the evaluation of the impedance of the rectangle shows that the contributions to the contour integrals by the two longer sides of length s are completely independent of the contributions by the shorter sides of length b. This is due to the fact that the longer sides are perpendicular to the shorter ones and the entire rectangle is sufficiently small so that the current may be assumed to be the same at all points. If the shorter sides are replaced by impedances other than the straight wires, and these impedances are so constructed that their currents do not contribute significantly to the vector potential *along* the two-wire line nor alter the uniform amplitude of the currents in the line, the contributions to the impedance by the two long sides are unchanged. The inductance due to the line is given by the following part of (10.2):

$$L_{\text{line}}^c = 2(L_{dd} + L_{df}) \tag{2}$$

The contribution to the external self-inductance by one of the wires is

$$L_{dd} = \frac{s}{2\pi\nu_0}\left\{\ln\frac{2s}{a} - 1 + \frac{a}{s}\right\} \tag{3}$$

The mutual inductance of one of the wires due to the current in the other is

$$L_{df} = -\frac{s}{2\pi\nu_0}\left\{\ln\left(\frac{s+D}{b}\right) + \frac{b}{s} - \frac{D}{s}\right\} \tag{4}$$

Upon combining to form (2)

$$L_{\text{line}}^c = \frac{1}{\pi\nu_0}\left\{s\ln\frac{2bs}{a(s+D)} + a + D - b - s\right\} \tag{5}$$

For a line that is long compared with the distance between the wires so that the following conditions are satisfied

$$s^2 >> b^2; \qquad D \doteq s \tag{6}$$

$$L_{\text{line}}^c = \frac{1}{\pi\nu_0}\left\{s\ln\frac{b}{a} + a - b\right\} \tag{7}$$

If, in addition to (6), the following usually somewhat severer restriction is true:

$$s \ln \frac{b}{a} >> (b - a) \tag{8}$$

the external inductance reduces to

$$L_{\text{line}}^e = \frac{s}{\pi \nu_0} \ln \frac{b}{a} \tag{9}$$

It is possible now to define an external inductance l_0^e per unit length of the two wires or per loop unit length of the line. It is

$$l_0^e = \frac{L_{\text{line}}^e}{s} = \frac{1}{\pi \nu_0} \ln \frac{b}{a} \tag{10}$$

Formulas (9) and (10) apply to a line that is so long compared with the distance between the wires that the vector potential at any point along one of the wires is practically the same as though the line were infinitely long in both directions except near the ends. For a line that satisfies (8), this is a good approximation. This is discussed in greater detail and in more general terms in the analysis of the two-wire transmission line in Sec. 23 and in Volume III. Although (9) and (10) apply specifically to a line that is long compared with the distance between the wires, they are not correct for spacings that are small compared with the radius of the wire. It has been assumed that the two wires are sufficiently far apart so that it is immaterial around what contour along its surface the integration is performed with respect to the unprimed coordinates. This means, in effect, that the radius of the wires must be small compared with the distance between them so that rotational symmetry in the interior of the conductor may be assumed. It is shown in the next section that this is a good approximation when

$$b^2 >> a^2 \tag{11}$$

Since it has been shown that the external inductances of the two pairs of sides of a rectangle are independent, it is interesting to determine whether it is possible to define the inductance of one side alone in a way that is independent of the opposite side. Since the two sides are parallel, the vector potential tangent to the surface of one side always includes a contribution

due to the current in this side itself, as well as a contribution
due to the current in the other side. These contributions
are independent if (1) is satisfied, so that the first is properly
called the self-inductance of the one side, the other the mutual
inductance. For example, for the side e in Fig. 10.1, they are,
respectively,

$$L_{ee} = \frac{b}{2\pi\nu_0}\left[\sinh^{-1}\left(\frac{b}{a}\right) + \frac{a}{b} - \sqrt{1 + \frac{a^2}{b^2}}\right] \qquad (12)$$

$$L_{ec} = -\frac{b}{2\pi\nu_0}\left[\sinh^{-1}\left(\frac{b}{s}\right) + \frac{s}{b} - \sqrt{1 + \frac{s^2}{b^2}}\right] \qquad (13)$$

In order that L_{ec} be negligible compared with L_{ee} it is necessary
to impose the following conditions:

$$a^2 << b^2 << s^2 \qquad (14)$$

With (14), L_{ec} reduces to

$$L_{ec} \doteq -\frac{b}{2\pi\nu_0}\frac{b}{s} \qquad (15)$$

Clearly L_{ec} will be negligible compared with L_{ee} if

$$\sinh^{-1}\frac{b}{a} >> \frac{b}{s} \qquad (16)$$

or

$$\frac{b}{a} >> \sinh\frac{b}{s} \doteq \frac{b}{s} \qquad (17)$$

This is true if

$$a << s \qquad (18)$$

Since (18) is equivalent to (14), it follows that subject to (14)
it is possible to write

$$L_{ee} >> L_{ec} \qquad (19)$$

Subject to (14), (12) reduces to

$$L^e = \frac{b}{2\pi\nu_0}\left[\ln\frac{2b}{a} + \frac{a}{b} - 1\right] \qquad (20)$$

In most cases it is possible to write

$$a << b \qquad (21)$$

so that

$$L^e = \frac{b}{2\pi\nu_0}\left[\ln\frac{2b}{a} - 1\right] \qquad (22)$$

This formula gives the external inductance of a straight piece of wire of length b and radius a, specifically when it is a short side in a long rectangle. It is well to note that (22) does not apply to *every* straight piece of wire of this length and radius when arbitrarily placed in an unrestricted circuit.

If the evaluation of the external resistance R_1^e of the rectangle described in Sec. 10 is carried out explicitly using the general formula (8.13) (for a quasi-conventional loop of wire of any shape with a sensibly uniform current) around the rectangle instead of using the general integrated result (8.23), the following individual contributions or self- and mutual impedances of parts of the circuit are obtained. Referring to Fig. 10.1,

$$R_1^e = 2(R_{cc} + R_{ce} + R_{dd} + R_{df}) \tag{23}$$

with

$$R_{cc} = -5\beta_0^4 \int_0^b \int_0^b [a^2 + (x - x')^2]dx'dx = -5\beta_0^4 a^2 b^2 \tag{24}$$

$$R_{dd} = -5\beta_0^4 \int_0^s \int_0^s [a^2 + (z - z')^2]dz'dz = -5\beta_0^4 a^2 s^2 \tag{25}$$

$$R_{ce} = 5\beta_0^4 \int_0^b \int_0^b [s^2 + (x - x')^2]dx'dx = 5\beta_0^4 s^2 b^2 \tag{26}$$

$$R_{df} = 5\beta_0^4 \int_0^s \int_0^s [b^2 + (z - z')^2]dz'dz = 5\beta_0^4 b^2 s^2 \tag{27}$$

The reversal in sign in (26) and (27) is due to the fact that the primed and unprimed elements of integration are oppositely directed. Since it has been assumed that a^2 is negligible compared with b^2, the external resistance of the rectangle is

$$R_1^e \doteq 2(R_{ce} + R_{df}) = 20\beta_0^4 b^2 s^2 \tag{28}$$

in agreement with (10.19). The contribution to (28) by each *pair* of parallel sides, and hence of the long ones forming the line, is

$$R_{line}^e = 10\beta_0^4 b^2 s^2 \tag{29}$$

Since the two equal contributions to (28) by the currents in the line and in the short ends are independent because the two pairs of conductors are mutually perpendicular, it is possible to assign the radiation resistance (29) to the two-wire line and an equal value to the two short ends. If the attempt is made to divide R_1^e for each pair of conductors into two parts that are the independent contributions of each conductor, this is found to be impossible. R_1^e for an opposite pair is determined

almost entirely by the terms R_{ce} and R_{df} (which involve *one* integration along *each* of the two parallel conductors in a pair) and only to a negligible extent by the double integration along one conductor.   It is not possible, therefore, to define an independent radiation resistance for each of the short ends as it is possible to define an independent external reactance.   The external or radiation resistance of all quasi-near-zone circuits is, in general, a property of the circuit as a whole, in particular, a function of the area enclosed by it.

   12. Proximity Effect in a Two-wire Line.—In the analysis of the long rectangle in the preceding section, it was assumed that the radius of the wire is sufficiently small compared with the distances between parallel conductors that the distribution of current in each cross-sectional plane of each conductor may be assumed to depart in a negligible degree from rotational symmetry.   For wires with finite separation, this is true exactly only if the parallel wires themselves have zero radius.   If the wires have a nonvanishing cross section, rotational symmetry exists rigorously only for wires that are infinitely far apart. It is assumed throughout the analysis of electric circuits in this chapter that there is no circulation of current around the axis of the conductors.   That is, in a straight wire, $i_\theta = 0$ so that $E_\theta = 0$, $A_\theta = 0$.   Since

$$E_\theta = -\frac{j\omega}{\beta_0^2}\left(\frac{\partial}{r\partial\theta}\operatorname{div}\mathsf{A} + \beta_0^2 A_\theta\right) = 0 \tag{1}$$

it follows that with A_r small so that $\mathsf{A} \doteq \hat{z}A_z$,

$$\frac{\partial A_z}{\partial\theta} = 0 \tag{2}$$

This means that the surface of each conductor is an equipotential surface.   If the two conductors are moved so close together that rotational symmetry is no longer a good approximation for the cross-sectional distribution of current, the requirement that $E_\theta = 0$ at the surface must still be true, so that (2) must be satisfied.   Accordingly, any circumference of each of the parallel wires is necessarily an equipotential surface no matter how close together the parallel conductors are brought.   That is, the distribution of axial current in each cross section of the parallel wires must be such as to keep the circumference of each wire a ring of constant potential.   As the conductors are moved

closer together the vector potential along conductor 1 due to the current in conductor 2, for example, becomes more and more significant and also increasingly unequal at near and far points around the circumference of conductor 1. This necessitates a change in the distribution of current in both conductors, so that the sum of the vector potentials A_s due to both currents remains constant around any circumference.

Since it is assumed that the dimensions of the rectangle s and b satisfy the inequality

$$s^2 >> b^2 \tag{3}$$

the vector potential has only the single significant component A_s directed parallel to the long sides and to the z axis except near the ends. At points in space, it satisfies the equation

$$\frac{\partial^2 A_s}{\partial x^2} + \frac{\partial^2 A_s}{\partial y^2} + \frac{\partial^2 A_s}{\partial z^2} + \beta_0^2 A_s = 0 \tag{4a}$$

The rectangle previously analyzed was required to be of such size that all its parts were in the quasi-near zone with respect to each other so that the equation for the quasi-stationary state applied. It is

$$\frac{\partial^2 A_s}{\partial x^2} + \frac{\partial^2 A_s}{\partial y^2} + \frac{\partial^2 A_s}{\partial z^2} = 0 \tag{4b}$$

In this case, since the current at all points around the rectangle has the same amplitude, the variation of A_s with respect to z is very small compared with the variation with respect to the transverse variables except near the ends. Hence, a good approximation for determining the transverse distribution of potential in any cross-sectional plane not too near the ends of the rectangle is

$$\frac{\partial^2 A_s}{\partial x^2} + \frac{\partial^2 A_s}{\partial y^2} = 0 \tag{4c}$$

Actually, the conditions leading to (4c) are more severe than required. Clearly if each transverse plane satisfies the condition for the near zone and the variation of A_s in a direction parallel to the long sides is very small compared with the variation in the transverse plane, it is still possible to write (4c) with

$$\frac{\partial^2 A_s}{\partial z^2} + \beta_0^2 A_s = 0 \tag{5}$$

If (4c) and (5) are independently satisfied, (4a) is also satisfied.

Hence, (4c) is a satisfactory approximation for the transverse distribution for a rectangle that is entirely in the near zone, or for a rectangle of any length in which the transverse and axial distributions are for all practical purposes mutually independent. This is strictly true for infinitely long perfectly conducting wires that are extremely close together; it is an excellent approximation for all good conductors that are sufficiently long to satisfy (3) and so close together that it is possible to write

$$(\beta_0 b)^2 << 1 \tag{6}$$

For the present, the transverse distribution calculated from (4c) will be applied specifically to the rectangle previously analyzed which satisfies the condition for the quasi-stationary state in length as well as in width. In Volume III use is made of the fact that this distribution is also a good approximation for suitably restricted two-wire lines of any length.

The solution of the two-dimensional Laplace equation (4) that satisfies the boundary condition that A_z have a constant but different value on each of two circles is

$$A_z = \bar{A}_z \ln \frac{r_2}{r_1} \tag{7}$$

where $\bar{A}_z$ is independent of the coordinates x and y, locating points on the circle,

$$r_2 = \sqrt{\left(x + \frac{c}{2}\right)^2 + y^2} \tag{8}$$

$$r_1 = \sqrt{\left(x - \frac{c}{2}\right)^2 + y^2} \tag{9}$$

and c is a constant. It is readily verified by direct substitution that (7) satisfies (4). It is proved below that the curves

$$\ln\left(\frac{r_2}{r_1}\right) = \text{const.} \tag{10a}$$

or

$$\frac{r_2}{r_1} = K \tag{10b}$$

are circles with K an arbitrary real and positive constant. Using (8) and (9), (10b) when squared leads to

$$\left(x + \frac{c}{2}\right)^2 + y^2 = K^2\left[\left(x - \frac{c}{2}\right)^2 + y^2\right] \tag{11a}$$

which can be rearranged as follows:

$$x^2 + y^2 + xc\left(\frac{1 + K^2}{1 - K^2}\right) + \frac{c^2}{4} = 0 \tag{11b}$$

This equation defines two families of circles, 1 and 2, respectively, with $K = K_1 > 1$ and with $K = K_2 < 1$. Their centers are at

$$x_1 = \frac{c}{2}\left(\frac{K_1^2 + 1}{K_1^2 - 1}\right); \qquad y_1 = 0; \qquad \text{with } K_1 > 1 \tag{12a}$$

and at

$$x_2 = -\frac{c}{2}\left(\frac{1 + K_2^2}{1 - K_2^2}\right); \qquad y_2 = 0; \quad \text{with } K_2 < 1 \tag{12b}$$

and with radii given by

$$a_1 = \frac{cK_1}{K_1^2 - 1}; \qquad\qquad K_1 > 1 \tag{13a}$$

$$a_2 = \frac{cK_2}{1 - K_2^2}; \qquad\qquad K_2 < 1 \tag{13b}$$

Fig. 12.1.—Equipotential surfaces.

The two families of circles are divided by the straight line $x = 0$ characterized by $K_1 = K_2 = 1$. Circles and straight line are shown in Fig. 12.1.

A pair of circles with equal radii is defined by

$$a_1 = a_2 = a \tag{14}$$

From (13a,b), with (14)

$$K_2 = \frac{1}{K_1} \tag{15}$$

for circles with equal radii, and

$$K_1 = \frac{c}{2a} + \sqrt{\frac{c^2}{4a^2} + 1}; \qquad K_2 = -\frac{c}{2a} + \sqrt{\frac{c^2}{4a} + 1} \tag{16}$$

The distance between the centers of the two circles with equal radii is by definition b.   Using (12a,b) with (15), b is determined.

$$b = x_1 - x_2 = c \left[\frac{K_1^2 + 1}{K_1^2 - 1} \right] \tag{17}$$

For $K_1 = \infty$ (as when $a = 0$), $b = c$.   Or, using (13a) solved for c in (17),

$$b = a \left[\frac{K_1^2 + 1}{K_1} \right] \tag{18}$$

This relation can be solved for K_1 in terms of the radius a and distance between centers b.   It is

$$K_1 = \frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - 1} = \frac{b}{2a} \left[1 + \sqrt{1 - \left(\frac{2a}{b}\right)^2} \right] \tag{19}$$

The positive sign is chosen for the radical because K_1 must be greater than unity.   The negative sign gives $K_2 = 1/K_1$.

If two identical parallel wires of radius a and separated a distance b between centers carry equal and opposite currents across a given transverse plane, the potentials on their surfaces must be equal and opposite.   That is,

$$A_{z1} = -A_{z2} \tag{20}$$

Since each potential is given by (7), it follows that if

$$A_{z2} = \bar{A}_z \ln K_1 \tag{21}$$

$$A_{z1} = -\bar{A}_z \ln K_1 = \bar{A}_z \ln K_2 \tag{22}$$

The external inductance of the two wires is defined by

$$L_{\text{line}}^e = \frac{2}{I_z} \int_0^z A_z dz \tag{23}$$

The inductance per unit length along which A_z is sensibly con-

stant subject to (3) is

$$l_0^s = \frac{2A_s}{I_s} = \frac{2\bar{A}_s}{I_s} \ln K_1 \tag{24}$$

Since this must reduce to (11.10) for $b^2 >> a^2$, it follows that

$$\bar{A}_s = \frac{I_s}{2\pi\nu_0} \tag{25}$$

and with (19)

$$l_0^s = \frac{1}{\pi\nu_0} \ln\left[\frac{b + \sqrt{b^2 - (2a)^2}}{2a}\right] = \frac{1}{\pi\nu_0} \cosh^{-1}\left(\frac{b}{2a}\right) \tag{26}$$

This inductance per unit length is correct for all separations of the line that satisfy (3) and (6). It reduces to (11.10) when

$$b^2 >> (2a)^2 \tag{27}$$

It may be written in the form

$$l_0^s = \frac{1}{\pi\nu_0} \ln\left(\frac{b_s}{a}\right) \tag{28a}$$

with

$$b_s = \frac{b}{2}\left[1 + \sqrt{1 - \left(\frac{2a}{b}\right)^2}\right] \tag{28b}$$

It is to be noted that (28a) is the same in form as (11.10) with the effective separation b_s in (28b) substituted for the actual distance between centers b. This is of considerable importance in the more complicated analysis of the two-wire line that is not restricted in length by the condition for the near zone. If the line is sufficiently closely spaced so that it may be assumed to a first approximation that the transverse distribution of current in each cross section is independent of the axial distribution, it is possible to simplify the problem by treating a line with wires that are far enough apart that rotational symmetry may be assumed for this transverse distribution. The solution is then at once generalized to closer spacing by writing b_s for b wherever b occurs.

The change in the internal impedance per unit length with proximity of a parallel conductor of the same radius with an

equal current in the opposite direction is intricate, and a general solution in closed form is not available. With rotational symmetry a simple formula applies (V.12.6) when the skin depth is small and most of the current is in a thin layer near the surface as defined by

$$\beta_1 a \geq 10 \qquad (29)$$

The formula is

$$z^i = \frac{1+j}{2\pi a} \sqrt{\frac{\omega}{2\sigma\nu}} \qquad (30)$$

When a second conductor of the same size and with equal current in the opposite direction is near and (29) is true, the current not only is in a thin layer, but its density is greater in the adjacent parts of the parallel conductors. This is equivalent to decreasing the radius of a conductor with rotational symmetry so that a conductor of radius a in proximity with the second conductor has the same internal impedance per unit length as a conductor with rotational symmetry and an effective radius a_o smaller than a. The value of a_o is written without proof.[1] It is

$$a_o = a\left(\frac{c}{b}\right) = a\sqrt{1 - \left(\frac{2a}{b}\right)^2} \qquad (31)$$

The impedance per unit of each conductor subject to (29) is

$$z^i = \frac{1+j}{2\pi a} \sqrt{\frac{\omega}{2\sigma\nu\left[1 - \left(\frac{2a}{b}\right)^2\right]}} \qquad (32)$$

This reduces to (30) when (27) is satisfied.

13. Mutual Impedance of Coaxial Rings of Wire.—The calculation of the impedance of loosely wound single-layer coils depends on the previous determination of the self-impedance of a single circular ring of wire and the mutual impedance of two such rings arranged coaxially. Since the self-impedance can be obtained directly from the integral for mutual impedance by a simple specialization of the parameters, it is convenient to solve first the problem of mutual impedance of two coaxial rings of different size.

Consider two coaxial circular wire rings of radii b_1 and b_2. The radii b_1 and b_2 as well as the axial separation h of the rings are sufficiently small so that all elements of both rings are in the

[1] See, for example, J. R. Carson, *Phil. Mag.* **41**, 607 (1921).

near zone with respect to each other. That is,

$$\beta_0 R_{12} << 1; \qquad \beta_0 R_{11} << 1; \qquad \beta_0 R_{22} << 1 \qquad (1)$$

As shown in Fig. 13.1, R_{12} is the distance between an element ds_1 on the inner edge of one ring and an element ds_2' on the axis of the wire of the other ring; R_{11} is the distance between an element ds_1 on the inner edge and an element ds_1' on the axis of the wire of one and the same ring. R_{22} is defined like R_{11}

FIG. 13.1.—Coaxial rings of wire.

but for the second ring. Subject to (1) each ring has a current that is sensibly uniform in amplitude around the ring. Therefore, the distribution functions $f_1(s_1)$ and $f_2(s_2)$ for the currents I_{1s} and I_{2s} may be set equal to unity, and the subscript s may be omitted. Let the positive direction for current in each ring be taken in the direction of increasing θ. Each ring may be assumed driven by a slice generator at $\theta = 0$. The quasi-near-zone formula for the mutual impedance $Z_{12} = R_{12} + jX_{12}$ of ring 1 due to the current I_2 in ring 2 is obtained using (9.10) and (9.11). It is

$$Z_{12} = R_{12} + jX_{12} = R_{12} + j\omega L_{12} \qquad (2)$$

with

$$L_{12} = \frac{1}{4\pi\nu_0} \oint_{s_1} \oint_{s_1} \frac{(ds_1, ds_2')}{R_{12}} \tag{3}$$

$$R_{12} = -\frac{\zeta_0 \beta_0^4}{24\pi} \oint_{s_1} \oint_{s_1} R_{12}^3 (ds_1, ds_2') \tag{4}$$

The cylindrical coordinates r, θ, z of the two elements are

$$ds_1(b_1, \theta_1, 0) \tag{5a}$$
$$ds_2'(b_2, \theta_2, h) \tag{5b}$$

The radii a_1, a_2 of the wires are assumed to be small compared with b_1, b_2, and h. That is,

$$a_1^2 << b_1^2; \qquad a_2^2 << b_2^2; \qquad a_{1,2}^2 << h^2 \tag{6}$$

The distance between the elements ds_1 and ds_2' is most easily obtained by determining first the distance r_{12} between ds_1

Fig. 13.2.—Construction for determining r_{12} for coaxial rings.

and the projection of ds_2' onto the plane of ring 1. This plane is shown in Fig. 13.2 from which

$$r_{12} \doteq \sqrt{b_1^2 + b_2^2 - 2b_1 b_2 \cos (\theta_1 - \theta_2)} \tag{7}$$

The distance R_{12} between ds_1 and ds_2' follows directly.

$$R_{12} = \sqrt{h^2 + r_{12}^2} = \sqrt{h^2 + b_1^2 + b_2^2 - 2b_1 b_2 \cos (\theta_1 - \theta_2)} \tag{8}$$

Since the elements ds_1 and ds_2' are tangent to the circles and hence perpendicular to the radii drawn from the common axis

of the circles to the elements, it is clear that the angle between ds_1 and ds_2' is the same as the angle between these radii. Hence

$$(ds_1, ds_2') = ds_1 ds_2' \cos (\theta_1 - \theta_2) \tag{9}$$

Also,

$$ds_1 = b_1 d\theta_1 \tag{10}$$
$$ds_2' = b_2 d\theta_2 \tag{11}$$

after substituting (8) to (11) in (3) and (4), these become

$$L_{12} = \frac{1}{4\pi\nu_0} \int_0^{2\pi} b_1 d\theta_1 \int_0^{2\pi} \frac{b_2 \cos (\theta_1 - \theta_2) d\theta_2}{\sqrt{h^2 + b_1^2 + b_2^2 - 2b_1 b_2 \cos (\theta_1 - \theta_2)}} \tag{12}$$

$$R_{12} = -\frac{\zeta_0 \beta_0^4}{24\pi} \int_0^{2\pi} b_1 d\theta_1 \int_0^{2\pi} [h^2 + b_1^2 + b_2^2 - 2b_1 b_2 \cos (\theta_1 - \theta_2)]$$
$$b_2 \cos (\theta_1 - \theta_2) d\theta_2 \tag{13}$$

The evaluation of (12) may be carried out as follows. Let

$$\phi = \theta_2 - \theta_1; \qquad d\phi = d\theta_2 \tag{14}$$

Since the integration is completely around the ring, the starting point is of no consequence so that the limits are unchanged. With (14) and $(1 + \cos \phi) = 2 \cos^2 \frac{1}{2}\phi$, (12) becomes

$$L_{12} = \frac{b_1 b_2}{4\pi\nu_0} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} \frac{(2 \cos^2 \frac{1}{2}\phi - 1) d\phi}{\sqrt{h^2 + (b_1 + b_2)^2 - 4b_1 b_2 \cos^2 \frac{1}{2}\phi}} \tag{15}$$

Let the following change in variable and in limits be made in the second integral in (15) which will be denoted by A.

$$\phi = \pi - 2\psi; \qquad d\phi = -2d\psi. \tag{16}$$

At

$$\phi = 0, \qquad \psi = \frac{\pi}{2}; \qquad \text{at } \phi = 2\pi, \qquad \psi = -\frac{\pi}{2} \tag{17}$$

Thus, with $\cos \frac{1}{2}\phi = \cos (\frac{1}{2}\pi - \psi) = \sin \psi$,

$$A = 2 \int_{-\pi/2}^{\pi/2} \frac{(2 \sin^2 \psi - 1) d\psi}{\sqrt{h^2 + (b_1 + b_2)^2 - 4b_1 b_2 \sin^2 \psi}} \tag{18}$$

Here the denominator can be factored into

$$\sqrt{h^2 + (b_1 + b_2)^2} \sqrt{1 - k^2 \sin^2 \psi} \tag{19}$$

with the parameter k^2 given by

$$k^2 \equiv \frac{4b_1b_2}{h^2 + (b_1 + b_2)^2} \tag{20}$$

Using (19) in (18) and noting that the integrand is independent of a change in sign of ψ,

$$A = \frac{4}{\sqrt{h^2 + (b_1 + b_2)^2}}$$
$$\left[\int_0^{\pi/2} \frac{2 \sin^2 \psi \, d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} - \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} \right] \tag{21}$$

The integrand of the first integral in (21) is equivalent to

$$\frac{2}{k^2} \left[\frac{1}{\sqrt{1 - k^2 \sin^2 \psi}} - \sqrt{1 - k^2 \sin^2 \psi} \right] \tag{22}$$

as can be verified directly. Hence (21) is equivalent to

$$A = \frac{4}{\sqrt{h^2 + (b_1 + b_2)^2}} \left[-\frac{2}{k^2} \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \psi} \, d\psi \right.$$
$$\left. + \left(\frac{2}{k^2} - 1 \right) \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} \right] \tag{23}$$

The two integrals in (23) are of a standard type known as *complete elliptic integrals of the first and second kinds.* They are denoted by

$$K\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}} \tag{24}$$

$$E\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \psi} \, d\psi \tag{25}$$

The symbol F is sometimes used instead of K in (24). The parameter k is often represented in the form

$$k = \sin \alpha \tag{26}$$

so that

$$K\left(\frac{\pi}{2}, \alpha\right) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - \sin^2 \alpha \sin^2 \psi}} \tag{27}$$

$$E\left(\frac{\pi}{2}, \alpha\right) = \int_0^{\pi/2} \sqrt{1 - \sin^2 \alpha \sin^2 \psi} \, d\psi \tag{28}$$

For every value of the parameter k or α, each integral defines

a definite number. These numbers are extensively listed in standard tables.[1] Using the notations (24) and (25) in (21), this may be substituted for the second integral in (15). The integration with respect to θ_1 can be performed directly to give 2π.

$$L_{12} = \frac{2b_1b_2}{v_0\sqrt{h^2 + (b_1 + b_2)^2}}\left[-\frac{2}{k^2}E\left(\frac{\pi}{2},k\right) + \left(\frac{2}{k^2} - 1\right)K\left(\frac{\pi}{2},k\right)\right] \quad (29)$$

With (20) this reduces to

$$L_{12} = \frac{\sqrt{b_1b_2}}{v_0}\left[-\frac{2}{k}E\left(\frac{\pi}{2},k\right) + \left(\frac{2}{k} - k\right)K\left(\frac{\pi}{2},k\right)\right] \quad (30)$$

where

$$k = \frac{2\sqrt{b_1b_2}}{\sqrt{h^2 + (b_1 + b_2)^2}} \quad (31)$$

This is the final formula for the mutual inductance of two circular rings entirely within the near zone with respect to each other. If the rings have the same radius, $b_1 = b_2 = b$ and $k = 2b/\sqrt{h^2 + (2b)^2}$.

The mutual resistance R_{12} is quickly determined from (13). With the change in variable $\phi = \theta_1 - \theta_2$,

$$R_{12} = -\frac{\zeta_0\beta_0^4}{24\pi}\int_0^{2\pi} b_1 d\theta_1 \int_0^{2\pi}[(h^2 + b_1^2 + b_2^2)b_2\cos\phi$$
$$- 2b_1b_2^2\cos^2\phi]d\phi \quad (32)$$

Since

$$\int_0^{2\pi}\cos\phi\, d\phi = 0 \quad (33)$$

$$\int_0^{2\pi}\cos^2\phi\, d\phi = 4\int_0^{\pi/2}\cos^2\phi\, d\phi = \pi \quad (34)$$

$$R_{12} = \frac{\zeta_0}{6\pi}\beta_0^4\pi b_1^2\pi b_2^2 = \frac{\zeta_0}{6\pi}\beta_0^4 S_1 S_2 \quad (35)$$

where $S_1 = \pi b_1^2$ and $S_2 = \pi b_2^2$ are the areas enclosed by the two rings. Using $\zeta_0 \doteq 120\pi$ ohms

$$R_{12}^4 = 20\beta_0^4 S_1 S_2 \quad (36)$$

The mutual resistance of two coaxial rings with uniform currents depends on the area of each ring but not on their axial separation.

[1] For example, JAHNKE-EMDE, "Tables of Functions," p. 85, 1938 ed.

14. Self-impedance of a Circular Ring.—The external self-impedance of a single ring of wire of radius b measured to the inner edge and driven by a point generator at $\theta = 0$ is readily obtained using (13.30) and (13.36). The external self-impedance is given by

$$Z_1^e = R_1^e + jX_1^e = R_1^e + j\omega L_1^e \tag{1}$$

with

$$L_1^e = \frac{1}{4\pi\nu_0} \oint_{s_1} \oint_{s_1} \frac{(ds_1, ds_1')}{R_{11}} \tag{2}$$

$$R_1^e = -\frac{\zeta_0 \beta_0^4}{24\pi} \oint_{s_1} \oint_{s_1} R_{11}^3 (ds_1, ds_1') \tag{3}$$

These expressions are the same as (13.3) and (13.4) except that the primed element is on the axis in ring 1 with wire of radius $b_1 + a_1$ instead of in ring 2. The unprimed element is on the inner edge of the ring with radius b_1. Accordingly, (2) and (3) are like (13.3) and (13.4) with $b_1 + a_1$ written for b_2 and $h = 0$. It follows from (13.30) that

$$L_1^e = \frac{\sqrt{b_1(b_1 + a_1)}}{\nu_0}\left[-\frac{2}{k} E\left(\frac{\pi}{2}, k\right) + \left(\frac{2}{k} - k\right) K\left(\frac{\pi}{2}, k\right) \right] \tag{4}$$

with

$$k = \frac{2\sqrt{b_1(b_1 + a_1)}}{2b_1 + a_1} = \sqrt{1 - \left(\frac{a_1}{2b_1 + a_1}\right)^2} \tag{5}$$

The quantity

$$k' = \sqrt{1 - k^2} = \left(\frac{a_1}{2b_1 + a_1}\right) \tag{6}$$

is small because it has been required that the following inequality be satisfied:

$$a_1^2 << b_1^2 \tag{7}$$

The elliptic integrals may be expanded in series in powers of the small quantity k'. The series expansions are[1]

$$K = \ln\frac{4}{k'} + \frac{1}{4}\left(\ln\frac{4}{k'} - 1\right)k'^2 + \cdots \tag{8}$$

$$E = 1 + \frac{1}{2}\left(\ln\frac{4}{k'} - \frac{1}{2}\right)k'^2 + \cdots \tag{9}$$

Subject to (7), which is equivalent to

$$(k')^2 << 1 \tag{10}$$

[1] JAHNKE-EMDE, "Tables of Functions," p. 73, 1938 ed.

it is sufficient to retain the leading terms. Thus

$$K \doteq \ln \frac{4}{k'} = \ln \left[\frac{8b_1}{a_1} - 4 \right] \tag{11}$$

$$E \doteq 1 \tag{12}$$

Using these values in (4) and neglecting a_1^2 compared with b_1^2,

$$L_1^e = \frac{\sqrt{b_1(b_1 + a_1)}}{\nu_0} \left[\ln \left(\frac{8b_1}{a_1} - 4 \right) - 2 \right] \tag{13}$$

In most cases b_1 is sufficiently large compared with a_1 to permit writing

$$b_1 >> a_1 \tag{14}$$

so that

$$L_1^e \doteq \frac{b_1}{\nu_0} \left[\ln \frac{8b_1}{a_1} - 2 \right] \tag{15}$$

This may be written in terms of the total length of wire in the ring, $s_t = 2\pi b_1$.

$$L_1^e = \frac{s_t}{2\pi \nu_0} \left[\ln \frac{4s_t}{\pi a_1} - 2 \right] = \frac{s_t}{2\pi \nu_0} \left[\ln \frac{s_t}{a_1} - 1.76 \right] \tag{16}$$

Comparison of (16) with the expression (10.28) for a square shows that the external self-inductance of a loop of wire of length s_t is very nearly the same if the loop is a circle as when it is a square if s_t/a is large. It may be concluded that loops with shapes that do not differ greatly in shape from a circle or a square have external self-inductances that are approximated by (16) if the total length of wire is s_t.

The external or radiation self-resistance R_1^e is obtained directly from (13.36) after writing $(b_1 + a_1)$ for b_2 or $\pi(b_1 + a_1)^2$ for S_2. In most cases, a_1 is sufficiently small to permit writing

$$\pi(b_1 + a_1)^2 \doteq \pi b_1^2 = S_1 \tag{17}$$

Then

$$R_1^e = 20\beta_0^4 S_1^2 \tag{18}$$

in exact agreement with the general formula (8.22) for a plane loop of any shape. It is interesting to note that for loops which are confined to the near zone, the self-inductance depends primarily upon the *total length of wire* in a way that is roughly independent of the shape of the loop if this is not too different

from a circle or square, whereas the radiation resistance depends entirely on the *area* in a way that is completely independent of shape. For a given length of wire the radiation resistance of a quasi-conventional loop is greatest for a circle, least for a narrow rectangle.

The total self-impedance of a circular loop is

$$Z_{11} = Z_1^i + Z_1^e = (R_1^i + R_1^e) + j(X_1^i + \omega L_1^e) \qquad (19)$$

R_1^i and X_1^i are given by

$$R_1^i = r_1^i s_t \qquad (20)$$

$$X_1^i = x_1^i s_t \qquad (21)$$

where r_1^i and x_1^i are the internal resistance and reactance per unit length of a circular conductor with rotationally symmetrical current. For a loop which is large compared with the radius of the wire, the distribution of current in a cross section departs only in a negligible degree from rotational symmetry. Formulas for r_1^i and x_1^i are in Sec. V.7.

As a numerical example consider a circular loop of copper wire of circumference 0.245 meter and radius of wire 4×10^{-4} meter at a frequency of 159×10^6 hertz. This satisfies the condition for the near zone $\beta_0 R_{11} << 1$, since $\beta_0 R_{11} \doteq 0.03$ with R_{11} the diameter. Using (16) and (18)

$$L_1^e = \frac{s_t}{2\pi \nu_0} \left\{ \ln \frac{s_t}{a} - 1.76 \right\} = \frac{0.245 \times 4\pi}{2\pi \times 10^7} \{ \ln 612.5 - 1.76 \}$$
$$= 0.228 \times 10^{-6} \text{ henry}$$

$$X_1^e = \omega L_1^e = 228 \text{ ohms}$$

$$R_1^e = 20\beta_0^4 S_1^2 = 20\pi^2 \left(\frac{0.039}{0.3} \right)^4 = 0.056 \text{ ohm}$$

Since the length and size of the wire are the same as in the square in Sec. 10, the values for R_1^i and X_1^i there computed apply. They are

$$R_1^i = X_1^i = 0.321 \text{ ohm}$$

Hence,

$$Z_{11} = (R_1^i + R_1^e) + j(X_1^i + X_1^e) = (0.321 + 0.056)$$
$$+ j(0.321 + 228)$$

$$Z_{11} = 0.38 + j228$$

The value for the square made of the same wire is

$$Z_{11} = 0.35 + j209$$

The difference in $R_1^i + R_1^i$ is due to the greater external resistance of the circle; the difference in X_{11} is due to the greater external self-inductance of the circle.

15. Impedance of a Helical Coil.—The circuit of Fig. 15.1 consists of a point generator between A and B; of a helical coil of wire of total length s_c, radius of turns b, and pitch of turns p between C and D; and of a rectangle of connecting wire. The

Fig. 15.1.—A circuit including a helical coil. A point generator is between A and B.

dimensions of the rectangle are s_1 and s_2. All the wire is the same in size and material; its radius is a. The circuit as a whole satisfies the condition for the quasi-near zone so that the distribution function is unity and the current uniform. The impedance of the circuit is

$$Z_{11} = (R_1^i + R_1^i) + j(X_1^i + X_1^i) \tag{1}$$

The internal resistance and reactance R_1^i and X_1^i may be computed from the formulas for r_1^i and x_1^i of Chapter V for a straight wire of length s_t if the turns of the helix are far enough apart so that the distribution of current in the interior of the conductor is approximately rotationally symmetrical. If the turns are close together, more intricate formulas must be derived for r_1^i and x_1^i taking into account the departure from rotational symmetry; this involves the proximity effect in a more complicated form than for the two parallel conductors as considered in Sec. 12. For simplicity, it will be assumed below that the coil is loosely wound whenever numerical values of r_1^i and x_1^i are required. The internal impedance of the circuit of total length of wire s_t is

$$Z_1^i = R_1^i + jX_1^i; \qquad R_1^i = r_1^i s_t; \qquad X_1^i = x_1^i s_t \tag{2}$$

The contribution due to the coil is

$$Z_{1c}^i = R_{1c}^i + jX_{1c}^i; \qquad R_{1c}^i = r_1^i s_c; \qquad X_{1c}^i = x_1^i s_c \tag{3}$$

The external reactance and resistance in (1) are defined by (8.11)

and (8.12), *viz.*,

$$X_1^q = \omega L_1^q; \qquad L_1^q = \frac{1}{4\pi\nu_0} \oint_{s_1} \oint_{s_1} \frac{(ds_1, ds_1')}{R_{11}} \tag{4}$$

$$R_1^q = -\frac{\zeta_0 \beta_0^4}{24\pi} \oint_{s_1} \oint_{s_1} R_{11}^3 (ds_1, ds_1') \tag{5}$$

As usual, ds_1' is an element along the axis of the wire; ds_1 is an element along the surface of the wire at its inner edge; R_{11} is the distance between ds_1 and ds_1'. Because the wire forming the coil is not in the same plane as the rest of the circuit, the exact evaluation of (4) and (5) is difficult. The problem is very much simplified if certain approximations are made. These depend upon the fact that the uniform current in the helix may be considered to be equivalent approximately to a current directed along the axis of the coil from C to D (Fig. 15.1) and a series of equal currents in identical circular rings spaced at axial distances p. That is, in evaluating the *external* imped-

Fig. 15.2.—Circuit approximately equivalent to that of Fig. 15.1. For simplicity, the cross section of the wire is not shown.

ance, the circuit of Fig. 15.1 is replaced by that of Fig. 15.2. It is assumed that the gaps in the axial conductor between C and D are very small, as are those in the circular rings. The two radial conductors extending from the axis to the circumference of each ring are very close together, almost parallel, and have equal and opposite currents. Since they are either at right angles to other conductors or are symmetrically placed with respect to conductors carrying equal and opposite currents, all integrations along them cancel. That is, the currents in these pairs of conductors contribute nothing to the external impedance. In the arrangement of Fig. 15.2 contributions to the contour integral when one of the elements ds_1 and ds_1' is on a circle of wire and the other is not are *zero* because either the elements are mutually perpendicular so that (ds_1, ds_1') vanishes or another pair of elements exists with an equal contribution of opposite sign. The integral for the complete circuit thus separates into two independent parts in this quasi-near-zone case. The first part is the rectangle of

wire; the second part consists of the circular loops of wire. The contour integrals for the circuit thus reduce to components that have already been evaluated, *viz.*, a rectangle and coaxial circular rings. It is necessary merely to add together all the contributions. These include the external self-impedance of the rectangle Z_r^e, the external self-impedance of n identical coaxial rings, and the sum of all mutual impedances between ring i and rings $j = 1$ to $n - 1$, with i taking on all values from 1 to n. The external impedance of the rectangle is Z_r^e, the external self-impedance of ring number i is Z_{ii}^e, the mutual impedance between ring i and ring j is Z_{ij}^e. Assuming the radius a of the wire to be sufficiently small, these impedances are as given below.

$$Z_r^e = R_r^e + jX_r^e = R_r^e + j\omega L_r^e \tag{6}$$

with

$$R_r^e = 20\beta_0^4 s_1^2 s_2^2 \tag{7}$$

$$L_r^e = \frac{1}{\pi v_0}\left\{ s_1 \ln \frac{2s_1 s_2}{a(s_1 + \sqrt{s_1^2 + s_2^2})} + s_2 \ln \frac{2s_1 s_2}{a(s_2 + \sqrt{s_1^2 + s_2^2})} \right.$$
$$\left. + 2(a + \sqrt{s_1^2 + s_2^2} - s_1 - s_2) \right\} \tag{8}$$

as given in (10.19) and (10.14) with appropriate changes in symbols.

$$Z_{ii}^e = R_{ii}^e + jX_{ii}^e = R_{ii}^e + j\omega L_{ii}^e \tag{9}$$

with

$$R_{ii}^e = 20\pi^2(\beta_0 b)^4 \tag{10}$$

$$L_{ii}^e = \frac{b}{v_0}\left\{ \ln \frac{8b}{a} - 2 \right\} \tag{11}$$

as given in (14.18) and (14.15).

$$Z_{ij}^e = R_{ij}^e + jX_{ij}^e = R_{ij}^e + j\omega L_{ij}^e \tag{12}$$

where, using (13.36) and (13.30) with $b_1 = b_2 = b$,

$$R_{ij}^e = 20\pi^2(\beta_0 b)^4 \tag{13}$$

$$L_{ij}^e = \frac{b}{v_0}\left\{ -\frac{2}{k_{ij}} E\left(\frac{\pi}{2}, k_{ij}\right) + \left(\frac{2}{k_{ij}} - k_{ij}\right) K\left(\frac{\pi}{2}, k_{ij}\right) \right\} \tag{14}$$

and

$$k_{ij} = \frac{2b}{\sqrt{h_{ij}^2 + 4b^2}} = \frac{1}{\sqrt{1 + \left(\frac{h_{ij}}{2b}\right)^2}} \tag{15}$$

The axial distance between coil i and coil j is denoted by h_{ij}.

Since the pitch is uniform,

$$h_{ij} = p|i - j| \tag{16}$$

Let the ratio of pitch to diameter be denoted by

$$\delta \equiv \frac{p}{2b} \tag{17}$$

With (16) and (17), (15) becomes

$$k_{ij} = \frac{1}{\sqrt{1 + \delta^2(i - j)^2}}; \qquad i \neq j \tag{18}$$

The external impedance of the circuit is given by

$$Z_{11}^a = \left(R_r^c + \sum_{j=1}^{n} \sum_{i=1}^{n} R_{ij}^c \right) + j\omega \left(L_r^c + \sum_{j=1}^{n} \sum_{i=1}^{n} L_{ij}^a \right) \tag{19}$$

The evaluation of the double sum of the L_{ij}^a is straightforward but tedious if there are many turns. If the pitch and diameter are given, the elliptic integrals for successive values of k_{ij} as defined in (18) may be obtained from tables or curves, and L_{ij} evaluated for $i \neq j$ using (14). The number of terms may be halved if it is recalled that according to the reciprocal theorem $L_{ij} = L_{ji}$. That is,

$$\sum_{j=1}^{n} \sum_{i=1}^{n} L_{ij} = \sum_{j=1}^{n} L_{jj} + 2\left\{ \sum_{j=2}^{n} L_{1j} + \sum_{j=3}^{n} L_{2j} + \cdots L_{(n-1)n} \right\} \tag{20}$$

If the pitch is large compared with the radius of the wire so that

$$p^2 >> a^2 \tag{21}$$

the contributions to the double sum by all but the first two turns on each side of every coil are negligible. The condition (20) is also the one required to permit calculation of the internal impedance using formulas that imply rotational symmetry in the interior of the wire. The determination of the double sum of the R_{ij}^c is simple because $R_{ii}^c = R_{ij}^c = 20\pi^2(\beta_0 b)^4$. Hence

$$R_{11}^c = R_r^c + n^2 R_{ii}^c = 20\beta_0^4(s_1^2 s_2^2 + n^2\pi^2 b^4) \tag{22}$$

As an example, consider a coil of five turns with radius equal to pitch

$$b = p \tag{23}$$

so that

$$\delta = \frac{p}{2b} = \frac{1}{2} \tag{24}$$

The radius of the wire satisfies (21). The possible values of $(i - j)$, $i \neq j$ include 1, 2, 3, 4, so that L_{ij} for the corresponding four values of k_{ij} are required. These values are tabulated below:

$i - j$	k_{ij}	$\dfrac{2}{k_{ij}}$	$\left(\dfrac{2}{k_{ij}} - k_{ij}\right)$	E	K	L_{ij}	
1	0.895	2.24	1.34	1.18	2.25	$0.87b/\nu_0$	
2	0.707	2.83	2.12	1.35	1.85	$0.10b/\nu_0$	(25)
3	0.555	3.60	3.04	1.44	1.72	$0.06b/\nu_0$	
4	0.447	4.46	4.01	1.48	1.66	$0.03b/\nu_0$	

$$\sum_{j=2}^{5} L_{1j} = \frac{b}{\nu_0}\{0.87 + 0.10 + 0.06 + 0.03\} \qquad = 1.06\frac{b}{\nu_0}$$

$$\sum_{j=3}^{5} L_{2j} = \frac{b}{\nu_0}\{0.87 + 0.10 + 0.06\} \qquad = 1.03\frac{b}{\nu_0}$$

$$\sum_{j=4}^{5} L_{3j} = \frac{b}{\nu_0}\{0.87 + 0.10\} \qquad = 0.97\frac{b}{\nu_0}$$

$$L_{45} = \qquad\qquad = 0.87\frac{b}{\nu_0}$$

$$2\left\{\sum_{j=2}^{5} L_{1j} + \sum_{j=3}^{5} L_{2j} + \sum_{j=4}^{5} L_{3j} + L_{45}\right\} = 2(3.93)\frac{b}{\nu_0} = 7.86\frac{b}{\nu_0} \tag{26}$$

The external inductance of the entire circuit including the five-turn coil is

$$L_{11}^i = L_r^i + \frac{b}{\nu_0}\left\{5\left(\ln\frac{8b}{a} - 2\right) + 7.86\right\}$$

$$= L_r^i + \frac{b}{\nu_0}\left\{5\ln\frac{8b}{a} - 2.14\right\} \tag{27}$$

To continue the numerical data, let the length of wire in the coil be $s_c = 0.245$ meter so that the length of each turn is 0.049 meter. Since this length is $\sqrt{p^2 + (2\pi b)^2}$ and $p = b$, it follows that

$$b\sqrt{1 + 4\pi^2} = 0.049 \text{ meter}$$

or

$$p = b = 0.77 \text{ centimeter} \qquad (28)$$

Let the radius of the wire be

$$a = 0.04 \text{ centimeter} \qquad (29)$$

Let the rectangle be a square of side 3.85 centimeters so that the coil replaces one of the four sides.   Hence, with (10.27),

$$L_r^e = \frac{0.077 \times 4\pi}{\pi \times 10^7} \left\{ \ln \left(\frac{3.85}{0.04} \right) - 0.77 \right\} = 0.151 \times 10^{-6} \text{ henry} \qquad (30)$$

Using (27)

$$L_{11}^e = L_r^e + \frac{0.77 \times 10^{-2} \times 4\pi}{10^7} \left\{ 5 \ln \left(\frac{6.16}{0.04} \right) - 2.14 \right\} \qquad (31)$$

$$L_{11}^e = (0.151 + 0.223) \times 10^{-6} = 0.374 \times 10^{-6} \text{ henry} \qquad (32)$$

It is interesting to compare this value with a square with the same total length of wire *i.e.*, $s_t = s_c + 3 \times 3.85 = 36.05$ centimeters.   The external self-inductance of this square is

$$L_{11}^e = \frac{0.1802 \times 4\pi}{\pi \times 10^7} \left\{ \ln \frac{9.01}{0.04} - 0.77 \right\} = 0.336 \times 10^{-6} \text{ henry} \qquad (33)$$

The small rectangle with one side replaced by a loosely wound coil of five turns is seen to have a somewhat larger external self-inductance than the larger square with the same total length of wire.   The difference is of order of magnitude of 10 per cent.

If the rectangle is very long compared with its width and both are great compared with the radius of the wire, it is possible to write as in (11.14)

$$a^2 << s_2^2 << s_1^2 \qquad (34)$$

As discussed in Sec. 11, the self-inductance of the straight piece of wire forming one of the short sides of a long rectangle is independent of the rest of the rectangle.   If the helical coil between C and D in Fig. 15.1 and the approximate equivalent in Fig. 15.2 replaces one of the short sides of a long rectangle, the external inductance of the coil between C and D is also independent of the rest of the rectangle.   The axial wire through the center is equivalent to the side of the rectangle, and the circular rings are symmetrically placed as discussed above.

Hence, the independent self-inductance of a helical coil of length $CD = s_2$ is

$$L_h^i = L_{s_2}^i + \sum_{j=1}^{n} \sum_{i=1}^{n} L_{ij}^o \tag{35}$$

The value of $L_{s_2}^i$ is obtained from (11.22). It is

$$L_{s_2}^i = \frac{s_2}{2\pi\nu_0}\left[\ln\frac{2s_2}{a} - 1\right] \tag{36}$$

For the five-turn helix analyzed above

$$L_h^i = \frac{s_2}{2\pi\nu_0}\left[\ln\frac{2s_2}{a} - 1\right] + \frac{b}{\nu_0}\left[5\ln\frac{8b}{a} - 2.14\right] \tag{37}$$

If the rectangle is very long with shorter sides $s_2 = 3.85$ centimeters and the coil is dimensioned as above, the following inductances are computed:

$$L_{s_2}^i = \frac{0.0385 \times 4\pi}{2\pi \times 10^7}\left[\ln\frac{7.7}{0.04} - 1\right] = 0.0405 \times 10^{-6} \text{ henry} \tag{38}$$

The contribution to the inductance by the circular rings is the same as before so that the external self-inductance of the loosely wound helical coil is

$$L_h^i = (0.0405 + 0.223) \times 10^{-6} = 0.264 \times 10^{-6} \text{ henry} \tag{39}$$

This value may be compared with

$$L_1^i = 0.228 \times 10^{-6} \text{ henry} \tag{40}$$

for a circular loop with the same total length of wire and with

$$L_1^i = 0.209 \times 10^{-6} \text{ henry} \tag{41}$$

for a square with the same length of wire.

16. Inductance of a Long, Closely Wound Coil.—If a helical coil of radius b and axial length h_c has a single layer of n closely wound turns, the double sum $\sum_{j=1}^{n} \sum_{i=1}^{n} L_{ij}$, in (15.35), may be evaluated approximately by a double integral. The integral is obtained by replacing the unit of summation 1 by $(n/h_c)dz$ in forming the integral. The limits in the sum, 1 and n, become 0 and h_c in the integral. Thus, the following approximation is made:

$$L_c^e = \sum_{j=1}^{n} \sum_{i=1}^{n} L_{ij} \doteq \left(\frac{n}{h_c}\right)^2 \int_0^{h_c} \int_0^{h_c} L_{z_1 z_2} dz_1 dz_2 \qquad (1)$$

This approximation improves as the coil becomes more like a uniform sheet of current around a cylinder of the same diameter as the coil. In (1)

$$L_{z_1 z_2} = \frac{b}{\nu_0}\left[-\frac{2}{k_z} E\left(\frac{\pi}{2}, k_z\right) + \left(\frac{2}{k_z} - k_z\right) K\left(\frac{\pi}{2}, k_z\right)\right] \qquad (2)$$

with

$$k_z \equiv \frac{2b}{\sqrt{(z_2 - z_1)^2 + 4b^2}} \qquad (3)$$

The axial distance between the two elements dz_1 and dz_2 is $(z_2 - z_1)$. The evaluation of the integrals in (1) is tedious but can be carried out in closed form[1] to give

$$L_c^e = \frac{n^2 b}{\nu_0} \frac{2}{3} \left\{ \frac{K\left(\frac{\pi}{2}, k\right) + (\tan^2 \alpha - 1) E\left(\frac{\pi}{2}, k\right)}{\sin \alpha} - \tan^2 \alpha \right\} \qquad (4)$$

where

$$k = \frac{2b}{\sqrt{h_c^2 + 4b^2}} = \sin \alpha \qquad (5)$$

and

$$\tan \alpha = \frac{2b}{h_c} \qquad (6)$$

The term L_{s2} in (15.35) contributes a negligible amount so that (4) is the entire external self-inductance of the coil. The factor

$$\frac{2\pi L_c^e \nu_0}{n^2 b} = \frac{4\pi}{3} \left\{ \frac{K\left(\frac{\pi}{2}, k\right) + (\tan^2 \alpha - 1) E\left(\frac{\pi}{2}, k\right)}{\sin \alpha} - \tan^2 \alpha \right\} \qquad (7)$$

is tabulated in the Jahnke-Emde "Tables of Functions," pp. 88 and 89, 1938 ed., for ratios of diameter to length $(2b/h_c)$ from zero to 1 and for ratios of length to diameter $(h_c/2b)$ from zero to 1. The internal impedance of a long, closely wound coil is not easily determined because rotational symmetry certainly does not obtain even approximately in the interior of the wires forming the winding. It will not be analyzed.

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[1] F. OLLENDORFF, "Potentialfelder der Elektrotechnik," pp. 115ff.

17. Impedance of a Circular Condenser in a Quasi-conventional Circuit.—The internal impedance Z^i of a circular condenser of radius b was evaluated in Chapter V. It was defined to be the impedance between the outer edges of the adjacent surfaces of the plates. In Fig. 17.1, the condenser is shown symmetrically connected in a rectangular loop of wire with a point generator at AB. The internal impedance Z^i determined in Chapter V

FIG. 17.1.—Circuit with condenser.

is the impedance between the edges D and E. In order to determine the self-impedance of the entire circuit as observed at AB, it is necessary to evaluate (2.1) around the circuit.

$$V^e_{10} = \oint_{s_1} z^i_1 I_{1s} ds_1 + j\omega \oint_{s_1} (\mathbf{A}_{11}, ds_1) \tag{1}$$

The contour integration for the first term in (1), the internal impedance, may be carried out around a closed contour along the surface of the wire forming the rectangle from B to C, then radially outward along the outer surface of the condenser plate, and around its edge to E, straight across from D to E, along the edge and radially inward to F, and finally back to A along the rectangle. From A to C and from F back to A, the circuit is like any other rectangle. For uniform wire of internal impedance z^i_w and length s_w, the resulting integral reduces to

$$I_1 Z^i_{1w} = I_1 z^i_w s_w \tag{2}$$

The evaluation of the radial integral from C to D can be expressed directly in terms of the surface impedance defined in Sec. V.8 provided the thickness of the plates is great compared with the skin thickness. This is usually true except at very low frequencies when the impedance of the plates is of the same order of magnitude as the d.c. resistance of the plates, a negligible quantity compared with the d.c. resistance r_0 of the connecting

wires. If a quasi-surface current l'_r and a surface impedance Z^s can be defined (Sec. V.8) such that

$$Z^s = \frac{E_r}{l'_r} \tag{3}$$

the internal impedance z^i_r of a ring of radius r and width dr is

$$z^i_r = \frac{E_r}{I_r} = \frac{Z^s}{2\pi r} \tag{4}$$

This follows because rotational symmetry obtains so that the total radial quasi-surface current is

$$I_r = 2\pi r l^i_r \tag{5}$$

The internal impedance of the circular plate from $r = a$ to $r = b$ is

$$Z^i_p = \int_a^b z^i_r dr = \frac{Z^s}{2\pi} \int_a^b \frac{dr}{r} = \frac{Z^s}{2\pi} \ln \frac{b}{a} \tag{6}$$

Since the two outer surfaces of the plates have equal and opposite currents, the total internal impedance of the condenser plates for quasi-surface currents on the outer surfaces is

$$2Z^i_p = \frac{Z^s}{\pi} \ln \frac{b}{a} \tag{7}$$

If this is combined with the internal impedance between E and D (Fig. 16.1), the internal impedance of the condenser is

$$Z^i_c = Z^i + 2Z^i_p \tag{8}$$

It is to be noted that Z^i was evaluated in Chapter V under the assumption that the inner surfaces of the plates may be treated as though perfect conductors. Actually, $2Z^i_p$ in (8) is always so small compared with Z^i_w for the connecting wires that it may be neglected. This is equivalent to assuming perfectly conducting condenser plates. In the following, it is assumed that the entire internal impedance of a condenser is given by

$$Z^i_c \doteq Z^i \tag{9}$$

where Z^i is the internal impedance defined in Chapter V.

In order to determine the contribution to the external impedance of the rectangle in Fig. 17.1 by the condenser, it is necessary

to examine the second contour integral in (1). If the rectangular loop is sufficiently large so that to a good approximation rotational symmetry exists on the outer surfaces of the condenser plates, the only contributions to the vector potential directed radially along the plates are due to radial currents in the plates themselves. The vector potential tangent to the outer surface of a condenser plate and directed along a particular radius must be determined from all currents in both plates that have components parallel to the radius chosen. Since rotational symmetry

is assumed to be a good approximation, the vector potential $\mathbf{A}$ at a point P (Fig. 17.2) on the outer surface of one of the plates is directed from P toward Q if the current everywhere on the plate is radially outward. $\mathbf{A}$ is made up of contributions directed from P toward Q due to all currents in the upper semicircle and smaller contributions due to the equal but more distant currents in the lower semicircle. Clearly, *only a small fraction* of the total radial current $I_r = I_a$ on the surface is effective in maintaining a radial vector potential at any point such as P. This is in contrast to the axial vector potential on the surface of the cylindrical connecting wire joining the plate normally at its center, which is due to the *entire* current I_s in the conductor. Furthermore, the contributions to the vector potential at P due to the currents on the inner surface of the same plate and on both surfaces of the second plate have a resultant that is opposite to and smaller than A_r as determined from the current on the outer surface alone. The vector potential at P is the resultant of equal and opposite currents that are at different distances from P. Since the total axial current I_s in the connecting wire is equal to the total radial current $I_r = I_a = I_b$ on the outer surface of the condenser plate if the circuit is confined to the quasi-near zone as assumed, it is clear that the radial vector potential at any point in the plate is extremely small compared with the axial vector potential along the connecting wires. Consequently, the integral of the radial

FIG. 17.2.—Condenser plate.

vector potential from $r = a$ to $r = b$ can lead only to a quantity that is negligibly small compared with a similar integral along a section of the rectangle of connecting wires of length $s = b - a$. The combined thickness of plates and dielectric is necessarily very much smaller than any reasonable length of connecting wire. It follows that the contributions to the second line integral in (1) due to integrations radially along the plates and axially across their combined edges is negligible compared with the integration around the rest of the circuit. Therefore, the external impedance of the condenser may be neglected, and an excellent approximation of its entire impedance is the internal impedance Z^i between the edges of the inner surfaces of the plates as determined in Chapter V. The impedance of a condenser is independent of the rest of the quasi-near-zone circuit provided connecting wires are perpendicular to the plates.

18. "Lumped" Constants in Near-zone Circuits.—The rigorous analysis of the impedance of a closed circuit consisting of coils, condensers, resistors, and connecting wires involves a double integration around the contour of the circuit. In near-zone circuits characterized by a distribution function $f(s) = 1$ for each closed loop, the contribution to the impedance by a condenser has been shown to be independent of the rest of the circuit. The external impedance of a coil depends on the rest of the circuit only in the same degree as would a single wire along the axis of the coil. The external impedance of a resistor, consisting of a straight or specially wound resistance wire, also depends on the circuit as a whole only in the same degree as a straight wire, either the actual wire if single or a wire as long as the axis of a wound resistor. The complete impedance of a circuit thus reduces to the independent contributions of condensers, coils, and resistors in combination with the impedance of a loop of wire consisting of the actual wires connecting the several elements and fictitious wires along the axes of coils and wound resistors. In near-zone circuits it is usually possible to neglect *all* contributions to the complete contour integral for the *external* impedance or for mutual impedances arising from integration around the contour of the loops of connecting wires. Analytically this is equivalent to neglecting the vector potential in a contour integral with a path of integration that follows the connecting wires *between* coils (or other

impedors) and that cuts *directly* across the air gaps from terminal to terminal of each coil (or other impedor). The coil, or other impedance between the terminals, is thus mathematically equivalent to a *discontinuity in scalar potential* across the terminals. Physically, the impedance of the circuit reduces to a good approximation to the sum of the self-impedances of coils, condensers, and resistors, which may be assumed to be individually independent, and mutual inductances between suitably placed coils. The inductance, capacitance, and resistance of the several elements are called "lumped" constants or parameters because they are associated exclusively with particular elements rather than with the circuit as a whole. Furthermore, with the near-zone condition of a distribution function $f(s) = 1$, these lumped parameters are simply functions of the geometry of the elements and of the constitutive parameters of the conductors and dielectrics. It is to be noted that in neglecting the contribution to the impedance from the loops of connecting wires the external or radiation resistance involving the principal area enclosed by the circuit is disregarded. In near-zone circuits, it is always negligible compared with the internal or ohmic resistance but is never zero except in the stationary state. In most cases, conventional or near-zone electric circuits may be analyzed in terms of independent elements with lumped constants; radiation may be neglected. Accordingly, the impedance of a circuit consisting, for example, of a series combination of a coil, a condenser, and a resistor may be written in the conventional form

$$Z_{11} = R_1 + j\omega L_1 - \frac{j}{\omega C_1}$$

where R_1 is the internal or ohmic series resistance of coil, condenser, resistor, and, if required, of connecting wires; $j\omega L_1$ is the external impedance of the coil; $1/j\omega C_1$ is the internal impedance of the condenser. The internal inductances of the coil and of the condenser are usually negligible.

It is well to recall that a circuit for alternating currents may satisfy the condition for the near zone $\beta_0 R << 1$ for the external circuit, whereas it does not satisfy the corresponding condition for the very much smaller distances inside the conductors. This is due to the great difference in the order of magnitude of the real phase constant β in a dielectric or β_0 in space and of β_1 in a

metallic conductor. A wire that has an internal or ohmic resistance r^i per unit length that differs appreciably from the d.c. resistance r_0^i per unit length does not satisfy the condition for the near zone for the *internal* field even in its cross-sectional dimensions. On the other hand, its length, whether formed into a rectangle or wound into a coil, may be short enough to satisfy the condition for the near zone for the *external* field. In speaking of conventional or near-zone circuits, reference is always made to distances and phase constants measured in the dielectric or in space outside the conductors.

In the analysis of circuits involving a coil, it frequently happens that the condition

$$\beta_0^2 R_{11}^2 << 1 \tag{1}$$

is very well satisfied for all parts of the circuit including the coil, while the condition

$$f_1(s_1) \doteq 1 \tag{2}$$

is satisfied in all parts of the circuit *except the coil.* In such cases the current is sensibly uniform everywhere except around the turns of the coil where, for sufficiently loose winding, it is well represented by (8.3) in the form

$$I_s = I\frac{\cos \beta_0(\frac{1}{2}s_e - s)}{\cos \frac{1}{2}\beta_0 s_e} = I(\cos \beta_0 s + \sin \beta_0 s \tan \frac{1}{2}\beta_0 s_e) \tag{3}$$

I_s is the current at a distance s along the wire of the coil measured from one end; I is the current entering or leaving the ends of the coil and is the same as the current of uniform amplitude at all points in the rest of the circuit; s_e is the total length of wire around the turns of the coil. The coil is assumed to be symmetrical with respect to its center. If the length of wire is sufficiently short to satisfy the inequality

$$\beta_0 s_e < 1 \tag{4}$$

the distribution function is the same as in (8.5). Thus

In the coil: $f_1(s_1) \doteq 1 + \beta_0^2 s(s_e - s) \cdots$ (5a)
In the circuit: $f_1(s_1) \doteq 1$ (5b)

If the contribution to the reactance of the entire circuit by the connecting wires is negligible, the external reactance of the coil

can be defined independently to be

$$X_L^e = \frac{\omega}{4\pi\nu_0} \int_0^{s_e} \left(ds_1, \int_0^{s_e} \frac{f_1(s_1')}{R_{11}} ds_1' \right) \tag{6}$$

With (5a) this becomes

$$X_L^e = \frac{\omega}{4\pi\nu_0} \int_0^{s_e} \int_0^{s_e} \frac{(ds_1, ds_1')}{R_{11}} + \frac{\omega\beta_0^2}{4\pi\nu_0} \int_0^{s_e} \int_0^{s_e} \frac{s'(s_e - s')}{R_{11}} (ds_1, ds_1') \tag{7}$$

The first term in (7) is the reactance of the coil with a uniform current in all turns equal to that entering and leaving its ends. The second term takes account of the actual nonuniformity of current.   The current at the center of the coil is larger than that entering or leaving.

Noting that

$$\beta_0^2 = \frac{\omega^2}{v_0^2} = \frac{\omega^2\epsilon_0}{\nu_0} \tag{8}$$

the reactance (7) may be written in the form

$$X_L^e = \omega L^e (1 + \omega^2 L^e C_L) \tag{9}$$

where C_L is defined by

$$C_L = \frac{\epsilon_0}{4\pi(\nu_0 L^e)^2} \int_0^{s_e} \int_0^{s_e} \frac{s'(s_e - s')}{R_{11}} (ds_1, ds_1') \tag{10}$$

and where

$$L^e = \frac{1}{4\pi\nu_0} \int_0^{s_e} \int_0^{s_e} \frac{(ds_1, ds_1')}{R_{11}} \tag{11}$$

The quantity C_L is a capacitance and the reactance (9) is that of a parallel combination of an inductance L^e with a capacitance C_L in the special case when the following inequality is satisfied:

$$(\omega^2 L^e C_L)^2 << 1 \tag{12}$$

This condition corresponds to (4).   With (12)

$$X_L^e = \omega L^e (1 + \omega^2 L^e C_L) \doteq \frac{\omega L^e}{(1 - \omega^2 L^e C_L)} \tag{13}$$

It appears, therefore, that a coil with a distribution of current that is not quite uniform, but that is symmetrical with respect to the center of the coil, where the amplitude is maximum, may be treated as though the current were uniform and equal to the

entering current if an appropriate "lumped" capacitance is
assumed connected in parallel with the entire coil.

If the distribution of current in a coil is nonuniform and of the
form (3), the equation of continuity

$$\frac{dI_s}{ds} + j\omega q_s = 0 \tag{14}$$

requires a distribution of charge q_s per unit length of wire of the
form

$$q_s = \frac{-jI}{v_0} \frac{\sin \beta_0(\tfrac{1}{2}s_o - s)}{\cos \tfrac{1}{2}\beta_0 s_o} \tag{15}$$

This shows that when the wire in the coil is positively charged in
the half from $s = 0$ to $s = \tfrac{1}{2}s_o$, the other half is negatively
charged in the same manner. The charge that is distributed
along the conductors according to (15) because the wire is too
long to satisfy the condition for the near zone is commonly said
to charge the "distributed capacitance" of the coil. This
"distributed capacitance" is then replaced analytically by an
equivalent "lumped" capacitance (10). In this connection it
is well to bear in mind that the concept of *distributed charge* is
more fundamental than that of *distributed capacitance*. In
fact, the need for the latter is the result of an attempt to describe
general electromagnetic phenomena in terms of the special theory
of the near zone.

If the condition (4) is not satisfied for the total length of wire
in the loosely wound helix, an adequate representation of the
reactance of a coil with nonuniformly distributed current is not
possible in terms of a coil with a uniform current connected in
parallel with a lumped capacitance. On the other hand, if (4)
is satisfied with s_o replaced by the length of a *single turn*, a repre-
sentation is possible in which *each turn* carries a uniform current
equal to that at its center, and an appropriate lumped capacitance
is connected across the ends of each turn. If (4) is not satisfied
even for each turn, the coil cannot be treated analytically in
terms of wires carrying uniform currents with lumped capaci-
tances in parallel. That is, the reactance of the coil cannot be
represented even approximately by an "equivalent circuit" of
"lumped" inductances and capacitances.

If the coil is a part of a circuit which as a whole does not satisfy
(1), so that in general the current entering one end of the coil

is not equal to and in the *same direction* around the circuit the current leaving the other end of the coil, a *reactance of the cc* cannot be defined.

BALANCED TWO-WIRE AND FOUR-WIRE TRANSMISSION LINI

19. Formulation of the Problem of the Two-wire Line.—C cuits that are not confined to the near or quasi-near zone a analyzed in Volumes II and III under the names of antennas ar transmission circuits. One of the most important of the latt is the balanced two-wire line consisting of two parallel conducto that are unrestricted in length but are so close together th opposite points are well within the near zone with respect each other. Although the rigorous derivation of the differenti equation for the distribution of current and charge along tl two-wire line is properly a subject of the first chapter in Volun III, it is conveniently included here as a concluding illustratio of the application of general electromagnetic principles in tl analysis of electric circuits. In this way, a background usefi in both Volumes II and III is provided. Furthermore, the two wire line as used in transmission involves electromagnetic theor only in the derivation of the differential equations and in tl formulation of the approximations implied in their use, and n at all in the solution and application of the equations to mos practical problems. By substituting an approximate derivatio of the equations of the transmission line for that based on electr magnetic principles, and supplementing this by conclusion drawn from the rigorous analysis, the study of transmission lin of conventional type can be carried out without previous stud of electromagnetic principles. Accordingly, the first part Volume III need not be made to depend directly on Volume I.

In order to derive from electromagnetic equations the differen tial equations for the two-wire line immersed in an imperfect bu simple dielectric, let the following conditions be imposed.

1. The line is isolated and so driven that currents at opposit points in the two conductors are equal in magnitude and oppo sitely directed. Practical arrangements to accomplish this ar described in Volumes II and III.

2. The distance *b* between centers and the radius *a* of the tw identical, highly conducting, parallel wires each of length

satisfy the following inequalities:

$$a < b << s \tag{1}$$
$$\beta_s b << 1 \tag{2}$$

where β_s is defined in (20.13).

Subject to (1) and (2) the following assumptions are good approximations.

3. The distribution of current in the cross section of each conductor (skin effect and proximity effect) is approximately

Fig. 19.1.—Section of two-wire line.

independent of the axial distribution of current *along* the conductors. This assumption is justified by the fact that the principal forces acting on the electric charges in an element of length dz_2 at z in a particular cross section of one of the wires are due primarily to the charges and currents in both conductors which are at distances from z that are *not large* compared with the distance between the wires. The charges and currents in the opposite elements dz_1' and dz_2' (Fig. 19.1) exert retarded forces on the moving charges on the surface of the element dz_2 which decrease with increasing distances $\sqrt{(z'-z)^2 + b^2}$ and $\sqrt{(z'-z)^2 + a^2}$. If $(z' - z)$ exceeds b in magnitude, the distances from dz_1' to dz_2 and from dz_2' to dz_2 become equal rapidly as $|z' - z|$ is increased, and forces exerted on charges in dz_2 become more and more nearly *equal* and *opposite*. Forces acting on charges in dz_2 due to distant charges and currents in the two conductors practically cancel and are, therefore, negligible compared with forces due to neighboring charges that are not only nearer but also largely uncanceled. It follows that the cross-sectional distribution may be assumed to be the same as in an infinitely long conductor so that the formulas for internal impedance per unit length either for approximately rotational symmetry (Chapter V) or as corrected for proximity (Sec. 12) may be used. Furthermore, the problem may be analyzed for

wires sufficiently far apart and so small in radius that rotational symmetry in the interior and on the surface of each conductor is a good approximation, and then generalized to closely spaced conductors of large radius simply by writing

$$a_e = a \sqrt{1 - \left(\frac{2a}{b}\right)^2}; \qquad b_e = \frac{b}{2}\left[1 + \sqrt{1 - \left(\frac{2a}{b}\right)^2}\right]$$

instead of a and b in formulas for the external impedance. However, since small variations in effective values of b or a (due to proximity) necessarily have a negligible effect on all functions that depend primarily on s, such as $\sqrt{s^2 + a^2}$, $\sqrt{s^2 + b^2}$, effective values of a and b are not required in such functions.

4. The conductors are completely immersed in a homogeneous isotropic simple medium characterized by the complex parameters $\dot{\sigma}$, ε_r and the real parameter $\nu_r = 1/\mu_r$.

20. Differential Equations for Potential Functions.—Since every circumference of each conductor is an equipotential circle, it is possible to define the potentials ϕ_1 and A_1 on the surface of conductor 1, the potentials ϕ_2 and A_2 on the surface of conductor 2 at any cross section z. The corresponding electric fields are E_1 and E_2. The first differential equation governing the potential functions is derived by proceeding from the general expression defining the complex amplitude of the scalar potential

$$-\text{grad } \phi = E + j\omega A \tag{1}$$

It follows directly that

$$-\frac{\partial \phi_1}{\partial z} = E_{1s} + j\omega A_{1s} \tag{2}$$

$$-\frac{\partial \phi_2}{\partial z} = E_{2s} + j\omega A_{2s} \tag{3}$$

Subtracting (3) from (2) and defining scalar and vector potential differences according to

$$V \equiv \phi_1 - \phi_2 \tag{4}$$
$$W_s \equiv A_{1s} - A_{2s} \tag{5}$$

gives

$$-\frac{\partial V}{\partial z} = E_{1s} - E_{2s} + j\omega W_s \tag{6}$$

On the surface of each conductor,

$$E_{1s} = z_1^i I_1 \tag{7}$$
$$E_{2s} = z_2^i I_2 \tag{8}$$

where z^i is the internal impedance per unit length and I the total axial current in each conductor as indicated by the subscripts. By hypothesis,

$$I_2 = -I_1 = -I_s \tag{9}$$

Also let

$$z^i \equiv z_1^i + z_2^i \tag{10}$$

be the internal impedance per *loop* unit length, *i.e.*, of the two conductors. If they are of the same material, $z_1^i = z_2^i$ and $z^i = 2z_1^i$.

With (7) to (10), (6) becomes

$$-\frac{\partial V}{\partial z} = z^i I_s + j\omega W_s \tag{11}$$

This is one of the fundamental circuit equations for the line.

The second differential equation is obtained from the equation of continuity for potential functions

$$\text{div } \mathbf{A} + j\frac{\beta^2}{\omega}\phi = 0 \tag{12}$$

Here

$$\beta = \beta_s - j\alpha_s = \omega\sqrt{\frac{\xi}{\nu}} = \omega\sqrt{\mu\left(\epsilon_s - j\frac{\sigma_s}{\omega}\right)} \tag{13}$$

characterizes the medium in which the conductors are immersed. It is assumed that the reluctivity $\nu = 1/\mu$ is real. It will be assumed that rotational symmetry is a good approximation on the surface of each conductor. As discussed in Sec. 19, a departure from rotational symmetry has a negligible effect on the axial distribution of current and charge, and its effect on the cross-sectional distribution may be taken into account at any time by using an effective radius a_e and wire separation b_e. With rotational symmetry, (12) may be written in cylindrical coordinates as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} + j\frac{\beta^2}{\omega}\phi = 0 \tag{14}$$

The vector and scalar potentials in (14) are to be calculated entirely in terms of the distributions of current and charge in the conductors. The contribution to the potentials due to charges in the simple medium in which they are immersed is taken into

account in the parameters ν, ϵ_0, and σ_e. As in the analysis of the single straight conductor, the only significant component of the vector potential is the axial component calculated from the total axial current I_s in each conductor. The radial component A_r due to the small radial current in each cross section is negligible both because $\overline{\rho_m v_r}$ is small in the conductor and because elements of current can always be grouped in equal and opposite pairs very close together so that A_r is very small as a result of partial cancellation. A_θ is zero because $\overline{\rho_m v_\theta}$ is assumed to be zero. It is assumed, therefore, that the following inequalities are good approximations at all points outside the conductors:

$$A_r <<< A_s; \qquad A_\theta = 0; \qquad \mathbf{A} \doteq 2A_s \qquad (15)$$

Accordingly, (14) reduces to

$$\frac{\partial A_s}{\partial z} + j \frac{\beta^2}{\omega} \phi = 0 \qquad (16)$$

On the respective surfaces of the two conductors

$$\frac{\partial A_{1s}}{\partial z} + j \frac{\beta^2}{\omega} \phi_1 = 0 \qquad (17)$$

$$\frac{\partial A_{2s}}{\partial z} + j \frac{\beta^2}{\omega} \phi_2 = 0 \qquad (18)$$

Subtracting and using (4) and (5),

$$\frac{\partial W_s}{\partial z} + j \frac{\beta^2}{\omega} V = 0 \qquad (19)$$

This is the second fundamental differential equation for the potential functions.

21. The Perfectly Conducting Pair.—In the idealized case of perfect conductors, the two fundamental equations (20.11) and (20.19) reduce to

$$\frac{\partial V}{\partial z} + j\omega W_s = 0 \qquad (1)$$

$$\frac{\partial W_s}{\partial z} + j \frac{\beta^2}{\omega} V = 0 \qquad (2)$$

By differentiation, the dependent variables may be separated to give

$$\frac{\partial^2 V}{\partial z^2} + \beta^2 V = 0 \tag{3}$$

$$\frac{\partial^2 W_s}{\partial z^2} + \beta^2 W_s = 0 \tag{4}$$

These are one-dimensional wave equations with general solutions

$$V = C_1 \exp(j\beta z) + C_2 \exp(-j\beta z) \tag{5}$$

$$W_s = \frac{j}{\omega}\frac{\partial V}{\partial z} = -\frac{\beta}{\omega}[C_1 \exp(j\beta z) - C_2 \exp(-j\beta z)] \tag{6}$$

The C's are arbitrary constants of integration. For a line that is infinitely long in the positive z direction or is terminated in such a way that $C_1 = 0$,

$$V = C_2 e^{-(\alpha_s + j\beta_s)z} \tag{7}$$

$$W_s = \frac{\beta C_2}{\omega} e^{-(\alpha_s + j\beta_s)z} \tag{8}$$

The general solutions (5) and (6) may also be written

$$V = D_1 \cosh(\alpha_s + j\beta_s)z + D_2 \sinh(\alpha_s + j\beta_s)z \tag{9}$$

$$W_s = \frac{-\beta_s + j\alpha_s}{\omega}[D_1 \sinh(\alpha_s + j\beta_s)z + D_2 \cosh(\alpha_s + j\beta_s)z] \tag{10}$$

where the D's are arbitrary constants of integration. The hyperbolic functions are defined by

$$\cosh x = \tfrac{1}{2}[\exp(x) + \exp(-x)];$$
$$\sinh x = \tfrac{1}{2}[\exp(x) - \exp(-x)] \tag{11}$$

They are discussed in detail in Volume III. If the medium is perfectly nonconducting, in particular if it is empty space with $\alpha_s = 0$, $\beta_s = \beta_0$,

$$V = D_1 \cos \beta_0 z + D_2' \sin \beta_0 z \tag{12}$$

$$W_s = \frac{j\beta_0}{\omega}[-D_1 \sin \beta_0 z + D_2' \cos \beta_0 z] \tag{13}$$

where $D_2' = jD_2$. It is to be noted that there is no *attenuation in the potential functions* along a perfectly conducting line in space in spite of the fact that the external or radiation resistance referred to maximum current is not zero as will be shown specifically in a later section.

22. Vector Potential Difference.—The general equations (20.11) and (20.19) for the two-wire line are written in terms of

scalar and vector potential differences as well as the total current and not in terms of scalar potential difference and total current alone as is customary. It is possible to eliminate the vector potential only if the condition (19.2) $\beta_s b << 1$ is a good approximation. The vector potential due to a total axial current I_s in a straight conductor extending from $z = 0$ to $z = s$ is (IV.6.4)

$$A_s = \frac{1}{4\pi\nu} \int_0^s I_s' \frac{\exp{(-j\beta R)}}{R} \, dz' \tag{1}$$

where

$$R = \sqrt{(z' - z)^2 + r^2} \tag{2}$$

is the distance from the element dz' on the axis of the conductor to the point where A_s is calculated. This expression is a good approximation on the surface of the conductor.

On the equipotential surface $r = a$ of conductor 1, the vector potential due to the total axial currents I_1 and I_2 in both conductors is

$$A_{1s} = \frac{1}{4\pi\nu} \int_0^s \left\{ I_1' \frac{\exp{(-j\beta R_a)}}{R_a} + I_2' \frac{\exp{(-j\beta R_b)}}{R_b} \right\} \, dz' \tag{3}$$

with

$$R_a = \sqrt{(z' - z)^2 + a^2} \tag{4}$$

$$R_b = \sqrt{(z' - z)^2 + b^2} \tag{5}$$

The primed coordinates locate the element of integration; the unprimed the point where A_{1s} is calculated. The primed currents are at z'. On conductor 2,

$$A_{2s} = \frac{1}{4\pi\nu} \int_0^s \left\{ I_2' \frac{\exp{(-j\beta R_a)}}{R_a} + I_1' \frac{\exp{(-j\beta R_b)}}{R_b} \right\} \, dz' \tag{6}$$

The vector potential difference is

$$W_s = A_{1s} - A_{2s}$$
$$= \frac{1}{4\pi\nu} \int_0^s (I_1' - I_2') \left\{ \frac{\exp{(-j\beta R_a)}}{R_a} - \frac{\exp{(-j\beta R_b)}}{R_b} \right\} \, dz' \tag{7}$$

Since it is assumed that

$$I_2' = -I_1' = -I_s' \tag{8}$$

$$W_s = \frac{1}{2\pi\nu} \int_0^s I_s' \left\{ \frac{\exp{(-j\beta R_a)}}{R_a} - \frac{\exp{(-j\beta R_b)}}{R_b} \right\} \, dz' \tag{9}$$

Now let a distribution function $f(z')$ for current be introduced

and the current I'_s at z' referred to the current I_s at z where W_s is calculated.   Thus

$$I'_s = I_s f(z') \tag{10}$$

and I_s is independent of the primed variables of integration. Using (10) in (9), this may be written

$$W_s = \frac{I_s}{2\pi\nu} \int_0^s f(z') \left\{ \frac{\exp\,(-j\beta R_a)}{R_a} - \frac{\exp\,(-j\beta R_b)}{R_b} \right\} dz' \tag{11}$$

For convenience let

$$j\omega W_s = I_s z^s \tag{12}$$

where

$$z^s \equiv \frac{j\omega}{2\pi\nu} \int_0^s f(z') \left\{ \frac{\exp\,(-j\beta R_a)}{R_a} - \frac{\exp\,(-j\beta R_b)}{R_b} \right\} dz' \tag{13}$$

is dimensionally an impedance per unit length.   With this notation, (20.11) becomes

$$-\frac{\partial V}{\partial z} = (z^i + z^s)I_s \tag{14}$$

but z^s is a function of the coordinate z.

An approximate evaluation of the function z^s in (13) may be carried out as follows.   By adding and subtracting $\left(\dfrac{1}{R_a} - \dfrac{1}{R_b}\right)$, (13) becomes

$$z^s = \frac{j\omega}{2\pi\nu} \left\{ \int_0^s \left(\frac{1}{R_a} - \frac{1}{R_b} \right) dz' \right.$$
$$\left. + \int_0^s \left(\frac{f(z')\,\exp\,(-j\beta R_a) - 1}{R_a} - \frac{f(z')\,\exp\,(-j\beta R_b) - 1}{R_b} \right) dz' \right\} \tag{15}$$

The first integral may be evaluated as follows.   It is $j\omega l_0^s$ with

$$l_0^s \equiv \frac{1}{2\pi\nu} \left\{ \int_0^s \frac{dz'}{\sqrt{(z'-z)^2 + a^2}} - \int_0^s \frac{dz'}{\sqrt{(z'-z)^2 + b^2}} \right\} \tag{16}$$

Using standard tables and rearranging, the integrals reduce to

$$l_0^s = \frac{1}{2\pi\nu} \left\{ 2 \ln \frac{b}{a} - \ln F \right\} \tag{17}$$

where

$$F = \frac{[\sqrt{(s-z)^2 + b^2} + (s-z)][\sqrt{z^2 + b^2} + z]}{[\sqrt{(s-z)^2 + a^2} + (s-z)][\sqrt{z^2 + a^2} + z]} \tag{18}$$

At distances z or $(s - z)$ from each end of the line which are large compared with the distance b between centers, *i.e.*, when

$$(s - z)^2 > b^2 \tag{19}$$

$$z^2 > b^2 \tag{20}$$

it is a good approximation to neglect $\ln F$ compared with $2 \ln \dfrac{b}{a}$ so that the *external inductance* per unit length of the line reduces to

$$l_0^s \doteq \frac{1}{\pi \nu} \ln \frac{b}{a} \tag{21}$$

This is true specifically when

$$(s - z)^2 >> \frac{b^2 - a^2}{8 \ln \dfrac{b}{a}} \tag{22}$$

$$z^2 >> \frac{b^2 - a^2}{8 \ln \dfrac{b}{a}} \tag{23}$$

For lines that satisfy the condition $s >> b$, these conditions are satisfied for the entire line except for short sections near the ends. At the ends where $z = 0$ or $z = s$

$$F \doteq \frac{b}{a} \tag{24}$$

so that an error of 50 per cent is made in l_0^s at the ends when the term in F is neglected. Since F is a function of z, l_0^s is not a convenient parameter except when F is negligible. *Conventional transmission-line theory always neglects F*, so that it is a good approximation only when applied to lines that are sufficiently long ($s >> b$) to make the large error near the ends small for the line as a whole. Alternatively, the error in F may be transferred to the terminal impedance as discussed in Volume III. The parameter l_0^s in (21) is readily identified with the external inductance per loop unit length of the line as in (11.10) if the latter is modified for a simple medium of relative reluctivity ν_r by writing $\nu = \nu_r \nu_0$ for ν_0.

The second integral in (15) is

$$z_r^c = r_r^c + jx_r^c$$
$$= \frac{j\omega}{2\pi\nu} \int_0^s \left\{ \frac{f(z') \exp(-j\beta R_a) - 1}{R_a} - \frac{f(z') \exp(-j\beta R_b) - 1}{R_b} \right\} dz' \tag{25}$$

cannot be evaluated without a prior knowledge of the distribution function $f(z')$. This is known to be equal to unity if the line satisfies the condition for the near zone. It can be determined approximately if z_r^c is itself so small compared with $j\omega l_0^s$ that it is a satisfactory approximation to write

$$z^s \doteq j\omega l_0^s \tag{26}$$

With (26), the distribution function $f(z')$ is approximately independent of z_r^c and can be determined by writing $z_r^c = 0$ in the equations. The value of $f(z')$ so obtained then can be substituted in (25) to determine z_r^c. This is done in a later section. It is to be noted that z_r^c is zero if the conditions of the near zone are imposed, viz., $f(z') \doteq 1$; $\exp(-j\beta R_b) \doteq 1$.

23. Long-line Equations; Parameters of the Line.—Let it be assumed that (22.26) is a good approximation so that the equation (22.14) becomes

$$-\frac{\partial V}{\partial z} = (z^i + j\omega l_0^c)I_z = [r^i + j(x^i + \omega l_0^c)]I_z \tag{1}$$

with z^i and l_0^c constant parameters per loop unit length. This is one of the long-line equations in conventional form. It is customary to write ωl^i for x^i even though l^i is a function of frequency, so that

$$x^i + \omega l_0^c = \omega(l^i + l_0^c) = \omega l \tag{2}$$

The total inductance per loop unit length is denoted by l.

With (22.12) and (22.26)

$$j\omega W_z = z^c I_z \doteq j\omega l_0^c I_z \tag{3}$$

Upon substituting (3) in (21.2), this becomes

$$-\frac{\partial I_z}{\partial z} = \frac{j\beta^2}{\omega l_0^c} V \tag{4}$$

Since

$$\beta^2 = \frac{\omega^2}{\nu}\left(\epsilon_a - \frac{j\sigma_a}{\omega}\right) \tag{5}$$

the factor of V in the right in (4) reduces to

$$\frac{j\beta^2}{\omega l_0^s} = \frac{j\omega\epsilon_s}{\nu l_0^s} + \frac{\sigma_s}{\nu l_0^s} \equiv j\omega c + g \tag{6}$$

With (22.21), the newly defined parameters g and c become

$$g \equiv \frac{\sigma_s}{\nu l_0^s} = \frac{\pi\sigma_s}{\ln\dfrac{b}{a}} \tag{7}$$

$$c \equiv \frac{\epsilon_s}{\nu l_0^s} = \frac{\pi\epsilon_s}{\ln\dfrac{b}{a}} \tag{8}$$

The parameter g is the *leakage conductance* per unit length of the line; the parameter c is the *capacitance* per unit length of the line. With this notation, (4) at once reduces to the conventional form

$$-\frac{\partial I_s}{\partial z} = (g + j\omega c)V \tag{9}$$

If the ratio b/a is not so large that proximity effect is negligible, the expressions for l_0^s, g, and c are readily generalized by writing b_s for b as discussed in Secs. 12 and 20. Thus,

$$l_0^s = \frac{1}{\pi\nu}\ln\left(\frac{b_s}{a}\right) = \frac{1}{\pi\nu}\cosh^{-1}\frac{b}{2a} \tag{10}$$

$$g = \frac{\pi\sigma_s}{\ln\dfrac{b_s}{a}} = \frac{\pi\sigma_s}{\cosh^{-1}\dfrac{b}{2a}} \tag{11}$$

$$c = \frac{\pi\epsilon_s}{\ln\dfrac{b_s}{a}} = \frac{\pi\epsilon_s}{\cosh^{-1}\dfrac{b}{2a}} \tag{12}$$

The internal impedance per loop unit length has a simple form only for large values of $\beta_c a$ where β_c applies to the conductor as discussed in Sec. 12. In this case, for a loop unit length of the line

$$r^i + jx^i = \frac{1}{\pi a}\sqrt{\frac{\omega}{2\sigma_c\nu_c\left[1 - \left(\dfrac{2a}{b}\right)^2\right]}}; \qquad \beta_c a \geq 10 \tag{13}$$

The subscript c designates a material parameter defined for the conductor as distinct from parameters for the dielectric which are written without subscript.

The dependent variables in (1) and (9) are easily separated by differentiation to give

$$\frac{\partial^2 V}{\partial z^2} - \gamma^2 V = 0 \tag{14}$$

$$\frac{\partial^2 I_z}{\partial z^2} - \gamma^2 I_z = 0 \tag{15}$$

with

$$\gamma^2 \equiv (r^i + j\omega l)(g + j\omega c) \tag{16}$$

General solutions of (14) and (15) involving two arbitrary constants of integration C_1 and C_2 are

$$V = C_1 \exp(\gamma z) + C_2 \exp(-\gamma z) \tag{17}$$

$$I = \left(\frac{\gamma}{r^i + j\omega l}\right) [-C_1 \exp(\gamma z) + C_2 \exp(-\gamma z)] \tag{18}$$

A symbol Z_c, called the characteristic impedance, is usually introduced as follows:

$$Z_c \equiv \frac{r^i + j\omega l}{\gamma} = \sqrt{\frac{r^i + j\omega l}{g + j\omega c}} \tag{19a}$$

Its approximate magnitude on low-loss lines is

$$R_c \doteq \sqrt{\frac{l_0^i}{c}} = \frac{\zeta}{\pi} \cosh^{-1} \frac{b}{2a} = \frac{120}{\sqrt{\nu_r \epsilon_{cr}}} \cosh^{-1} \frac{b}{2a} \tag{19b}$$

With (19a), (17) and (18) become

$$V = D_1 \cosh \gamma z + D_2 \sinh \gamma z \tag{20}$$

$$I = -\frac{1}{Z_c} [D_1 \sinh \gamma z + D_2 \cosh \gamma z] \tag{21}$$

where D_1 and D_2 are complex constants of integration.

The application of the above solutions to practical problems constitutes a principal part of Volume III on Transmission Circuits. It is shown there that a good approximation for the distribution of current in a long two-wire line that is highly conducting and immersed in a good dielectric is

$$I_z = I_s \frac{\cos [\beta_z(s - z) + \Phi_s']}{\cos \Phi_s'} \tag{22}$$

I_s is the current at the end $z = s$ entering and leaving a terminating impedance Z_s that is characterized by a phase function Φ_s'.

If the termination is a conducting bridge of the same wire as the line,

$$\Phi'_s = \beta_s k_s; \qquad k_s \doteq \frac{b}{2} \tag{23}$$

These formulas are written down at this point without proof or discussion merely in order that the distribution (22) may be used to evaluate the radiation resistance of a resonant two-wire line with resonance defined by

$$\beta_s s + \Phi'_0 + \Phi'_s = n\pi \tag{24}$$

Φ'_0 is a phase function characteristic of the termination at $z = 0$. If this is a piece of the same wire as the line, Φ'_0 has the same value as Φ'_s in (23). The phase function of a termination is defined and described in detail in Volume III.[1]

If a slice generator is inserted in the termination at $z = 0$, the distribution function for current referred to I_0 is obtained from (22). It is

$$f(z) = \frac{I_s}{I_0} = \frac{\cos\,[\beta_s(s - z) + \Phi'_s]}{\cos\,[\beta_s s + \Phi'_s]} \tag{25}$$

It is this distribution function that will be used to determine the external or radiation resistance of a resonant line.

24. Radiation Resistance of a Balanced Two-wire Line.—The external or radiation resistance of a resonant line *in space* may be determined using the general formula for the external self-impedance.

$$Z^e_{11} = R^e_1 + jX^e_1 = \frac{j\omega}{4\pi\nu_0} \oint_s \left(ds, \oint_s f(s') \frac{e^{-j\beta_0 R_{11}}}{R_{11}} ds'\right) \tag{1}$$

Since it is assumed that R^e_1 is sufficiently small so that it plays no significant part in determining the distribution function for current, the impedance of the line as seen by the point generator at the center of the conducting bridge at $z = 0$ is approximately the input impedance of a section of line as determined neglecting radiation, plus a small—and actually negligible—R^e_1. That is

$$Z_{11} = Z_{1n} \text{ (line theory)} + R^e_1 \tag{2}$$

The input impedance Z_{1n} as determined from line theory is considered in detail in Volume III. The purpose in this section

[1] See also R. KING, *J. Applied Phys.*, **14**, 577 (1943).

is to evaluate R_1^c in order to learn its order of magnitude and the specific conditions under which it actually is negligible in (2). Therefore, only the real part of (1) need be determined. The evaluation of R_1^c from (1) can be carried out in two parts because the line and the terminations are mutually perpendicular. Thus,

$$R_1^c = R_{\text{term}}^c + R_{\text{line}}^c \tag{3}$$

The distribution function for current is given by (23.25) for the line. It is

$$f(z) = \frac{\cos\left[\beta_0(s - z) + \Phi_s'\right]}{\cos\left[\beta_0 s + \Phi_s'\right]} \tag{4}$$

The terminations will be assumed to be either straight conductors or loosely wound coils of only a few turns. The current in the two terminations may be assumed essentially uniform under these conditions and given by I_0 and I_s where

$$I_s = I_0 \frac{\cos \Phi_s'}{\cos(\beta_0 s + \Phi_s')} \tag{5}$$

Accordingly, at $z = 0$ in CD, Fig. 10.1,

$$f(x) = 1 \cdot \tag{6}$$

at $z = s$ in EF,

$$f(x) = \frac{\cos \Phi_s'}{\cos(\beta_0 s + \Phi_s')} \tag{7}$$

The contribution to R_1^c by the terminations is given by four integrals. Referring to Fig. 10.1, they are

$$R_{\text{term}}^c = R_{cc} + R_{ce} + R_{ee} + R_{ee} \tag{8a}$$

The integrals are evaluated as follows:

$$R_{cc} = \frac{\omega}{4\pi\nu_0} \int_0^b dx \int_0^b \sin \frac{\beta_0 R_{11}}{R_{11}} dx' \tag{8b}$$

with $R_{11} = \sqrt{(x' - x)^2 + a^2}$. Since $\beta_0 R_{11}$ in (8b) cannot exceed $\beta_0 b$ significantly and the inequality $\beta_0 b << 1$ is postulated,

$$\sin \beta_0 R_{11} \doteq \beta_0 R_{11}.$$

The integration is carried out directly to give

$$R_{cc} = \frac{\omega\beta_0}{4\pi\nu_0} b^2 = \frac{\zeta_0}{4\pi} \beta_0^2 b^2 = 30\beta_0^2 b^2 \text{ ohms} \tag{9}$$

where

$$\zeta_0 = \frac{1}{\sqrt{\epsilon_0 \nu_0}} = 376.7 \text{ ohms} \doteq 120\pi \text{ ohms} \tag{10}$$

so that $\zeta_0/4\pi \doteq 30$ ohms. . Similarly,

$$R_{es} = \frac{\zeta_0}{4\pi} \beta_0^2 b^2 \frac{\cos \Phi_s'}{\cos (\beta_0 s + \Phi_s')} \tag{11}$$

(It is important to note that the first power terms in $\beta_0 R_{11}$ do not vanish in this case as in the rectangle confined entirely to the near zone where the terms in $(\beta_0 R_{11})^3$ were the leading terms contributing to R_1^e.)

In R_{ce}, $ds = -dx$; $ds' = dx'$, so that

$$R_{ce} = -\frac{\omega}{4\pi\nu_0} \frac{\cos \Phi_s'}{\cos (\beta_0 s + \Phi_s')} \int_0^b dx \int_0^b \frac{\sin \beta_0 R_{11}}{R_{11}} dx' \tag{12}$$

with

$$R_{11} = \sqrt{s^2 + (x' - x)^2} \tag{13}$$

Since x cannot exceed b and the inequalities $b^2 << s^2$ and $\beta_0 b << 1$ are assumed, it is possible to write

$$R_{11} \doteq s \tag{14}$$

in both argument and denominator in (13). With (10) the result is

$$R_{ce} = -\frac{\zeta_0}{4\pi} \beta_0^2 b^2 \frac{\sin \beta_0 s}{\beta_0 s} \frac{\cos \Phi_s'}{\cos (\beta_0 s + \Phi_s')} \tag{15}$$

Similarly

$$R_{ec} = -\frac{\zeta_0}{4\pi} \beta_0^2 b^2 \frac{\sin \beta_0 s}{\beta_0 s} \tag{16}$$

Upon adding (9), (11), (15), and (16),

$$R_{term}^e = \frac{\zeta_0}{4\pi} \beta_0^2 b^2 \left(1 + \frac{\cos \Phi_s'}{\cos (\beta_0 s + \Phi_s')} \right) \left(1 - \frac{\sin \beta_0 s}{\beta_0 s} \right) \tag{17}$$

The contribution to the external resistance by the line is more difficult to evaluate, but a result in closed form can be obtained as follows. Using (4) in (1) together with the requirement that currents at opposite points be in the same direction around the contour (in opposite directions along the line), the complete integral is

$$R_{\text{line}}^{a} = \frac{\omega}{2\pi\nu_0} \int_0^s dz \int_0^s \frac{\cos\,[\beta_0(s - z') + \Phi_s']}{\cos\,(\beta_0 s + \Phi_s')}$$
$$\left\{\frac{\sin\,\beta_0 R_a}{R_a} - \frac{\sin\,\beta_0 R_b}{R_b}\right\} dz' \quad (18)$$

where

$$R_a = \sqrt{(z' - z)^2 + a^2} \quad (19)$$
$$R_b = \sqrt{(z' - z)^2 + b^2} \quad (20)$$

Note that Φ_s' is a constant independent of primed and unprimed variables.   Let the following symbols be introduced temporarily:

$$U \equiv \beta_0(z' - z); \quad \psi \equiv \beta_0(s - z) + \Phi_s' \quad (21)$$
$$B \equiv \beta_0 b; \quad A \equiv \beta_0 a \quad (22)$$

With

$$\cos\,[\beta_0(s - z') + \Phi_s'] = \cos U \cos \psi + \sin U \sin \psi \quad (23)$$

and the appropriate changes in limits, (18) becomes

$$R_{\text{line}}^{a} = \frac{\omega}{2\pi\nu_0 \cos\,(\beta_0 s + \Phi_s')} \left\{ \left[\int_0^s \cos \psi \, dz \int_{-\beta_0 z}^{\beta_0(s-z)} \right. \right.$$
$$\cos U \left[\frac{\sin \sqrt{U^2 + A^2}}{\sqrt{U^2 + A^2}} - \frac{\sin \sqrt{U^2 + B^2}}{\sqrt{U^2 + B^2}}\right] dU + \int_0^s \sin \psi \, dz$$
$$\left. \int_{-\beta_0 z}^{\beta_0(s-z)} \sin U \left[\frac{\sin \sqrt{U^2 + A^2}}{\sqrt{U^2 + A^2}} - \frac{\sin \sqrt{U^2 + B^2}}{\sqrt{U^2 + B^2}}\right] dU \right\} \quad (24)$$

Subject to the inequalities

$$B^2 \equiv \beta_0^2 b^2 << 1; \quad A^2 \equiv \beta_0^2 a^2 << 1 \quad (25)$$

the differences between the trigonometric functions in (24) may be simplified.   By expanding in series and neglecting all fourth power and higher terms in B^2 and A^2, the following is obtained:

$$\frac{\sin \sqrt{U^2 + A^2}}{\sqrt{U^2 + A^2}} - \frac{\sin \sqrt{U^2 + B^2}}{\sqrt{U^2 + B^2}}$$
$$\doteq (B^2 - A^2)\left[\frac{1}{3!} - \frac{2U^2}{5!} + \frac{3U^4}{7!} - \frac{4U^6}{9!} + \cdots \right] \quad (26)$$

The series in square brackets in (26) is given exactly by the following:

$$\frac{\sin \sqrt{U^2 + A^2}}{\sqrt{U^2 + A^2}} - \frac{\sin \sqrt{U^2 + B^2}}{\sqrt{U^2 + B^2}}$$
$$= \frac{B^2 - A^2}{2}\left[\frac{\sin U}{U^3} - \frac{\cos U}{U^2}\right] \quad (27)$$

Accordingly, (24) reduces to

$$
R_{\text{line}}^{s} = \frac{\omega\beta_0^2(b^2 - a^2)}{4\pi\nu_0 \cos{(\beta_0 s + \Phi_s')}} \left\{ \int_0^s \cos\psi\, dz \int_{-\beta_0 s}^{\beta_0(s-z)} \left[\frac{\sin 2U}{2U^3} \right.\right.
$$
$$
\left.\left. - \frac{\cos^2 U}{U^2} \right] dU + \int_0^s \sin\psi\, dz \int_{-\beta_0 s}^{\beta_0(s-z)} \left[\frac{\sin^2 U}{U^3} - \frac{\sin 2U}{2U^2} \right] dU \right\}
$$

$$(28)$$

These integrals are all readily reduced to tabulated forms. The result is

$$
R_{\text{line}}^{s} = \frac{\omega\beta_0^2(b^2 - a^2)}{4\pi\nu_0 \cos{(\beta_0 s + \Phi_s')}} \left\{ \int_0^s \cos{[\beta_0(s - z) + \Phi_s']} \left[\frac{1}{2\beta_0(s - z)} \right.\right.
$$
$$
\left. + \frac{1}{2\beta_0 z} - \frac{\sin 2\beta_0(s - z)}{4\beta_0^2(s - z)^2} - \frac{\sin 2\beta_0 z}{(2\beta_0 z)^2} \right] dz + \int_0^s \sin{[\beta_0(s - z) + \Phi_s']}
$$
$$
\left. \left[-\frac{1}{4\beta_0^2(s - z)^2} + \frac{1}{(2\beta_0 z)^2} + \frac{\cos 2\beta_0(s - z)}{4\beta_0^2(s - z)^2} - \frac{\cos 2\beta_0 z}{(2\beta_0 z)^2} \right] dz \right\} \quad (29)
$$

With a little manipulation, (29) can be brought into the following form:

$$
R_{\text{line}}^{s} = \frac{\omega\beta_0(b^2 - a^2)}{8\pi\beta_0 \cos{(\beta_0 s + \Phi_s')}} \left\{ [\cos{(\beta_0 s + \Phi_s')} + \cos\Phi_s'] \int_0^{\beta_0 s} \right.
$$
$$
\left. \left[\frac{\cos v}{v} - \frac{\sin v}{v^2} \right] dv + [\sin{(\beta_0 s + \Phi_s')} - \sin\Phi_s'] \int_0^{\beta_0 s} \frac{\sin v}{v}\, dv \right\} \quad (30)
$$

Since

$$
\int \frac{\sin v}{v^2}\, dv = -\frac{\sin v}{v} + \int \frac{\cos v}{v}\, dv \tag{31}
$$

and

$$
\int_0^x \frac{\sin x}{x}\, dx \equiv Si(x) \tag{32}
$$

where $Si(x)$ is the integral sine (tabulated and discussed in Volume II and in standard tables), (30) integrates into

$$
R_{\text{line}}^{s} = \frac{\zeta_0}{8\pi} \frac{\beta_0^2(b^2 - a^2)}{\cos{(\beta_0 s + \Phi_s')}} \left\{ \left(\frac{\sin\beta_0 s}{\beta_0 s} - 1 \right) [\cos{(\beta_0 s + \Phi_s')} + \cos\Phi_s'] \right.
$$
$$
\left. + Si(\beta_0 s) [\sin{(\beta_0 s + \Phi_s')} - \sin\Phi_s'] \right\} \quad (33)
$$

The complete expression for the external or radiation resistance of the terminated line as seen by a generator symmetrically placed at the center of the terminal impedance Z_0 at $z = 0$ is given by (3) with (17) and (33). It is

$$R_1^e = \frac{\zeta_0}{8\pi} \frac{\beta_0^2}{\cos(\beta_0 s + \Phi_s')} \left\{ (b^2 + a^2)\left(1 - \frac{\sin \beta_0 s}{\beta_0 s}\right)[\cos(\beta_0 s + \Phi_s') \right.$$
$$\left. + \cos \Phi_s'] + (b^2 - a^2)Si(\beta_0 s)[\sin(\beta_0 s + \Phi_s') - \sin \Phi_s'] \right\} \quad (34)$$

This is the final expression; it applies specifically to a line for which it is possible to write $b^2 >> a^2$, so that the terms in a^2 vanish, and which has terminations equivalent to straight conductors for which Φ_0' and Φ_s' are small. It is a good approximation for lines with $b^2 > a^2$; it may also be used for lines with reactive terminations consisting of coils or condensers for which the assumption of a current or an equivalent current of sensibly uniform amplitude from one line to the other at $z = s$ is possible. Resistive terminations with resistances near $|Z_c|$ are not included because the distribution function (4) is then not a good approximation as described in Volume III. Open ends also are *not* included. The analysis of a line with one or two open ends must be carried out using (1.17) rather than (1.18) because the circuit is neither a closed conducting path nor a quasi-closed path with only a narrow gap in a condenser. A line with open ends is, in effect, two open circuits or antennas close together. It is analyzed in Volume II as a special case of two parallel antennas close together.

25. Radiation Resistance of a Resonant Two-wire Line.—A resonant line is defined by the condition (Volume III)

$$\beta_0 s = n\pi - \Phi_0' - \Phi_s' \quad (1)$$

where n is any integer and Φ_0' and Φ_s' are phase functions of the terminations. For use in (24.34), Φ_0' and Φ_s' must be small. If (1) is solved for $\beta_0 s + \Phi_s'$ and substituted in (24.34), the following trigonometric functions are encountered:

$$\cos(\beta_0 s + \Phi_s') = (-1)^n \cos \Phi_0' \quad (2a)$$
$$\sin(\beta_0 s + \Phi_s') = -(-1)^n \sin \Phi_0' \quad (2b)$$
$$\sin \beta s = -(-1)^n \sin(\Phi_0' + \Phi_s') \quad (2c)$$

With (1) and (2), (24.34) becomes

$$R_1^e = \frac{\zeta_0}{8\pi} \frac{\beta_0^2}{(-1)^n \cos \Phi_0'}$$

$$\left\{ (b^2 + a^2) \left[1 + \frac{(-1)^n \sin (\Phi_0' + \Phi_s')}{n\pi - \Phi_0' - \Phi_s'} \right] [\cos \Phi_s' + (-1)^n \cos \Phi_0'] \right.$$

$$\left. - (b^2 - a^2) Si(n\pi - \Phi_0' - \Phi_s')[\sin \Phi_s' + (-1)^n \sin \Phi_0'] \right\} \quad (3)$$

This formula is applicable when Φ_0' and Φ_s' are small. It does not apply when Φ_s' and Φ_0' are $\pi/2$ (open end).

It is clear that for all cases with

$$\Phi_0' = \Phi_s'; \qquad n \text{ odd}; \qquad R_1^e = 0 \qquad (4)$$

Since higher order terms than $\beta_0^2 b^2$ and $\beta_0^2 a^2$ were neglected in deriving (3), the conclusion (4) means that all resonant lines with identical terminations at each end and with currents through them in *opposite* directions *around* the contour, as is necessarily true with n odd in (1), have radiation resistances proportional to no lower powers of $\beta_0 b$ than $(\beta_0 b)^4$. Actually if fourth-power terms are included in (24.9), (24.11), and (24.27) and the explicit formula evaluated all terms involving b and a raised to any power are found to have the expressions in small square brackets in (3) as factors, so that (4) is identically true.

If the currents in the two terminations are in the same direction around the contour and the terminal impedances are identical,

$$\Phi_0' = \Phi_s'; \qquad n \text{ even} \qquad (5)$$

$$R_1^e = \frac{\zeta_0 \beta_0^2}{4\pi} \left\{ (b^2 + a^2) \left(1 + \frac{\sin 2\Phi_0'}{n\pi - 2\Phi_0'} \right) \right.$$

$$\left. - (b^2 - a^2) Si(n\pi - 2\Phi_0') \tan \Phi_0' \right\} \quad (6)$$

If the terminations consist of straight conductors,

$$\Phi_0' = \Phi_s' = \beta_0 k \doteq \frac{\beta_0 b}{2} \qquad (7)$$

and

$$\sin 2\Phi_0' \doteq \beta_0 b \qquad (8)$$
$$\tan \Phi_0' \doteq \tfrac{1}{2}\beta_0 b \qquad (9)$$

For simplicity let it be assumed that

$$b^2 >> a^2 \qquad (10)$$

With (7) to (10), (6) becomes

$$R_1^e = \frac{\zeta_0 \beta_0^2 b^2}{4\pi}\left[\left(1 + \frac{\beta_0 b}{n\pi - \beta_0 b}\right) - \frac{1}{2}\beta_0 b\, Si(n\pi - \beta_0 b)\right]$$

$$(n \text{ even}) \quad (11)$$

In particular, with $\zeta_0 \doteq 120\pi$

$$R_1^e = 30\beta_0^2 b^2(1 - 0.55\beta_0 b); \qquad n = 2 \qquad (12a)$$

$$R_1^e = 30\beta_0^2 b^2(1 - 0.75\beta_0 b); \qquad n = 4 \qquad (12b)$$

$$R_1^e = 30\beta_0^2 b^2\left(1 - \frac{\pi}{4}\beta_0 b\right); \qquad n \text{ even}; \quad n > 4 \quad (12c)$$

If $\beta_0 b$ is sufficiently small so that it is possible to write

$$\beta_0 b << 1 \tag{13}$$

all the above expressions reduce to

$$R_1^e \doteq 30\beta^2 b^2; \qquad n \text{ even} \tag{14}$$

Fɪɢ. 25.1.—Resonant two-wire line;   Fɪɢ. 25.2.—Resonant two-wire line;
n odd.   n even.

These expressions give the input radiation resistance (in this case also referred to maximum sinusoidal current) for the entire line including the terminations as shown on Fig. 25.1. The arrangements with n odd (Fig. 25.2) all have radiation resistances that are zero in the idealized case of perfect symmetry. Asymmetry introduced by attenuation presumably leads to a very small radiation resistance.

26. The Balanced Four-wire Line.—The differential equation and parameters of the balanced four-wire line shown in Fig. 26.1 in which the two wires at the end of one diagonal in any cross section have equal currents in one direction while the two wires at the ends of the other diagonal have equal currents in the opposite direction are easily derived. The four wires of the line

may be analyzed as two parallel two-wire transmission lines each of which differs from the isolated two-wire line only in the parameters and not in the form of the equations. The fundamental equations for the potential differences in each two-wire line are

$$-\frac{\partial V}{\partial z} = z'I + j\omega W_s \tag{1}$$

$$-\frac{\partial W_s}{\partial z} = -j\frac{\beta^2}{\omega} V \tag{2}$$

Driving terminals → ↗ ↘ ← Load terminals

Fig. 26.1.—Balanced four-wire line arranged at the corners of a square. Generator and load are connected across the centers of the diagonal end connections.

The symbols have the same meaning as in the two-wire line. It is assumed that $b^2 >> a^2$. The four wires are at the corners of a square of side b and diagonal $b\sqrt{2}$ in each cross section.

The vector potential difference corresponding to (22.9) for the isolated two-wire line is given for the four-wire line by

$$W_s = \frac{1}{2\pi\nu} \int_0^s I_1' \left\{ \frac{\exp(-j\beta R_a)}{R_a} - \frac{2\exp(-j\beta R_b)}{R_b} \right. $$
$$\left. + \frac{\exp(-j\beta R_c)}{R_c} \right\} dz' \tag{3}$$

where

$$R_a = \sqrt{(z'-z)^2 + a^2} \tag{4a}$$
$$R_b = \sqrt{(z'-z)^2 + b^2} \tag{4b}$$
$$R_c = \sqrt{(z'-z)^2 + 2b^2} \tag{4c}$$

The current I_1' is that in each conductor, and hence one-half the total current I_s' in each direction. That is,

$$I_1' = \tfrac{1}{2}I_s' \tag{5}$$

With (5) and a regrouping of terms, (3) becomes

$$W_s = \frac{1}{4\pi\nu} \int_0^s I_s' \left\{ \left[\frac{\exp(-j\beta R_a)}{R_a} - \frac{\exp(-j\beta R_b)}{R_b} \right] \right. $$
$$\left. - \left[\frac{\exp(-j\beta R_b)}{R_b} - \frac{\exp(-j\beta R_c)}{R_c} \right] \right\} dz' \tag{6}$$

This is exactly like (22.9) in each pair of terms. Hence the integration can be carried out as in Sec. 22. The principal term corresponding to (22.16) integrates into

$$l_0^c = \frac{1}{4\pi\nu}\left\{2\ln\frac{b}{a} - \ln F - 2\ln\frac{b\sqrt{2}}{b} + \ln G\right\} \qquad (7a)$$

or

$$l_0^c = \frac{1}{4\pi\nu}\left\{2\ln\frac{b}{a\sqrt{2}} - \ln\frac{F}{G}\right\} \qquad (7b)$$

where

$$\frac{F}{G} = \left[\frac{[\sqrt{(s-z)^2+b^2}+(s-z)]^2[\sqrt{z^2+b^2}+z]^2}{[\sqrt{(s-z)^2+a^2}+(s-z)][\sqrt{z^2+a^2}+z]}{[\sqrt{(s-z)^2+2b^2}+(s-z)][\sqrt{z^2+2b^2}+z]}\right] \qquad (8)$$

At distances from the end which are large compared with the diagonal $b\sqrt{2}$, $\ln(F/G)$ may be neglected, leaving

$$l_0^c \doteq \frac{1}{2\pi\nu}\ln\left(\frac{b}{a\sqrt{2}}\right) = \frac{1}{2\pi\nu}\left[\ln\frac{b}{a} - 0.3464\right] \qquad (9)$$

This formula corresponds to (22.21) for the isolated two-wire line.

The leakage conductance and capacitance per unit length of the four-wire line are defined as in (23.6) using (9).

$$g = \frac{\sigma_e}{\nu l_0^c} = \frac{2\pi\sigma_e}{\ln\frac{b}{a\sqrt{2}}} \qquad (10)$$

$$c = \frac{\epsilon_e}{\nu l_0^c} = \frac{2\pi\epsilon_e}{\ln\frac{b}{a\sqrt{2}}} \qquad (11)$$

The internal impedance per unit length of the four-wire line is one-half that for the two-wire line if the separation of the wires is sufficient to make rotational symmetry a good approximation in the interior of each conductor. The approximate magnitude of the characteristic impedance is

$$R_c \doteq \sqrt{\frac{l_0^c}{c}} = \frac{\zeta_0}{2\pi}\ln\frac{b}{a\sqrt{2}} = 60\ln\frac{b}{a} - 20.78 \qquad (12)$$

The external or radiation resistance R_1^c of the terminated four-

wire line connected in the completely symmetrical manner shown in Fig. 26.1 may be evaluated using the contour integral (24.1). Since there are two paths in parallel, the resistance R_i^s seen by the generator is one-half that obtained by integrating around one closed contour. It is readily verified that all terms in $\beta_0^2 b^2$ cancel in such an integration and that R_i^s contains no smaller powers of $\beta_0 b$ than $(\beta_0 b)^4$ for any lengths. The radiation resistance is, therefore, very small. It may be determined specifically by an evaluation like that in Sec. 24 with fourth-power terms retained.

The coaxial line is analyzed in Volume III. Unbalanced transmission lines and open circuits are antennas. They are studied in Volume II.

HISTORICAL AND CRITICAL RETROSPECT

In formulating the theoretical foundations of electromagnetic engineering a purely logical approach was chosen in the interest of simplicity and coherence. As a result, the historical development of the science of electricity was ignored completely, and only casual mention was made of those illustrious men of genius whose imagination, perseverance, and skill erected one of the truly great monuments to the human intellect. While a brief summary is not adequate to do justice to the amazing and fascinating account of intellectual adventure and achievement which is the true history of electrical science, a short outline of its high lights will, nevertheless, serve to illuminate the following important lessons which may be learned from the story of electromagnetism: (1) An entire field of engineering and of technological industry may owe its existence to abstract mathematical theory. (2) Preconception and tradition may be as important in determining the nonmathematical descriptive text of a physical theory as scientific fact.

The history of electromagnetism may be treated conveniently under three headings, which characterize the guiding principle in successive periods of its growth. From the purely mathematical point of view, the combined achievements in all three periods form a single, powerful tool for predicting pointer readings in intricate electrophysical and electrotechnical experiments. On the other hand, the physical theories and the philosophical implications that are associated with the mathematical symbolism differ greatly and fundamentally in successive periods. Indeed, it is amazing that different physical pictures or philosophical interpretations could be so irreconcilable and so positive when there is no conflict in the mathematics for which they seek to provide the explanatory text. The basic conflict in the several interpretations of electromagnetic action center about the fundamental physical and psychological differences between *action by contact* and *action at a distance*. Shall it be postulated that two bodies can interact only by direct contact with each other or with a continuous material medium that makes contact

with both? Or, shall it be assumed that nature may be so constituted that the interaction of two or more bodies requires no contact and may be accomplished when they are widely separated in a matterless void? In other words, do gravitational attraction, electric and magnetic attraction and repulsion actually require contact with a physically meaningful material medium between two interacting entities so that no experimentally verifiable theory can be devised without it? Or, is it adequate to proceed on the assumption that such a medium does not actually exist and, hence, is not required in a consistent and comprehensive theory? In studying the answer to this question, care must be taken to distinguish between arguments based on traditions of thought and evidence based on experimental observation.

The concept of action by contact or through a material medium is familiar to all in the sense of pushing a wheelbarrow or pressing on the pedal of a hydraulic brake. Historically it is as old as coordinated thought. Aristotle believed in action by contact. Newton in a letter to Bentley considered the very idea that one body could act on another through empty space absurd. Perhaps the concept of action at a distance might have been acceptable to early scientific thinkers if they had been more interested in natural law as revealed in visible nature than in systems of their own design based upon their preconceptions. For example, if they had merely taken for granted that the falling of a body is due to an interaction between it and the earth, and that the motion of the planets is possible because of mutual action between them ánd the sun, action at a distance might have been accepted as readily as action by contact. Actually, for the evolving scientific mind, an explanation for the process of *throwing* a stone was somehow so much nearer at hand than for just *dropping* the stone that action by contact became reasonable, action at a distance puzzling and absurd. Thus, if the stone was not set in motion by a *visible push*, there must be an invisible something to force it to the surface of the earth.

In spite of his own belief in action by contact or through a mechanical medium, it was Newton who was largely responsible for the first widespread acceptance of the concept of action at a distance. He expressed the law of gravitation in terms of a force acting at a distance to attract two masses. This formula-

tion was destined to plant in the minds of generations to come the idea that action at a distance between masses without reference to a medium might be a fundamental property of nature. But for more than half a century action by contact with a medium dominated scientific thought, and attempts were made, notably by the great mathematician Euler in the years 1746–1748, to formulate comprehensive theories of gravitation, of the transmission of light (following earlier work by Huygens), and of the interaction of permanent magnets in which an all-pervading material medium, the ether, played a dominant role. Euler described the ether as a perfectly elastic substance of very small density.

Action at a Distance between Charges and Currents.—In view of the widespread and traditional belief in action by contact in a medium in other fields, it is surprising that early work in electrostatics and electromagnetism developed in a way that led to the general acceptance of the conception of *action at a distance between electric charges and currents*. This may be traced directly to Newton's inverse square law of gravitation which served as a model in the formulation of the laws of force in electrostatics and magnetostatics. The inverse square law for electric charges was first stated by Priestley in 1766 using an electrometer. It was discussed by Cavendish in 1771 and finally formulated by Coulomb in 1785 following experiments using a torsion balance. The inverse square law for magnetic poles was first expressed by Michell in 1750. The magnetic effects of currents were investigated especially by Oersted, Biot, Savart, and Faraday in the time from 1820 to 1821. In this period, Laplace formulated a law of action at a distance between elements of current and magnetic poles. In 1823, Ampère conducted his celebrated experimental researches and formulated his fundamental law of force between currents, a law of action at a distance. Ohm's law was announced in 1826; Faraday formulated his law of induction in 1832; Lenz's law followed in 1834. In this period the theory of potential was developed by Gauss and independently by Green. Later work on induction due to moving conductors carrying currents and due to the rise and decay of currents was published by Neumann and Weber in 1845–1847. In 1845, Kirchhoff worked out his well-known laws for closed circuits.

In 1846, Weber began a systematic study of Ampère's funda-
mental law of instantaneous action at a distance from the point
of view of the interaction between discrete charges.   He took
into account both the velocities and accelerations of the charges.
Other important work in the formulation of a fundamental law
of electromagnetic action at a distance was done by Grassmann,
Riemann, and Clausius.   The first suggestion of a *propagation
of such action with a finite velocity* appears to be due to Gauss
(1845).   In 1858, the celebrated mathematician Riemann pre-
sented a paper in which he introduced a finite velocity of propaga-
tion of electromagnetic effects and showed this velocity to be
equal to that of light.   In 1867, the Danish physicist L. Lorenz
(not to be confused with the Dutch physicist H. A. Lorentz)
extended earlier work by Neumann to obtain formulas for
retarded scalar and vector potentials essentially like those which
are the basis of the modern theory of retarded action at a dis-
tance without an intervening medium.   Recognizing that light
was electromagnetic in nature, Lorenz concluded that a hypo-
thetical, all-pervading medium such as the ether was no more
required for the transmission of light than for other electro-
magnetic phenomena.

The actual formulation of the general laws of electromagnetism
in terms of a retarded action at a distance between charges and
currents without an accessory medium was thus within reach.
It was, however, not to be enunciated for another half-century
because the work of Lorenz was completely eclipsed by the
independent, brilliant, and comprehensive theory of Maxwell
which was destined to entangle all electromagnetic phenomena
in the ether hypothesis.

Action in an Elastic Ether.—In 1847, Kelvin exhibited analo-
gies between electric and elastic phenomena and made a first
attempt to treat electrical experiments by the equations of the
theory of elasticity.   This work, together with Faraday's con-
ceptions of lines and tubes of electric and magnetic force, inspired
Maxwell to begin in 1855 the investigation of electrical phe-
nomena from the point of view of conditions existing in a hypo-
thetical, continuous, all-pervading mechanical medium, rather
than in terms of interactions of discrete charges and currents.
As a first step, he showed that the magnetic B vector could be
represented formally by the velocity of flow in an incompressible

fluid. He ultimately devised a set of field equations governing all electromagnetic phenomena. They were so general that their solutions included Coulomb's law, Ampère's formula, the laws of induction, and the propagation of electromagnetic effects with the velocity of light. Maxwell, like Lorenz, identified light as an electromagnetic phenomenon, but he concluded that all electromagnetic effects must be propagated in the optical ether. Thus while Lorenz, proceeding from the laws of action between charges and currents and apparently completely unaware of the work three years earlier by Maxwell, wished to dispense with the optical ether as unnecessary, Maxwell's *initial assumption* that all electromagnetic effects resided in a mechanical medium necessarily led him to the opposite conclusion. Instead of abandoning the optical ether as unnecessary, Maxwell elevated it to the most fundamental role in electromagnetic theory. Indeed, electromagnetism following Maxwell became a mechanical theory of the state of an elastic medium.

After Maxwell's intricate analytical formulation had been simplified by Heaviside and Hertz, the field equations became the core of electromagnetic theory. Although the equations themselves in no way required the Maxwellian interpretation in terms of the ether, this was universally accepted, and the great success of the equations in predicting and coordinating diverse electromagnetic effects was assumed to substantiate the validity not only of the equations, but of the mechanical models in terms of which they had been derived. Faraday's lines and tubes of force became essential properties of the ether along with strains and displacements. Energy was pictured as stored in the strained medium, much as if this were filled with stretched rubber bands, and in 1884 Poynting defined the vector bearing his name which was assumed to govern the flow of energy distributed in the medium. In 1887, Hertz demonstrated experimentally the existence of what were interpreted to be electromagnetic waves *in the ether*. In a series of brilliant experiments he measured their velocity of propagation and showed their properties of reflection, refraction, and polarization. *Thus, the first radio transmission was achieved in order to confirm the mathematical predictions of the field equations of Maxwell.*

From the point of view of mechanically minded physicists, the entire Maxwellian theory together with all its assumptions and

interpretations was ideal, it fitted so perfectly into their precon-
ceptions. Every triumph of the field equations became new
proof of the correctness not only of the equations, but more
significantly of the physical pictures devised by Faraday and
Maxwell. In time, especially in the views of practical scien-
tists, the pictures rather than the intricate equations became
identified with the theory. The lines of force, the displacement,
and the stored energy became increasingly real in the minds of
students as texts and teachers used them and embellished them.
The mathematical predictions of Maxwell's equations thus
became completely enveloped in an ether the existence of which
had never been proved either real or necessary and the strange
properties of which were dogmatically inculcated into the minds
of students as truth itself.

Retarded Action at a Distance between Charges and Currents.
In 1895, the Dutch physicist H. A. Lorentz took a great step in
coordinating the early restricted theories of action between
charges and currents with Maxwell's general theory of the state
of the ether. Lorentz conceived matter to contain electric
charges (electrons) that act on each other in various ways to
produce all electromagnetic (including optical) effects. Instead
of acting on each other at a distance, Lorentz assumed the
charges and their motions to establish in Maxwell's ether pre-
cisely those conditions of strain and displacement required by
the Maxwellian interpretation of the field equations. In
Lorentz's theory these conditions constitute the so-called
electromagnetic field, and it is the field that characterizes and is
propagated by the ether. From its primary Maxwellian role
as ultimate seat of all electromagnetic and optical phenomena,
the ether is reduced to play only a secondary part in Lorentz's
theory where it is a mere means of transporting electrical effects
from one charge to another. Evidently, the ether was no longer
indispensable, it was needed only to satisfy the traditional pre-
conception that action at a distance is a priori inadmissible,
that natural law necessarily requires action by contact. Clearly,
a mere willingness to admit the possibility that electric charges
can exert forces directly on each other even when separated
in vacuum would make the ether completely unnecessary in the
electron theory. Thus, in Lorentz's theory the significance of
the ether is more psychological than physical.

In formulating his theory of the ether, Maxwell must have assumed that incontestable experimental evidence for the existence of the ether would be forthcoming in due course. But this has not been the case. Experiments skilfully devised to verify the existence of the ether, notably the ether-drift experiments, have without exception failed. Every experimental inquiry has given the same answer: all natural phenomena proceed exactly as if there were no ether. Furthermore, as a result of the theory of relativity, the ether hypothesis is faced with insurmountable logical difficulties. For example, the constancy of the electromagnetic velocity with respect to moving observers demands that each observer have his own ether that moves with him. Unless and until conclusive evidence is provided that a material, all-pervading ether exists in a true physical sense, it must be concluded that *there is no ether*. Therefore Maxwell's entire theory of the electromagnetic ether must necessarily become a part of the history and no longer a part of the practice of physical science. The Maxwell field equations, on the other hand, will continue to be the mathematical heart of macroscopic electromagnetic theory.

In spite of the negative answers of ether-drift experiments, and the general acceptance of the theory of relativity, the Maxwellian ether with all its implications is still made to play an apparently significant part in technical work. Those who continue to believe in the ether argue that nature may be so constituted as to conceal forever its existence. Since anything may be assumed to exist on the same basis, this is wishful, not scientific, thinking. Even engineers and physicists who are willing to agree that the ether is nonexistent and unnecessary, nevertheless continue to do all their electromagnetic thinking and teaching in terms of Maxwellian pictures. Such adherence to tradition in the face of contrary scientific evidence is carefully avoided in this book and in its sequel. Since the action of charge on charge requires no medium, none is introduced and all explanations are made without it. The fundamental law of macroscopic electromagnetism as expressed in the field and force equations is interpreted as a retarded action at a distance. The universal constants appearing in this law are assumed to be defined operationally in terms of experiments devised to measure them. They are, of course, not assigned a *localized* physical significance as

properties of a medium. The electromagnetic field and the fields of the potential functions serve merely as intermediate steps in a mathematical calculation of action between statistical distributions of charge and current.

Retarded action at a distance and the closely related space-time concept of relativity require an outlook that differs fundamentally from the mechanistic philosophy that has dominated scientific thought for so many centuries. The era of mechanical models of electromagnetic phenomena is past; the age of electrical models of matter is unfolding. It is reasonable to expect that this, too, will prove to be only another chapter in the great story of the evolution of the human mind. Later chapters may, for example, develop further the selective subjectivism of Eddington in which not only the formulas expressing natural laws but also the universal constants appearing in them are deducible from a priori reasoning without recourse to experiment.

By keeping abreast of scientific thought and its mathematical and experimental expression, technology guarantees its future.

APPENDIX I

DIFFERENTIAL OPERATORS; VECTOR FORMULAS AND IDENTITIES

The equations for potentials involve the differential operators

$$\nabla^2\phi \equiv \text{div grad } \phi \tag{1}$$
$$\nabla^2\mathsf{A} \equiv \text{grad div } \mathsf{A} - \text{curl curl } \mathsf{A} \tag{2}$$

and, hence,

$$\text{grad } \phi; \qquad \text{div } \mathsf{A}; \qquad \text{curl } \mathsf{A} \tag{3}$$

In order to express (1) and (2) in systems of orthogonal coordinates, it is necessary to obtain expressions for the operators grad, div, curl in general orthogonal curvilinear coordinates U, V, W and then specialize these to each system.

Let the curvilinear coordinate axes of U, V, W be drawn at a point A shown in Fig. I.1. At the point of intersection, the

Fig. I.1.—Orthogonal curvilinear coordinates.

tangents to these axes are mutually perpendicular. In advancing along the positive U axis a distance ds_U, from A to D, the coordinate U changes an amount dU. Similarly, traversing a distance ds_V in passing from A to B along the positive V axis, V changes by dV. In moving a distance ds_W from A to E along the W axis, W changes by dW. It is important to note that U, V, W do not necessarily have the dimension of a simple length as is true in the Cartesian system. However, dU, dV, dW are always functionally related, respectively, to ds_U, ds_V, ds_W. That is, for example,

$$ds_U = f(dU); \qquad ds_V = g(dV); \qquad ds_W = h(dW) \qquad (4)$$

By expanding the functions f, g, h in Maclaurin series about the origin at A,

$$ds_U = f(0) + f'(0)dU + f''(0)\frac{dU^2}{2!} + \cdots$$

$$ds_V = g(0) + g'(0)dV + g''(0)\frac{dV^2}{2!} + \cdots \qquad (5)$$

$$ds_W = h(0) + h'(0)dW + h''(0)\frac{dW^2}{2!} + \cdots$$

Here $f(0)$, $f'(0)$, $g(0)$, etc., are in general functions of all three variables U, V, W. In order to evaluate these functions, it is to be noted in the first place that

$$
\begin{array}{llll}
ds_U = 0, & \text{when } dU = 0 & \text{so that} & f(0) = 0 \\
ds_V = 0, & \text{when } dV = 0 & \text{so that} & g(0) = 0 \quad (6) \\
ds_W = 0, & \text{when } dW = 0 & \text{so that} & h(0) = 0
\end{array}
$$

If dU, dV, dW are sufficiently small, infinitesimals of higher order than the first may be neglected. In this case

$$
\begin{array}{lll}
ds_U = e_1 dU & \text{with} & e_1 = f'(0) \\
ds_V = e_2 dV & & e_2 = g'(0) \quad (7) \\
ds_W = e_3 dW & & e_3 = h'(0)
\end{array}
$$

The determination of e_1, e_2, e_3 is not difficult in Cartesian, cylindrical, and spherical coordinates. It is accomplished by inspection from appropriately specialized figures like Fig. I.1 by determining the factor by which each coordinate must be multiplied to give a length. The calculation in other cases is illustrated later, where the e factors are evaluated for rotationally symmetrical confocal coordinates.

The evaluation of grad ϕ in general curvilinear coordinates using e_1, e_2, e_3 is simple. Let $\hat{U}$, $\hat{V}$, $\hat{W}$ be unit vectors along the orthogonal curvilinear axes. Then,

$$\text{grad } \phi = (\hat{U}, \text{ grad } \phi)\hat{U} + (\hat{V}, \text{ grad } \phi)\hat{V} + (\hat{W}, \text{ grad } \phi)\hat{W} \qquad (8)$$

$$\text{grad } \phi = \frac{\partial \phi}{\partial s_U}\hat{U} + \frac{\partial \phi}{\partial s_V}\hat{V} + \frac{\partial \phi}{\partial s_W}\hat{W} \qquad (9)$$

$$\text{grad } \phi = \frac{\partial \phi}{e_1 \partial U}\hat{U} + \frac{\partial \phi}{e_2 \partial V}\hat{V} + \frac{\partial \phi}{e_3 \partial W}\hat{W} \qquad (10)$$

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$$\text{grad } \phi = \frac{\partial\phi}{e_1\partial U}\hat{U} + \frac{\partial\phi}{e_2\partial V}\hat{V} + \frac{\partial\phi}{e_3\partial W}\hat{W} \qquad\qquad (10)$$

The evaluation of div A is accomplished by direct calculation from the definition of the divergence. This is

$$\text{div } \mathsf{A} = \lim_{dr \to 0} \frac{\int(\mathsf{h},\mathsf{A})d\sigma}{d\tau} \tag{11}$$

Let dr be an element of volume in orthogonal curvilinear coordinates:

$$d\tau = ds_U \, ds_V \, ds_W = e_1 e_2 e_3 dU \, dV \, dW \tag{12}$$

The surface elements are of the form

$$d\sigma_{UV} = ds_U ds_V = e_1 e_2 dU \, dV \tag{13}$$

and similarly on the other surfaces. Consider the two opposite faces $ABCD$ and $EFGH$. On $ABCD$,

$$\int(\mathsf{h},\mathsf{A})d\sigma_{ABCD} = -A_W d\sigma_{UV} \tag{14}$$

The positive normal across $ABCD$ is taken along the positive axis of W. The value on $EFGH$ is obtained by expanding in a power series in ds_W and retaining only first-order terms.

$$\int (\mathsf{h},\mathsf{A})d\sigma_{EFGH} = A_W d\sigma_{UV} + \frac{\partial}{\partial s_W}(A_W d\sigma_{UV})ds_W \tag{15}$$

The value of $\int(\mathsf{h},\mathsf{A})d\sigma$ for $ABCD$ and $EFGH$ is the sum of (14) and (15). It is

$$\int (\mathsf{h},\mathsf{A})_{UV}d\sigma = \frac{\partial}{\partial s_W}(A_W d\sigma_{UV})ds_W = \frac{\partial}{\partial W}(A_W e_1 e_2 dU \, dV)dW \tag{16a}$$

By cyclic permutation,

$$\int (\mathsf{h},\mathsf{A})_{VW}d\sigma = \frac{\partial}{\partial U}(A_U e_2 e_3 dV \, dW)dU \tag{16b}$$

$$\int (\mathsf{h},\mathsf{A})_{WU}d\sigma = \frac{\partial}{\partial V}(A_V e_3 e_1 dW \, dU)dV \tag{16c}$$

Upon using (16) in (11),

$$\text{div } \mathsf{A} \equiv \lim_{dr \to 0} \frac{1}{e_1 e_2 e_3 dU \, dV \, dW} \left\{ \frac{\partial}{\partial U}(A_U e_2 e_3 dV \, dW)dU \right.$$
$$\left. + \frac{\partial}{\partial V}(A_V e_3 e_1 dW \, dU)dV + \frac{\partial}{\partial W}(A_W e_1 e_2 dU \, dV)dW \right\} \tag{17}$$

Since U, V, and W are independent of each other, cancellation leads to

$$\text{div } \mathsf{A} \equiv \frac{1}{e_1 e_2 e_3} \left\{ \frac{\partial}{\partial U} (e_2 e_3 A_U) + \frac{\partial}{\partial V} (e_3 e_1 A_V) + \frac{\partial}{\partial W} (e_1 e_2 A_W) \right\} \tag{18}$$

The evaluation of curl A is accomplished in a similar way from the alternative definition.

$$(\hat{\mathsf{N}}, \text{curl } \mathsf{A}) = \lim_{d\sigma \to 0} \frac{\oint (\mathsf{A}, \mathsf{ds})}{d\sigma} \tag{19}$$

Here $\hat{\mathsf{N}}$ is normal to the surface of the element $d\sigma$ around which ds is integrated.   Evidently,

$$(\hat{\mathsf{W}}, \text{curl } \mathsf{A}) = \lim_{d\sigma \to 0} \frac{\oint (\mathsf{A}, \mathsf{ds})}{d\sigma} \tag{20}$$

with $\hat{\mathsf{W}}$ normal to the surface $ABCD$ (Fig. I.1) around which ds is integrated.   The area of this element of surface is

$$d\sigma_{UV} = ds_U ds_V = e_1 e_2 dU \, dV \tag{21}$$

$$\int_{AB} (\mathsf{A}, \mathsf{ds}) = A_V ds_V \tag{22}$$

$$\int_{AD} (\mathsf{A}, \mathsf{ds}) = A_U ds_U \tag{23}$$

$$\int_{BC} (\mathsf{A}, \mathsf{ds}) = A_U ds_U + \frac{\partial}{\partial s_V} (A_U ds_U) ds_V \tag{24}$$

$$\int_{DC} (\mathsf{A}, \mathsf{ds}) = A_V ds_V + \frac{\partial}{\partial s_U} (A_V ds_V) ds_U \tag{25}$$

In order that the right-hand-screw relation be satisfied, the positive direction of integration must be $DCBA$.   Upon combining (22) to (25) algebraically, there remains only

$$\oint_{DCBA} (\mathsf{A}, \mathsf{ds}) = \frac{\partial}{\partial s_U} (A_V ds_V) ds_U - \frac{\partial}{\partial s_V} (A_U ds_U) ds_V \tag{26}$$

Upon substituting for ds_U and ds_V and remembering that U and V are independent,

$$\oint_{DCBA} (\mathsf{A}, \mathsf{ds}) = \left[\frac{\partial}{\partial U} (e_2 A_V) - \frac{\partial}{\partial V} (e_1 A_U) \right] dV \, dU \tag{27}$$

The evaluation of div A is accomplished by direct calculation from the definition of the divergence. This is

$$\text{div } \mathsf{A} = \lim_{d\tau \to 0} \frac{\int (\hbar, \mathsf{A}) d\sigma}{d\tau} \tag{11}$$

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Upon using (16) in (11),

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Upon substituting (27) in (20) and canceling,

$$(\hat{W}, \text{curl } A) = \frac{1}{c_1 c_2}\left[\frac{\partial}{\partial U}(c_2 A_V) - \frac{\partial}{\partial V}(c_1 A_U)\right] \tag{28a}$$

$$(\hat{U}, \text{curl } A) = \frac{1}{c_2 c_3}\left[\frac{\partial}{\partial V}(c_3 A_W) - \frac{\partial}{\partial W}(c_2 A_V)\right] \tag{28b}$$

$$(\hat{V}, \text{curl } A) = \frac{1}{c_3 c_1}\left[\frac{\partial}{\partial W}(c_1 A_U) - \frac{\partial}{\partial U}(c_3 A_W)\right] \tag{28c}$$

The general curvilinear formula for $\nabla^2\phi$ is obtained by forming div grad ϕ using (10) and (18). It is

$$\nabla^2\phi = \frac{1}{c_1 c_2 c_3}\left[\frac{\partial}{\partial U}\left(\frac{c_2 c_3}{c_1}\frac{\partial\phi}{\partial U}\right) + \frac{\partial}{\partial V}\left(\frac{c_3 c_1}{c_2}\frac{\partial\phi}{\partial V}\right) \right.$$
$$\left. + \frac{\partial}{\partial W}\left(\frac{c_1 c_2}{c_3}\frac{\partial\phi}{\partial W}\right)\right] \tag{29}$$

A formula for curl curl A may be obtained by substituting the expressions for $(\hat{U}, \text{curl } A)$, $(\hat{V}, \text{curl } A)$, $(\hat{W}, \text{curl } A)$, respectively, for A_U, A_V, A_W in (28). Since no simplification is obtained in this way, it is usually more convenient to evaluate curl curl directly as required.

In order to determine an expression for the quantity $\nabla^2 A$ [where A is a vector to be carefully distinguished from the scalar ϕ in (29)] which appears in the important Helmholtz equation for the vector potential, it is necessary to proceed from its definition

$$\nabla^2 A \equiv \text{grad div } A - \text{curl curl } A \tag{30}$$

The formula for $\nabla^2 A$ in general orthogonal coordinates is excessively complicated. In fact, in all systems of coordinates *except only rectangular ones*, the expressions for the components of $\nabla^2 A$ involve so many terms as to be practically useless. In Cartesian coordinates, on the other hand, its form is very simple and readily calculated. Thus, writing

$$A = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \tag{31}$$

and noting that

$$\nabla^2 A = \hat{x}(\hat{x},\nabla^2 A) + \hat{y}(\hat{y},\nabla^2 A) + \hat{z}(\hat{z},\nabla^2 A) \tag{32}$$

the x component of $\nabla^2 A$ must equal the x component of the

expression on the right in (30).   That is

$$(\hat{x}, \nabla^2 A) = \frac{\partial}{\partial x} \text{ div } A - \text{curl}_x \text{ curl } A \qquad (33a)$$

Using the formulas for divergence and curl derived above,

$$(\hat{x}, \nabla^2 A) = \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \frac{\partial}{\partial y}(\text{curl}_z A) + \frac{\partial}{\partial z}(\text{curl}_y A)$$

$$(33b)$$

Carrying out the differentiation in the terms associated with the divergence and writing in the expressions for the components of the curl leads to

$$(\hat{x}, \nabla^2 A) = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \, \partial y} + \frac{\partial^2 A_z}{\partial x \, \partial z}$$

$$- \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \quad (33c)$$

This reduces to

$$(\hat{x}, \nabla^2 A) = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = \nabla^2 A_x \qquad (34a)$$

Similarly

$$(\hat{y}, \nabla^2 A) = \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} = \nabla^2 A_y \qquad (34b)$$

$$(\hat{z}, \nabla^2 A) = \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} = \nabla^2 A_z \qquad (34c)$$

It thus appears that in Cartesian coordinates each rectangular component of the vector A in $\nabla^2 A$ satisfies precisely the same equation as the scalar ϕ.   It is especially important to remember that this is *not* true in cylindrical or spherical coordinates.

Upon substituting the values of e_1, e_2, e_3 for the coordinate systems as listed below in the formulas for grad ϕ, div A, curl A, and $\nabla^2 \phi$, the following tabulated results are obtained for Cartesian, cylindrical, and spherical coordinates. $\nabla^2 A$ is listed only in Cartesian coordinates as explained above.

Where it is not possible to determine the e factors by inspection, it is possible to proceed from the definitions (7).   Thus:

$$ds_U^2 = dx_U^2 + dy_U^2 + dz_U^2 = e_1^2 dU^2 \qquad (35)$$

Upon substituting (27) in (20) and canceling,

$$(\hat{W}, \text{curl } A) = \frac{1}{e_1 c_2}\left[\frac{\partial}{\partial U}(e_2 A_V) - \frac{\partial}{\partial V}(e_1 A_U)\right] \qquad (28a)$$

$$(\hat{U}, \text{curl } A) = \frac{1}{e_2 c_3}\left[\frac{\partial}{\partial V}(e_3 A_W) - \frac{\partial}{\partial W}(e_2 A_V)\right] \qquad (28b)$$

$$(\hat{V}, \text{curl } A) = \frac{1}{e_3 e_1}\left[\frac{\partial}{\partial W}(e_1 A_U) - \frac{\partial}{\partial U}(e_3 A_W)\right] \qquad (28c)$$

The general curvilinear formula for $\nabla^2 \phi$ is obtained by forming div grad ϕ using (10) and (18). It is

$$\nabla^2 \phi = \frac{1}{e_1 e_2 e_3}\left[\frac{\partial}{\partial U}\left(\frac{e_2 e_3}{e_1}\frac{\partial \phi}{\partial U}\right) + \frac{\partial}{\partial V}\left(\frac{e_3 e_1}{e_2}\frac{\partial \phi}{\partial V}\right) \right.$$
$$\left. + \frac{\partial}{\partial W}\left(\frac{e_1 e_2}{e_3}\frac{\partial \phi}{\partial W}\right)\right] \qquad (29)$$

A formula for curl curl A may be obtained by substituting the expressions for $(\hat{U}, \text{curl } A)$, $(\hat{V}, \text{curl } A)$, $(\hat{W}, \text{curl } A)$, respectively, for A_U, A_V, A_W in (28). Since no simplification is obtained in this way, it is usually more convenient to evaluate curl curl directly as required.

In order to determine an expression for the quantity $\nabla^2 A$ [where A is a vector to be carefully distinguished from the scalar ϕ in (29)] which appears in the important Helmholtz equation for the vector potential, it is necessary to proceed from its definition

$$\nabla^2 A \equiv \text{grad div } A - \text{curl curl } A \qquad (30)$$

The formula for $\nabla^2 A$ in general orthogonal coordinates is excessively complicated. In fact, in all systems of coordinates *except only rectangular ones*, the expressions for the components of $\nabla^2 A$ involve so many terms as to be practically useless. In Cartesian coordinates, on the other hand, its form is very simple and readily calculated. Thus, writing

$$A = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z \qquad (31)$$

and noting that

$$\nabla^2 A = \hat{x}(\hat{x}, \nabla^2 A) + \hat{y}(\hat{y}, \nabla^2 A) + \hat{z}(\hat{z}, \nabla^2 A) \qquad (32)$$

the x component of $\nabla^2 A$ must equal the x component of the

expression on the right in (30). That is

$$(\hat{x}, \nabla^2 A) = \frac{\partial}{\partial x} \text{div } A - \text{curl}_x \text{ curl } A \qquad (33a)$$

Using the formulas for divergence and curl derived above,

$$(\hat{x}, \nabla^2 A) = \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \frac{\partial}{\partial y}(\text{curl}_z A) + \frac{\partial}{\partial z}(\text{curl}_y A) \qquad (33b)$$

Carrying out the differentiation in the terms associated with the divergence and writing in the expressions for the components of the curl leads to

$$(\hat{x}, \nabla^2 A) = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \, \partial y} + \frac{\partial^2 A_z}{\partial x \, \partial z}$$
$$- \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \quad (33c)$$

This reduces to

$$(\hat{x}, \nabla^2 A) = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = \nabla^2 A_x \qquad (34a)$$

Similarly

$$(\hat{y}, \nabla^2 A) = \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} = \nabla^2 A_y \qquad (34b)$$

$$(\hat{z}, \nabla^2 A) = \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} = \nabla^2 A_z \qquad (34c)$$

It thus appears that in Cartesian coordinates each rectangular component of the vector A in $\nabla^2 A$ satisfies precisely the same equation as the scalar ϕ. It is especially important to remember that this is *not* true in cylindrical or spherical coordinates.

Upon substituting the values of e_1, e_2, e_3 for the coordinate systems as listed below in the formulas for grad ϕ, div A, curl A, and $\nabla^2\phi$, the following tabulated results are obtained for Cartesian, cylindrical, and spherical coordinates. $\nabla^2 A$ is listed only in Cartesian coordinates as explained above.

Where it is not possible to determine the e factors by inspection, it is possible to proceed from the definitions (7). Thus:

$$ds_U^2 = dx_U^2 + dy_U^2 + dz_U^2 = e_1^2 dU^2 \qquad (35)$$

DIFFERENTIAL OPERATORS

	Cartesian (x,y,z)	Cylindrical (r,θ,z)	Spherical* (Θ,Φ,R)
e_1	1	1	R
e_2	1	r	$R\sin\Theta$
e_3	1	1	1
grad ϕ	$\hat{x}\dfrac{\partial\phi}{\partial x} + \hat{y}\dfrac{\partial\phi}{\partial y} + \hat{z}\dfrac{\partial\phi}{\partial z}$	$\hat{r}\dfrac{\partial\phi}{\partial r} + \hat{\theta}\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} + \hat{z}\dfrac{\partial\phi}{\partial z}$	$\hat{\Theta}\dfrac{1}{R}\dfrac{\partial\phi}{\partial\Theta} + \hat{\Phi}\dfrac{1}{R\sin\Theta}\dfrac{\partial\phi}{\partial\Phi} + \hat{R}\dfrac{\partial\phi}{\partial R}$
$\nabla^2\phi$	$\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$	$\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2} + \dfrac{\partial^2\phi}{\partial z^2}$	$\dfrac{1}{R^2\sin\Theta}\dfrac{\partial}{\partial\Theta}\left(\sin\Theta\dfrac{\partial\phi}{\partial\Theta}\right) + \dfrac{1}{R^2\sin\Theta}\dfrac{\partial^2\phi}{\partial\Phi^2} + \dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2\dfrac{\partial\phi}{\partial R}\right)$
div A	$\dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_r) + \dfrac{1}{r}\dfrac{\partial A_\theta}{\partial\theta} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{R\sin\Theta}\dfrac{\partial}{\partial\Theta}(\sin\Theta\, A_\Theta) + \dfrac{1}{R\sin\Theta}\dfrac{\partial A_\Phi}{\partial\Phi} + \dfrac{1}{R^2}\dfrac{\partial}{\partial R}(R^2 A_R)$
curl A	$\text{curl}_x\,\mathsf{A} = \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}$ $\text{curl}_y\,\mathsf{A} = \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}$ $\text{curl}_z\,\mathsf{A} = \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}$	$\text{curl}_r\,\mathsf{A} = \dfrac{1}{r}\dfrac{\partial A_z}{\partial\theta} - \dfrac{\partial A_\theta}{\partial z}$ $\text{curl}_\theta\,\mathsf{A} = \dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r}$ $\text{curl}_z\,\mathsf{A} = \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{\partial A_r}{\partial\theta}\right)$	$\text{curl}_\Theta\,\mathsf{A} = \dfrac{1}{R}\left(\dfrac{1}{\sin\Theta}\dfrac{\partial A_R}{\partial\Phi} - \dfrac{\partial}{\partial R}(RA_\Phi)\right)$ $\text{curl}_\Phi\,\mathsf{A} = \dfrac{1}{R}\left(\dfrac{\partial}{\partial R}(RA_\Theta) - \dfrac{\partial A_R}{\partial\Theta}\right)$ $\text{curl}_R\,\mathsf{A} = \dfrac{1}{R\sin\Theta}\left(\dfrac{\partial}{\partial\Theta}(\sin\Theta\, A_\Phi) - \dfrac{\partial A_\Theta}{\partial\Phi}\right)$
$\nabla^2\mathsf{A}$	$\hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$ $\nabla^2 A_x = \dfrac{\partial^2 A_x}{\partial x^2} + \dfrac{\partial^2 A_x}{\partial y^2} + \dfrac{\partial^2 A_x}{\partial z^2}$ $\nabla^2 A_y = \dfrac{\partial^2 A_y}{\partial x^2} + \dfrac{\partial^2 A_y}{\partial y^2} + \dfrac{\partial^2 A_y}{\partial z^2}$ $\nabla^2 A_z = \dfrac{\partial^2 A_z}{\partial x^2} + \dfrac{\partial^2 A_z}{\partial y^2} + \dfrac{\partial^2 A_z}{\partial z^2}$		

* Note that the spherical coordinates are written in the order Θ, Φ, R, *not* R, Θ, Φ.

Then

$$e_1^2 = \left(\frac{ds_U}{dU}\right)^2 = \left(\frac{\partial x}{\partial U}\right)^2 + \left(\frac{\partial y}{\partial U}\right)^2 + \left(\frac{\partial z}{\partial U}\right)^2 \qquad (36a)$$

since $dx_U/dU = \partial x/\partial U$, etc.   Similarly,

$$e_2^2 = \left(\frac{ds_V}{dV}\right)^2 = \left(\frac{\partial x}{\partial V}\right)^2 + \left(\frac{\partial y}{\partial V}\right)^2 + \left(\frac{\partial z}{\partial V}\right)^2 \qquad (36b)$$

$$e_3^2 = \left(\frac{ds_W}{W}\right)^2 = \left(\frac{\partial x}{\partial W}\right)^2 + \left(\frac{\partial y}{\partial W}\right)^2 + \left(\frac{\partial z}{\partial W}\right)^2 \qquad (36c)$$

The calculation of e_1, e_2, e_3 in rotationally symmetrical confocal coordinates will now be carried out.

The equation of an ellipsoid in Cartesian coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (37)$$

An hyperboloid of one sheet is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (38)$$

An hyperboloid of two sheets is defined by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (39)$$

Here a, b, c are the semimajor axes; $a \geq b \geq c$. If rotational symmetry prevails, there are two possibilities: the prolate (stretched) ellipsoid with its related orthogonal hyperboloid defined by $b = c$, and the oblate (flattened) ellipsoid with its related orthogonal hyperboloid defined by $a = b$.

For the prolate surfaces with $a > b$ and $r^2 = y^2 + z^2$,

$$\frac{x^2}{a^2} + \frac{r^2}{b^2} = 1 \qquad (40)$$

$$\frac{x^2}{a^2} - \frac{r^2}{b^2} = 1 \qquad (41)$$

The hyperboloid has two sheets.

For the oblate surfaces with $a > c$ and $r^2 = x^2 + y^2$,

$$\frac{r^2}{a^2} + \frac{z^2}{c^2} = 1 \qquad (42)$$

$$\frac{r^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad (43)$$

DIFFERENTIAL OPERATORS

	Cartesian (x,y,z)	Cylindrical (r,θ,z)	Spherical* (Θ,Φ,R)
e_1	1	1	R
e_2	1	r	$R\sin\Theta$
e_3	1	1	1
grad ϕ	$\hat{x}\dfrac{\partial\phi}{\partial x} + \hat{y}\dfrac{\partial\phi}{\partial y} + \hat{z}\dfrac{\partial\phi}{\partial z}$	$\hat{r}\dfrac{\partial\phi}{\partial r} + \hat{\theta}\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} + \hat{z}\dfrac{\partial\phi}{\partial z}$	$\hat{\Theta}\dfrac{1}{R}\dfrac{\partial\phi}{\partial\Theta} + \hat{\Phi}\dfrac{1}{R\sin\Theta}\dfrac{\partial\phi}{\partial\Phi} + \hat{R}\dfrac{\partial\phi}{\partial R}$
$\nabla^2\phi$	$\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$	$\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2} + \dfrac{\partial^2\phi}{\partial z^2}$	$\dfrac{1}{R^2\sin\Theta}\dfrac{\partial}{\partial\Theta}\left(\sin\Theta\dfrac{\partial\phi}{\partial\Theta}\right) + \dfrac{1}{R^2\sin^2\Theta}\dfrac{\partial^2\phi}{\partial\Phi^2} + \dfrac{1}{R^2}\dfrac{\partial}{\partial R}\left(R^2\dfrac{\partial\phi}{\partial R}\right)$
div A	$\dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{r}\dfrac{\partial}{\partial r}(rA_r) + \dfrac{1}{r}\dfrac{\partial A_\theta}{\partial\theta} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{R\sin\Theta}\dfrac{\partial}{\partial\Theta}(\sin\Theta\, A_\Theta) + \dfrac{1}{R\sin\Theta}\dfrac{\partial A_\Phi}{\partial\Phi} + \dfrac{1}{R^2}\dfrac{\partial}{\partial R}(R^2 A_R)$
curl A	$\text{curl}_x\,\mathsf{A} = \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}$ $\text{curl}_y\,\mathsf{A} = \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}$ $\text{curl}_z\,\mathsf{A} = \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}$	$\text{curl}_r\,\mathsf{A} = \dfrac{1}{r}\dfrac{\partial A_z}{\partial\theta} - \dfrac{\partial A_\theta}{\partial z}$ $\text{curl}_\theta\,\mathsf{A} = \dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r}$ $\text{curl}_z\,\mathsf{A} = \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}(rA_\theta) - \dfrac{\partial A_r}{\partial\theta}\right)$	$\text{curl}_\Theta\,\mathsf{A} = \dfrac{1}{R}\left(\dfrac{1}{\sin\Theta}\dfrac{\partial A_R}{\partial\Phi} - \dfrac{\partial}{\partial R}(RA_\Phi)\right)$ $\text{curl}_\Phi\,\mathsf{A} = \dfrac{1}{R}\left(\dfrac{\partial}{\partial R}(RA_\Theta) - \dfrac{\partial A_R}{\partial\Theta}\right)$ $\text{curl}_R\,\mathsf{A} = \dfrac{1}{R\sin\Theta}\left(\dfrac{\partial}{\partial\Theta}(\sin\Theta\, A_\Phi) - \dfrac{\partial A_\Theta}{\partial\Phi}\right)$
$\nabla^2\mathsf{A}$	$\hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$ $\nabla^2 A_x = \dfrac{\partial^2 A_x}{\partial x^2} + \dfrac{\partial^2 A_x}{\partial y^2} + \dfrac{\partial^2 A_x}{\partial z^2}$ $\nabla^2 A_y = \dfrac{\partial^2 A_y}{\partial x^2} + \dfrac{\partial^2 A_y}{\partial y^2} + \dfrac{\partial^2 A_y}{\partial z^2}$ $\nabla^2 A_z = \dfrac{\partial^2 A_z}{\partial x^2} + \dfrac{\partial^2 A_z}{\partial y^2} + \dfrac{\partial^2 A_z}{\partial z^2}$		

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Then

$$e_1^2 = \left(\frac{ds_U}{dU}\right)^2 = \left(\frac{\partial x}{\partial U}\right)^2 + \left(\frac{\partial y}{\partial U}\right)^2 + \left(\frac{\partial z}{\partial U}\right)^2 \qquad (36a)$$

since $dx_U/dU = \partial x/\partial U$, etc.   Similarly,

$$e_2^2 = \left(\frac{ds_V}{dV}\right)^2 = \left(\frac{\partial x}{\partial V}\right)^2 + \left(\frac{\partial y}{\partial V}\right)^2 + \left(\frac{\partial z}{\partial V}\right)^2 \qquad (36b)$$

$$e_3^2 = \left(\frac{ds_W}{W}\right)^2 = \left(\frac{\partial x}{\partial W}\right)^2 + \left(\frac{\partial y}{\partial W}\right)^2 + \left(\frac{\partial z}{\partial W}\right)^2 \qquad (36c)$$

The calculation of e_1, e_2, e_3 in rotationally symmetrical confocal coordinates will now be carried out.

The equation of an ellipsoid in Cartesian coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (37)$$

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (38)$$

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$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (39)$$

Here a, b, c are the semimajor axes; $a \geq b \geq c$.   If rotational symmetry prevails, there are two possibilities: the prolate (stretched) ellipsoid with its related orthogonal hyperboloid defined by $b = c$, and the oblate (flattened) ellipsoid with its related orthogonal hyperboloid defined by $a = b$.

For the prolate surfaces with $a > b$ and $r^2 = y^2 + z^2$,

$$\frac{x^2}{a^2} + \frac{r^2}{b^2} = 1 \qquad (40)$$

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For the oblate surfaces with $a > c$ and $r^2 = x^2 + y^2$,

$$\frac{r^2}{a^2} + \frac{z^2}{c^2} = 1 \qquad (42)$$

$$\frac{r^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad (43)$$

For the prolate case the eccentricity e is defined by

$$e^2 = a^2 - b^2 \text{ (ellipse)} \qquad \text{or} \qquad b^2 = a^2 - e^2 \qquad (44)$$

$$e^2 = a^2 + b^2 \text{ (hyperbola)} \qquad \text{or} \qquad b^2 = e^2 - a^2 \qquad (45)$$

Accordingly, the equations for the prolate case are

$$\frac{x^2}{a^2} + \frac{r^2}{a^2 - e^2} = 1 \quad \text{(ellipsoid)} \qquad (46)$$

$$\frac{x^2}{a^2} - \frac{r^2}{e^2 - a^2} = 1 \quad \text{(hyperboloid of two sheets)} \quad (47)$$

The eccentricity for the oblate case is given by (44) and (45) with c written for b.   Hence

$$\frac{r^2}{a^2} + \frac{z^2}{a^2 - e^2} = 1 \quad \text{(ellipsoid)} \qquad (48)$$

$$\frac{r^2}{a^2} - \frac{z^2}{c^2 - a^2} = 1 \quad \text{(hyperboloid of one sheet)} \qquad (49)$$

The oblate case follows directly from the prolate one by interchanging the coordinates, writing r for x and z for r.

A family of confocal ellipsoids of revolution is defined by

$$\frac{x^2}{u^2} + \frac{r^2}{u^2 - e^2} = 1; \qquad u > e \qquad (50)$$

The associated orthogonal hyperboloids are

$$\frac{x^2}{v^2} - \frac{r^2}{e^2 - v^2} = 1; \qquad -e < v < e \qquad (51)$$

The variables u, v are ellipsoidal and hyperboloidal coordinates. The parameters c_1, c_2, c_3 may be calculated from

$$c_1 = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial r}{\partial u}\right)^2} \quad (52a)$$

with $r^2 = y^2 + z^2$

$$c_2 = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2} = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial r}{\partial v}\right)^2} \quad (52b)$$

$$c_3 = \sqrt{\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2} = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} \quad (52c)$$

The coordinate w is identically the angle of rotation θ about the axis of symmetry x.   It is related to y and z by

$$y = r \cos \theta; \qquad \frac{\partial y}{\partial \theta} = -r \sin \theta \qquad (53)$$

$$z = r \sin \theta; \qquad \frac{\partial z}{\partial \theta} = r \cos \theta$$

$$e_3 = \sqrt{\left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r; \qquad \frac{\partial x}{\partial \theta} = 0 \qquad (54)$$

Upon solving (50) and (51) for r and x,

$$x^2 = u^2 - \frac{u^2 r^2}{u^2 - e^2} = v^2 + \frac{v^2 r^2}{e^2 - v^2} \qquad (55a)$$

$$r^2 \left(\frac{v^2}{e^2 - v^2} + \frac{u^2}{u^2 - e^2}\right) = u^2 - v^2 \qquad (55b)$$

$$r^2 \left(\frac{e^2(u^2 - v^2)}{(e^2 - v^2)(u^2 - e^2)}\right) = u^2 - v^2 \qquad (55c)$$

$$r^2 = \frac{(e^2 - v^2)(u^2 - e^2)}{e^2} \qquad (55d)$$

$$x^2 = u^2 \left(1 - \frac{e^2 - v^2}{e^2}\right) = \frac{u^2 v^2}{e^2} \qquad (55e)$$

$$r = \frac{\sqrt{(e^2 - v^2)(u^2 - e^2)}}{e}; \qquad x = \frac{uv}{e} \qquad (56)$$

Hence,

$$\frac{\partial x}{\partial u} = \frac{v}{e}; \qquad \frac{\partial x}{\partial v} = \frac{u}{e} \qquad (57)$$

$$\frac{\partial r}{\partial u} = \frac{u}{e} \sqrt{\frac{e^2 - v^2}{u^2 - e^2}}; \qquad \frac{\partial r}{\partial v} = -\frac{v}{e} \sqrt{\frac{u^2 - e^2}{e^2 - v^2}} \qquad (58)$$

$$e_1 = \sqrt{\left(\frac{v}{e}\right)^2 + \frac{u^2}{e^2}\left(\frac{e^2 - v^2}{u^2 - e^2}\right)} = \sqrt{\frac{u^2 - v^2}{u^2 - e^2}} \qquad (59a)$$

$$e_2 = \sqrt{\left(\frac{u}{e}\right)^2 + \frac{v^2}{e^2}\left(\frac{u^2 - e^2}{e^2 - v^2}\right)} = \sqrt{\frac{u^2 - v^2}{e^2 - v^2}} \qquad (59b)$$

$$e_3 = r = \frac{\sqrt{(e^2 - v^2)(u^2 - e^2)}}{e} \qquad (59c)$$

Hence,

$$\text{grad } \phi = \sqrt{\frac{u^2 - e^2}{u^2 - v^2}} \frac{\partial \phi}{\partial u} \hat{u} + \sqrt{\frac{e^2 - v^2}{u^2 - v^2}} \frac{\partial \phi}{\partial v} \hat{v}$$
$$+ \frac{e}{\sqrt{(e^2 - v^2)(u^2 - e^2)}} \frac{\partial \phi}{\partial \theta} \hat{\theta} \qquad (60)$$

For the prolate case the eccentricity e is defined by

$$e^2 = a^2 - b^2 \text{ (ellipse)} \qquad \text{or} \qquad b^2 = a^2 - e^2 \qquad (44)$$
$$e^2 = a^2 + b^2 \text{ (hyperbola)} \qquad \text{or} \qquad b^2 = e^2 - a^2 \qquad (45)$$

Accordingly, the equations for the prolate case are

$$\frac{x^2}{a^2} + \frac{r^2}{a^2 - e^2} = 1 \quad \text{(ellipsoid)} \tag{46}$$

$$\frac{x^2}{a^2} - \frac{r^2}{e^2 - a^2} = 1 \quad \text{(hyperboloid of two sheets)} \tag{47}$$

The eccentricity for the oblate case is given by (44) and (45) with c written for b. Hence

$$\frac{r^2}{a^2} + \frac{z^2}{a^2 - e^2} = 1 \quad \text{(ellipsoid)} \tag{48}$$

$$\frac{r^2}{a^2} - \frac{z^2}{c^2 - a^2} = 1 \quad \text{(hyperboloid of one sheet)} \tag{49}$$

The oblate case follows directly from the prolate one by interchanging the coordinates, writing r for x and z for r.

A family of confocal ellipsoids of revolution is defined by

$$\frac{x^2}{u^2} + \frac{r^2}{u^2 - c^2} = 1; \qquad u > e \tag{50}$$

The associated orthogonal hyperboloids are

$$\frac{x^2}{v^2} - \frac{r^2}{c^2 - v^2} = 1; \qquad -c < v < e \tag{51}$$

The variables u, v are ellipsoidal and hyperboloidal coordinates. The parameters c_1, c_2, c_3 may be calculated from

$$c_1 = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial r}{\partial u}\right)^2} \tag{52a}$$

with $r^2 = y^2 + z^2$

$$c_2 = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2} = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial r}{\partial v}\right)^2} \tag{52b}$$

$$c_3 = \sqrt{\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2} = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial r}{\partial \theta}\right)^2} \tag{52c}$$

The coordinate w is identically the angle of rotation θ about the axis of symmetry x. It is related to y and z by

$$y = r \cos \theta; \qquad \frac{\partial y}{\partial \theta} = -r \sin \theta \tag{53}$$

$$z = r \sin \theta; \qquad \frac{\partial z}{\partial \theta} = r \cos \theta$$

$$e_3 = \sqrt{\left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = r; \qquad \frac{\partial x}{\partial \theta} = 0 \tag{54}$$

Upon solving (50) and (51) for r and x,

$$x^2 = u^2 - \frac{u^2 r^2}{u^2 - e^2} = v^2 + \frac{v^2 r^2}{e^2 - v^2} \tag{55a}$$

$$r^2 \left(\frac{v^2}{e^2 - v^2} + \frac{u^2}{u^2 - e^2} \right) = u^2 - v^2 \tag{55b}$$

$$r^2 \left(\frac{e^2(u^2 - v^2)}{(e^2 - v^2)(u^2 - e^2)} \right) = u^2 - v^2 \tag{55c}$$

$$r^2 = \frac{(e^2 - v^2)(u^2 - e^2)}{e^2} \tag{55d}$$

$$x^2 = u^2 \left(1 - \frac{e^2 - v^2}{c^2} \right) = \frac{u^2 v^2}{c^2} \tag{55e}$$

$$r = \frac{\sqrt{(e^2 - v^2)(u^2 - e^2)}}{e}; \qquad x = \frac{uv}{e} \tag{56}$$

Hence,

$$\frac{\partial x}{\partial u} = \frac{v}{e}; \qquad \frac{\partial x}{\partial v} = \frac{u}{e} \tag{57}$$

$$\frac{\partial r}{\partial u} = \frac{u}{e} \sqrt{\frac{e^2 - v^2}{u^2 - e^2}}; \qquad \frac{\partial r}{\partial v} = -\frac{v}{e} \sqrt{\frac{u^2 - e^2}{c^2 - v^2}} \tag{58}$$

$$e_1 = \sqrt{\left(\frac{v}{c}\right)^2 + \frac{u^2}{e^2}\left(\frac{e^2 - v^2}{u^2 - e^2}\right)} = \sqrt{\frac{u^2 - v^2}{u^2 - e^2}} \tag{59a}$$

$$e_2 = \sqrt{\left(\frac{u}{c}\right)^2 + \frac{v^2}{c^2}\left(\frac{u^2 - e^2}{e^2 - v^2}\right)} = \sqrt{\frac{u^2 - v^2}{c^2 - v^2}} \tag{59b}$$

$$e_3 = r = \frac{\sqrt{(e^2 - v^2)(u^2 - e^2)}}{e} \tag{59c}$$

Hence,

$$\text{grad } \phi = \sqrt{\frac{u^2 - c^2}{u^2 - v^3}} \frac{\partial \phi}{\partial u} \hat{u} + \sqrt{\frac{e^2 - v^2}{u^2 - v^2}} \frac{\partial \phi}{\partial v} \hat{v}$$

$$+ \frac{e}{\sqrt{(e^2 - v^2)(u^2 - e^2)}} \frac{\partial \phi}{\partial \theta} \hat{\theta} \tag{60}$$

$$\text{div } \mathsf{A} = \left[\frac{e}{u^2 - v^2} \right] \left[\frac{\partial}{\partial u} \left(\frac{\sqrt{(u^2 - v^2)(u^2 - e^2)}}{e} A_u \right) \right.$$

$$+ \frac{\partial}{\partial v} \left(\frac{\sqrt{(u^2 - v^2)(e^2 - v^2)}}{e} A_v \right)$$

$$\left. + \frac{\partial}{\partial \theta} \left(\frac{u^2 - v^2}{\sqrt{(u^2 - e^2)(e^2 - v^2)}} A_\theta \right) \right] \quad (61)$$

$$(\hat{u}, \text{curl } \mathsf{A}) = \frac{e}{\sqrt{(u^2 - v^2)(u^2 - e^2)}} \left\{ \frac{\partial}{\partial v} \left(\frac{\sqrt{(e^2 - v^2)(u^2 - e^2)}}{e} A_\theta \right) \right.$$

$$\left. - \frac{\partial}{\partial \theta} \left(\sqrt{\frac{u^2 - v^2}{e^2 - v^2}} A_v \right) \right\} \quad (62a)$$

$$(\hat{v}, \text{curl } \mathsf{A}) = \frac{e}{\sqrt{(u^2 - v^2)(e^2 - v^2)}} \left\{ \frac{\partial}{\partial \theta} \left(\sqrt{\frac{u^2 - v^2}{u^2 - e^2}} A_u \right) \right.$$

$$\left. - \frac{\partial}{\partial u} \left(\frac{\sqrt{(e^2 - v^2)(u^2 - e^2)}}{e} A_\theta \right) \right\} \quad (62b)$$

$$(\hat{\theta}, \text{curl } \mathsf{A}) = \frac{\sqrt{(u^2 - e^2)(e^2 - v^2)}}{u^2 - v^2} \left[\frac{\partial}{\partial u} \left(\sqrt{\frac{u^2 - v^2}{e^2 - v^2}} A_v \right) \right.$$

$$\left. - \frac{\partial}{\partial v} \left(\sqrt{\frac{u^2 - v^2}{u^2 - e^2}} A_u \right) \right] \quad (62c)$$

$$\nabla^2 \phi = \frac{e}{u^2 - v^2} \left\{ \frac{\partial}{\partial u} \left(\frac{u^2 - e^2}{e} \frac{\partial \phi}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{e^2 - v^2}{e} \frac{\partial \phi}{\partial v} \right) \right.$$

$$\left. + \frac{\partial}{\partial \theta} \left(\frac{e(u^2 - v^2)}{(u^2 - e^2)(e^2 - v^2)} \frac{\partial \phi}{\partial \theta} \right) \right\} \quad (63)$$

Vector formulas and identities with Cartesian equivalents. Scalar product

$$(\mathsf{A}, \mathsf{B}) = (\mathsf{B}, \mathsf{A}) = AB \cos (A, B) = A_x B_x + A_y B_y + A_z B_z$$

$$(\mathsf{A}, \mathsf{B} + \mathsf{C}) = (\mathsf{A}, \mathsf{B}) + (\mathsf{A}, \mathsf{C})$$

Vector product

$$[\mathsf{A}, \mathsf{B}] = -[\mathsf{B}, \mathsf{A}] = \mathsf{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

C is an axial vector perpendicular to the plane containing A and B and pointing in the direction of advance of a right-hand screw when A is turned into B through the shortest arc.

$$C = AB \sin (A, B)$$

$$[\mathsf{A}, \mathsf{B} + \mathsf{C}] = [\mathsf{A}, \mathsf{B}] + [\mathsf{A}, \mathsf{C}]$$

$$[\mathsf{A}, \mathsf{A}] = 0$$

Double products

$$(A,[B,C]) = (B,[C,A]) = (C[A,B])$$
$$(A,[A,B]) = 0$$
$$[A,[B,C]] = B(A,C) - C(A,B) \qquad \text{(polar vector)}$$

Differential operators

$$\text{grad } \phi \equiv \nabla\phi = \lim_{\Delta\tau\to 0} \frac{\int_{\Sigma} \hbar\phi\, d\sigma}{\Delta\tau} = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

is the rate of change $d\phi/ds$ in the direction of its maximum

$$\text{div } A \equiv (\nabla,A) = \lim_{\Delta\tau\to 0} \frac{\int_{\Sigma} (\hbar,A)d\sigma}{\Delta\tau} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{curl } A \equiv [\nabla,A] = \lim_{\Delta\tau\to 0} \frac{\int_{\Sigma} [\hbar,A]d\sigma}{\Delta\tau} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x}\,\text{curl}_x A + \hat{y}\,\text{curl}_y A + \hat{z}\,\text{curl}_z A$$

$$\nabla^2\phi \equiv \text{div grad } \phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla^2 A \equiv \text{grad div } A - \text{curl curl } A = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$$

Double operations

$$\text{grad div } A \equiv \nabla(\nabla,A) = \hat{x}\frac{\partial}{\partial x}(\text{div } A) + \hat{y}\frac{\partial}{\partial y}(\text{div } A)$$
$$+ \hat{z}\frac{\partial}{\partial z}(\text{div } A)$$

$$\text{div grad } \phi \equiv (\nabla,\nabla\phi) \equiv \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\text{div curl } A \equiv (\nabla,[\nabla A]) = 0$$

$$\text{curl grad } \phi = [\nabla,\nabla\phi] = 0$$

$$\text{curl curl } A \equiv [\nabla,[\nabla,A]] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{curl}_x A & \text{curl}_y A & \text{curl}_z A \end{vmatrix}$$

$$(A, \text{grad } \phi) \equiv (A, \text{grad})\phi = A_x\frac{\partial\phi}{\partial x} + A_y\frac{\partial\phi}{\partial y} + A_z\frac{\partial\phi}{\partial z} \text{ is the rate}$$

of change $d\phi/ds$ in the direction of A multiplied by A.

$$(\hat{a}, \text{grad } \phi) \equiv (\hat{a}, \text{grad})\phi \text{ is the rate of change } d\phi/ds \text{ in the}$$

direction of the unit vector $\hat{a}$.

$$\text{div } A = \left[\frac{e}{u^2 - v^2}\right]\left[\frac{\partial}{\partial u}\left(\frac{\sqrt{(u^2 - v^2)(u^2 - e^2)}}{e} A_u\right)\right.$$

$$+ \frac{\partial}{\partial v}\left(\frac{\sqrt{(u^2 - v^2)(e^2 - v^2)}}{e} A_v\right)$$

$$\left. + \frac{\partial}{\partial \theta}\left(\frac{u^2 - v^2}{\sqrt{(u^2 - e^2)(e^2 - v^2)}} A_\theta\right)\right] \quad (61)$$

$$(\hat{u}, \text{curl } A) = \frac{e}{\sqrt{(u^2 - v^2)(u^2 - e^2)}}\left\{\frac{\partial}{\partial v}\left(\frac{\sqrt{(e^2 - v^2)(u^2 - e^2)}}{e} A_\theta\right)\right.$$

$$\left. - \frac{\partial}{\partial \theta}\left(\sqrt{\frac{u^2 - v^2}{e^2 - v^2}} A_v\right)\right\} \quad (62a)$$

$$(\hat{v}, \text{curl } A) = \frac{e}{\sqrt{(u^2 - v^2)(e^2 - v^2)}}\left\{\frac{\partial}{\partial \theta}\left(\sqrt{\frac{u^2 - v^2}{u^2 - e^2}} A_u\right)\right.$$

$$\left. - \frac{\partial}{\partial u}\left(\frac{\sqrt{(e^2 - v^2)(u^2 - e^2)}}{e} A_\theta\right)\right\} \quad (62b)$$

$$(\hat{\theta}, \text{curl } A) = \frac{\sqrt{(u^2 - e^2)(e^2 - v^2)}}{u^2 - v^2}\left[\frac{\partial}{\partial u}\left(\sqrt{\frac{u^2 - v^2}{e^2 - v^2}} A_v\right)\right.$$

$$\left. - \frac{\partial}{\partial v}\left(\sqrt{\frac{u^2 - v^2}{u^2 - e^2}} A_u\right)\right], \quad (62c)$$

$$\nabla^2\phi = \frac{e}{u^2 - v^2}\left\{\frac{\partial}{\partial u}\left(\frac{u^2 - e^2}{e}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{e^2 - v^2}{e}\frac{\partial\phi}{\partial v}\right)\right.$$

$$\left. + \frac{\partial}{\partial \theta}\left(\frac{e(u^2 - v^2)}{(u^2 - e^2)(e^2 - v^2)}\frac{\partial\phi}{\partial\theta}\right)\right\} \quad (63)$$

Vector formulas and identities with Cartesian equivalents. Scalar product

$$(A,B) = (B,A) = AB \cos (A,B) = A_x B_x + A_y B_y + A_z B_z$$
$$(A,B + C) = (A,B) + (A,C)$$

Vector product

$$[A,B] = -[B,A] = C = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

C is an axial vector perpendicular to the plane containing A and B and pointing in the direction of advance of a right-hand screw when A is turned into B through the shortest arc.

$$C = AB \sin (A, B)$$
$$[A,B + C] = [A,B] + [A,C]$$
$$[A,A] = 0$$

Double products

$$(A,[B,C]) = (B,[C,A]) = (C[A,B])$$
$$(A,[A,B]) = 0$$
$$[A,[B,C]] = B(A,C) - C(A,B) \qquad \text{(polar vector)}$$

Differential operators

$$\text{grad } \phi \equiv \nabla\phi = \lim_{\Delta\tau\to 0} \frac{\int_\Sigma \hbar\phi\, d\sigma}{\Delta\tau} = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

is the rate of change $d\phi/ds$ in the direction of its maximum

$$\text{div } A \equiv (\nabla,A) = \lim_{\Delta\tau\to 0} \frac{\int_\Sigma (\hbar,A)d\sigma}{\Delta\tau} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{curl } A \equiv [\nabla,A] = \lim_{\Delta\tau\to 0} \frac{\int_\Sigma [\hbar,A]d\sigma}{\Delta\tau} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{x}\, \text{curl}_x A + \hat{y}\, \text{curl}_y A + \hat{z}\, \text{curl}_z A$$

$$\nabla^2\phi \equiv \text{div grad } \phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla^2 A \equiv \text{grad div } A - \text{curl curl } A = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$$

Double operations

$$\text{grad div } A \equiv \nabla(\nabla,A) = \hat{x}\frac{\partial}{\partial x}(\text{div } A) + \hat{y}\frac{\partial}{\partial y}(\text{div } A)$$
$$+ \hat{z}\frac{\partial}{\partial z}(\text{div } A)$$

$$\text{div grad } \phi \equiv (\nabla,\nabla\phi) \equiv \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\text{div curl } A \equiv (\nabla,[\nabla A]) = 0$$

$$\text{curl grad } \phi = [\nabla,\nabla\phi] = 0$$

$$\text{curl curl } A \equiv [\nabla,[\nabla,A]] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{curl}_x A & \text{curl}_y A & \text{curl}_z A \end{vmatrix}$$

$$(A, \text{grad } \phi) \equiv (A, \text{grad})\phi = A_x\frac{\partial\phi}{\partial x} + A_y\frac{\partial\phi}{\partial y} + A_z\frac{\partial\phi}{\partial z} \text{ is the rate}$$

of change $d\phi/ds$ in the direction of
A multiplied by A.

$(\hat{a}, \text{grad } \phi) \equiv (\hat{a}, \text{grad})\phi$ is the rate of change $d\phi/ds$ in the
direction of the unit vector $\hat{a}$.

$(A, grad)B = \hat{x}(A, grad\ B_x) + \hat{y}(A, grad\ B_y) + \hat{z}(A, grad\ B_z)$
    is the rate of change dB/ds in the direction of A
    multiplied by A.

$(\hat{a}, grad)B = \hat{x}(\hat{a}, grad\ B_x) + \hat{y}(\hat{a}, grad\ B_y) + \hat{z}(\hat{a}, grad\ B_z)$ is
    the rate of change dB/ds in the direction of the unit
    vector $\hat{a}$. It is the directional derivative of B in
    the direction $\hat{a}$.

Operations on products

$$grad\ \phi\psi = \phi\ grad\ \psi + \psi\ grad\ \phi$$
$$div\ \phi A = \phi\ div\ A + (grad\ \phi, A)$$
$$div\ [A,B] = (B,\ curl\ A) - (A,\ curl\ B)$$
$$curl\ \phi A = \phi\ curl\ A + [grad\ \phi, A]$$
$$curl\ [A,B] = A\ div\ B - B\ div\ A + (B,\ grad)A - (A,\ grad)B$$

APPENDIX II

TABLES OF THE FUNCTIONS

$$f(h) \equiv + \sqrt{\tfrac{1}{2}(\sqrt{1 + h^2} + 1)} = \cosh\left(\tfrac{1}{2} \sinh^{-1} h\right)$$
$$g(h) \equiv + \sqrt{\tfrac{1}{2}(\sqrt{1 + h^2} - 1)} = \sinh\left(\tfrac{1}{2} \sinh^{-1} h\right)$$

in the expression

$$\sqrt{1 \pm jh} = f(h) \pm jg(h)$$

as computed by PROFESSOR G. W. PIERCE*

h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)
.000	1.000	.0000	.072	1.001	.0360	.144	1.003	.0718	.216	1.006	.1074
.002	1.000	.0010	.074	1.001	.0370	.146	1.003	.0728	.218	1.006	.1084
.004	1.000	.0020	.076	1.001	.0380	.148	1.003	.0738	.220	1.006	.1094
.006	1.000	.0030	.078	1.001	.0390	.150	1.003	.0748	.222	1.006	.1104
.008	1.000	.0040	.080	1.001	.0400	.152	1.003	.0758	.224	1.006	.1114
.010	1.000	.0050	.082	1.001	.0410	.154	1.003	.0768	.226	1.006	.1124
.012	1.000	.0060	.084	1.001	.0420	.156	1.003	.0778	.228	1.006	.1133
.014	1.000	.0070	.086	1.001	.0430	.158	1.003	.0788	.230	1.007	.1143
.016	1.000	.0080	.088	1.001	.0440	.160	1.003	.0798	.232	1.007	.1153
.018	1.000	.0090	.090	1.001	.0450	.162	1.003	.0807	.234	1.007	.1163
.020	1.000	.0100	.092	1.001	.0460	.164	1.003	.0817	.236	1.007	.1172
.022	1.000	.0110	.094	1.001	.0470	.166	1.003	.0827	.238	1.007	.1182
.024	1.000	.0120	.096	1.001	.0480	.168	1.004	.0837	.240	1.007	.1192
.026	1.000	.0130	.098	1.001	.0490	.170	1.004	.0847	.242	1.007	.1202
.028	1.000	.0140	.100	1.001	.0499	.172	1.004	.0857	.244	1.007	.1211
.030	1.000	.0150	.102	1.001	.0509	.174	1.004	.0867	.246	1.007	.1221
.032	1.000	.0160	.104	1.001	.0519	.176	1.004	.0877	.248	1.007	.1230
.034	1.000	.0170	.106	1.001	.0529	.178	1.004	.0887	.250	1.008	.1240
.036	1.000	.0180	.108	1.001	.0539	.180	1.004	.0896	.252	1.008	.1250
.038	1.000	.0190	.110	1.002	.0549	.182	1.004	.0906	.254	1.008	.1260
.040	1.000	.0200	.112	1.002	.0559	.184	1.004	.0916	.256	1.008	.1270
.042	1.000	.0210	.114	1.002	.0569	.186	1.004	.0926	.258	1.008	.1280
.044	1.000	.0220	.116	1.002	.0579	.188	1.004	.0936	.260	1.008	.1289
.046	1.000	.0230	.118	1.002	.0589	.190	1.004	.0946	.262	1.008	.1299
.048	1.000	.0240	.120	1.002	.0599	.192	1.005	.0956	.264	1.008	.1309
.050	1.000	.0250	.122	1.002	.0609	.194	1.005	.0966	.266	1.009	.1319
.052	1.000	.0260	.124	1.002	.0619	.196	1.005	.0975	.268	1.009	.1329
.054	1.000	.0270	.126	1.002	.0629	.198	1.005	.0985	.270	1.009	.1338
.056	1.000	.0280	.128	1.002	.0639	.200	1.005	.0995	.272	1.009	.1348
.058	1.000	.0290	.130	1.002	.0649	.202	1.005	.1005	.274	1.009	.1358
.060	1.000	.0300	.132	1.002	.0659	.204	1.005	.1015	.276	1.009	.1368
.062	1.000	.0310	.134	1.002	.0669	.206	1.005	.1025	.278	1.009	.1377
.064	1.001	.0320	.136	1.002	.0679	.208	1.005	.1035	.280	1.010	.1387
.066	1.001	.0330	.138	1.002	.0689	.210	1.005	.1044	.282	1.010	.1397
.068	1.001	.0340	.140	1.002	.0698	.212	1.006	.1054	.284	1.010	.1406
.070	1.001	.0350	.142	1.003	.0708	.214	1.006	.1064	.286	1.010	.1416

h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)
.286	1.010	.1416	.358	1.015	.1762	.430	1.022	.2104	.502	1.029	.2438
.288	1.010	.1426	.360	1.016	.1772	.432	1.022	.2114	.504	1.029	.2447
.290	1.010	.1435	.362	1.016	.1782	.434	1.022	.2123	.506	1.030	.2456
.292	1.010	.1445	.364	1.016	.1792	.436	1.022	.2132	.508	1.030	.2465
.294	1.011	.1455	.366	1.016	.1801	.438	1.023	.2141	.510	1.030	.2475
.296	1.011	.1464	.368	1.016	.1811	.440	1.023	.2150	.512	1.030	.2484
.298	1.011	.1474	.370	1.016	.1820	.442	1.023	.2159	.514	1.031	.2493
.300	1.011	.1484	.372	1.017	.1829	.444	1.023	.2168	.516	1.031	.2502
.302	1.011	.1494	.374	1.017	.1839	.446	1.023	.2177	.518	1.031	.2510
.304	1.011	.1503	.376	1.017	.1848	.448	1.024	.2186	.520	1.031	.2520
.306	1.011	.1513	.378	1.017	.1858	.450	1.024	.2197	.522	1.031	.2529
.308	1.011	.1522	.380	1.017	.1867	.452	1.024	.2207	.524	1.032	.2538
.310	1.012	.1532	.382	1.017	.1877	.454	1.024	.2216	.526	1.032	.2547
.312	1.012	.1541	.384	1.018	.1886	.456	1.024	.2225	.528	1.032	.2556
.314	1.012	.1551	.386	1.018	.1896	.458	1.025	.2234	.530	1.032	.2566
.316	1.012	.1561	.388	1.018	.1905	.460	1.025	.2244	.532	1.033	.2575
.318	1.012	.1570	.390	1.018	.1915	.462	1.025	.2253	.534	1.033	.2584
.320	1.012	.1580	.392	1.018	.1924	.464	1.025	.2262	.536	1.033	.2593
.322	1.013	.1590	.394	1.018	.1934	.466	1.025	.2271	.538	1.033	.2602
.324	1.013	.1600	.396	1.019	.1943	.468	1.026	.2281	.540	1.034	.2612
.326	1.013	.1609	.398	1.019	.1953	.470	1.026	.2290	.542	1.034	.2621
.328	1.013	.1619	.400	1.019	.1962	.472	1.026	.2300	.544	1.034	.2630
.330	1.013	.1629	.402	1.019	.1972	.474	1.026	.2309	.546	1.034	.2639
.332	1.013	.1638	.404	1.019	.1981	.476	1.026	.2318	.548	1.034	.2648
.334	1.013	.1648	.406	1.020	.1991	.478	1.027	.2327	.550	1.035	.2658
336	1.014	.1658	.408	1.020	.2001	.480	1.027	.2337	.552	1.035	.2667
.338	1.014	.1667	.410	1.020	.2010	.482	1.027	.2346	.554	1.035	.2676
.340	1.014	.1677	412	1 020	.2020	.484	1.027	.2355	.556	1.035	.2685
.342	1.014	.1686	.414	1.020	.2029	.486	1.028	.2364	.558	1.036	.2694
.344	1.014	.1696	.416	1.020	.2038	.488	1.028	.2372	.560	1.036	.2703
.346	1.014	.1705	.418	1.021	.2048	.490	1.028	.2383	.562	1.036	.2712
.348	1.015	.1715	.420	1.021	.2057	.492	1.028	.2392	.564	1.036	.2721
.350	1.015	.1724	.422	1.021	.2067	.494	1.028	.2401	.566	1.036	.2730
.352	1.015	.1734	.424	1.021	.2076	.496	1.029	.2411	.568	1.037	.2739
.354	1.015	.1743	.426	1.022	.2085	.498	1.029	.2420	.570	1.037	.2748
.356	1.015	.1753	.428	1.022	.2095	.500	1.029	.2429	.572	1.037	.2757

h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)
.572	1.037	.2757	1.04	1.106	.4702	1.76	1.230	.715	2.48	1.356	.915
.574	1.038	.2766	1.06	1.109	.4779	1.78	1.233	.722	2.50	1.359	.920
.576	1.038	.2775	1.08	1.113	.4852	1.80	1.237	.728	2.52	1.362	.925
.578	1.038	.2784	1.10	1.116	.4933	1.82	1.241	.733	2.54	1.366	.930
.580	1.038	.2793	1.12	1.119	.5005	1.84	1.244	.740	2.56	1.369	.935
.582	1.039	.2802	1.14	1.122	.5080	1.86	1.248	.745	2.58	1.373	.939
.584	1.039	.2811	1.16	1.126	.5151	1.88	1.251	.751	2.60	1.376	.945
.586	1.039	.2820	1.18	1.129	.5225	1.90	1.255	.757	2.62	1.379	.950
.588	1.039	.2829	1.20	1.132	.530	1.92	1.258	.763	2.64	1.383	.955
.590	1.040	.2838	1.22	1.135	.538	1.94	1.262	.769	2.66	1.386	.960
.592	1.040	.2847	1.24	1.139	.544	1.96	1.265	.775	2.68	1.390	.964
.594	1.040	.2856	1.26	1.142	.552	1.98	1.269	.780	2.70	1.393	.969
.596	1.040	.2865	1.28	1.146	.559	2.00	1.272	.786	2.72	1.396	.974
.598	1.041	.2874	1.30	1.149	.565	2.02	1.275	.792	2.74	1.400	.979
.60	1.041	.2882	1.32	1.153	.572	2.04	1.279	.798	2.76	1.403	.984
.62	1.044	.2969	1.34	1.156	.580	2.06	1.283	.803	2.78	1.407	.989
.64	1.046	.3059	1.36	1.160	.586	2.08	1.286	.809	2.80	1.410	.993
.66	1.049	.3146	1.38	1.163	.593	2.10	1.290	.814	2.82	1.413	.998
.68	1.051	.3235	1.40	1.167	.600	2.12	1.293	.820	2.84	1.416	1.003
.70	1.054	.3321	1.42	1.170	.607	2.14	1.297	.825	2.86	1.420	1.007
.72	1.057	.3406	1.44	1.174	.613	2.16	1.300	.831	2.88	1.423	1.012
.74	1.060	.3491	1.46	1.178	.620	2.18	1.304	.836	2.90	1.426	1.017
.76	1.062	.3578	1.48	1.181	.627	2.20	1.307	.842	2.92	1.429	1.022
.78	1.065	.3662	1.50	1.184	.634	2.22	1.310	.847	2.94	1.433	1.026
.80	1.068	.3745	1.52	1.188	.640	2.24	1.314	.852	2.96	1.436	1.031
.82	1.071	.3828	1.54	1.191	.647	2.26	1.317	.858	2.98	1.440	1.035
.84	1.074	.3911	1.56	1.195	.653	2.28	1.321	.863	3.00	1.443	1.040
.86	1.077	.3993	1.58	1.198	.659	2.30	1.324	.869	3.02	1.446	1.044
.88	1.080	.4074	1.60	1.202	.666	2.32	1.328	.874	3.04	1.449	1.049
.90	1.083	.4155	1.62	1.205	.672	2.34	1.331	.879	3.06	1.453	1.053
.92	1.086	.4236	1.64	1.209	.678	2.36	1.335	.884	3.08	1.456	1.058
.94	1.089	.4316	1.66	1.212	.685	2.38	1.338	.889	3.10	1.459	1.062
.96	1.093	.4392	1.68	1.216	.691	2.40	1.342	.894	3.12	1.462	1.067
.98	1.096	.4471	1.70	1.219	.697	2.42	1.345	.900	3.14	1.465	1.072
1.0	1.099	.4552	1.72	1.223	.703	2.44	1.349	.904	3.16	1.469	1.076
1.02	1.102	.4628	1.74	1.226	.710	2.46	1.352	.910	3.18	1.472	1.080

h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)
3.18	1.472	1.080	3.90	1.585	1.230	4.62	1.692	1.365	5.34	1.794	1.490
3.20	1.475	1.085	3.92	1.588	1.234	4.64	1.695	1.369	5.36	1.797	1.493
3.22	1.478	1.089	3.94	1.591	1.238	4.66	1.698	1.372	5.38	1.800	1.497
3.24	1.482	1.093	3.96	1.595	1.241	4.68	1.701	1.376	5.40	1.802	1.499
3.26	1.485	1.098	3.98	1.598	1.245	4.70	1.704	1.379	5.42	1.804	1.502
3.28	1.489	1.102	4.00	1.601	1.249	4.72	1.706	1.383	5.44	1.807	1.506
3.30	1.492	1.106	4.02	1.604	1.253	4.74	1.709	1.387	5.46	1.810	1.509
3.32	1.495	1.110	4.04	1.607	1.257	4.76	1.712	1.390	5.48	1.813	1.513
3.34	1.498	1.115	4.06	1.610	1.261	4.78	1.715	1.394	5.50	1.815	1.515
3.36	1.502	1.119	4.08	1.613	1.265	4.80	1.718	1.397	5.52	1.818	1.518
3.38	1.505	1.123	4.10	1.616	1.269	4.82	1.721	1.400	5.54	1.821	1.522
3.40	1.508	1.127	4.12	1.619	1.273	4.84	1.723	1.403	5.56	1.824	1.526
3.42	1.511	1.132	4.14	1.622	1.276	4.86	1.726	1.407	5.58	1.827	1.529
3.44	1.514	1.136	4.16	1.625	1.280	4.88	1.729	1.411	5.60	1.829	1.531
3.46	1.517	1.140	4.18	1.628	1.284	4.90	1.732	1.415	5.62	1.831	1.535
3.48	1.520	1.145	4.20	1.631	1.288	4.92	1.735	1.418	5.64	1.834	1.537
3.50	1.523	1.149	4.22	1.634	1.291	4.94	1.738	1.422	5.66	1.837	1.540
3.52	1.526	1.153	4.24	1.637	1.295	4.96	1.741	1.425	5.68	1.840	1.546
3.54	1.529	1.158	4.26	1.640	1.299	4.98	1.743	1.429	5.70	1.842	1.548
3.56	1.533	1.161	4.28	1.642	1.303	5.00	1.746	1.432	5.72	1.844	1.550
3.58	1.536	1.165	4.30	1.645	1.307	5.02	1.749	1.435	5.74	1.847	1.553
3.60	1.539	1.170	4.32	1.648	1.310	5.04	1.752	1.439	5.76	1.850	1.557
3.62	1.542	1.174	4.34	1.651	1.314	5.06	1.755	1.442	5.78	1.853	1.560
3.64	1.545	1.178	4.36	1.654	1.318	5.08	1.758	1.445	5.80	1.855	1.562
3.66	1.549	1.181	4.38	1.657	1.322	5.10	1.760	1.449	5.82	1.857	1.564
3.68	1.552	1.186	4.40	1.660	1.325	5.12	1.763	1.452	5.84	1.860	1.568
3.70	1.555	1.190	4.42	1.663	1.328	5.14	1.766	1.455	5.86	1.863	1.571
3.72	1.558	1.194	4.44	1.666	1.332	5.16	1.769	1.459	5.88	1.866	1.575
3.74	1.561	1.198	4.46	1.669	1.336	5.18	1.772	1.463	5.90	1.868	1.577
3.76	1.564	1.202	4.48	1.672	1.340	5.20	1.774	1.465	5.92	1.871	1.581
3.78	1.567	1.206	4.50	1.675	1.343	5.22	1.777	1.468	5.94	1.874	1.584
3.80	1.570	1.210	4.52	1.678	1.346	5.24	1.780	1.473	5.96	1.877	1.588
3.82	1.573	1.214	4.54	1.681	1.350	5.26	1.783	1.477	5.98	1.880	1.592
3.84	1.576	1.218	4.56	1.684	1.354	5.28	1.786	1.480	6.00	1.882	1.594
3.86	1.579	1.222	4.58	1.686	1.358	5.30	1.788	1.482	6.02	1.885	1.598
3.88	1.582	1.226	4.60	1.689	1.362	5.32	1.791	1.486	6.04	1.888	1.602

h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)
6.04	1.888	1.602	6.76	1.979	1.708	7.48	2.066	1.808	8.20	2.151	1.904
6.06	1.890	1.604	6.78	1.981	1.710	7.50	2.069	1.811	8.22	2.153	1.906
6.08	1.892	1.606	6.80	1.984	1.713	7.52	2.071	1.813	8.24	2.156	1.909
6.10	1.895	1.609	6.82	1.986	1.716	7.54	2.074	1.816	8.26	2.158	1.912
6.12	1.898	1.613	6.84	1.989	1.719	7.56	2.076	1.819	8.28	2.160	1.915
6.14	1.900	1.616	6.86	1.991	1.722	7.58	2.078	1.822	8.30	2.163	1.918
6.16	1.903	1.619	6.88	1.994	1.725	7.60	2.081	1.825	8.32	2.165	1.920
6.18	1.905	1.621	6.90	1.996	1.727	7.62	2.083	1.828	8.34	2.167	1.923
6.20	1.908	1.625	6.92	1.999	1.730	7.64	2.086	1.830	8.36	2.170	1.926
6.22	1.911	1.628	6.94	2.001	1.733	7.66	2.088	1.833	8.38	2.172	1.928
6.24	1.913	1.630	6.96	2.004	1.736	7.68	2.090	1.836	8.40	2.174	1.930
6.26	1.916	1.633	6.98	2.006	1.739	7.70	2.093	1.838	8.42	2.176	1.933
6.28	1.918	1.636	7.00	2.009	1.743	7.72	2.095	1.841	8.44	2.179	1.935
6.30	1.921	1.639	7.02	2.011	1.745	7.74	2.097	1.844	8.46	2.181	1.938
6.32	1.924	1.641	7.04	2.014	1.748	7.76	2.100	1.847	8.48	2.183	1.941
6.34	1.926	1.645	7.06	2.016	1.751	7.78	2.102	1.850	8.50	2.186	1.944
6.36	1.929	1.649	7.08	2.019	1.754	7.80	2.104	1.852	8.52	2.188	1.946
6.38	1.931	1.652	7.10	2.021	1.756	7.82	2.106	1.854	8.54	2.190	1.949
6.40	1.934	1.655	7.12	2.023	1.758	7.84	2.109	1.857	8.56	2.193	1.952
6.42	1.936	1.657	7.14	2.026	1.761	7.86	2.112	1.860	8.58	2.195	1.954
6.44	1.939	1.661	7.16	2.028	1.764	7.88	2.114	1.863	8.60	2.197	1.956
6.46	1.941	1.664	7.18	2.030	1.767	7.90	2.117	1.866	8.62	2.199	1.959
6.48	1.944	1.667	7.20	2.033	1.770	7.92	2.119	1.868	8.64	2.202	1.962
6.50	1.946	1.669	7.22	2.035	1.772	7.94	2.122	1.871	8.66	2.204	1.965
6.52	1.949	1.672	7.24	2.038	1.775	7.96	2.124	1.873	8.68	2.207	1.967
6.54	1.951	1.675	7.26	2.040	1.778	7.98	2.126	1.876	8.70	2.209	1.970
6.56	1.954	1.678	7.28	2.042	1.781	8.00	2.128	1.878	8.72	2.211	1.972
6.58	1.956	1.680	7.30	2.045	1.784	8.02	2.130	1.881	8.74	2.214	1.975
6.60	1.959	1.684	7.32	2.047	1.786	8.04	2.133	1.884	8.76	2.216	1.977
6.62	1.962	1.689	7.34	2.050	1.789	8.06	2.135	1.886	8.78	2.218	1.980
6.64	1.964	1.690	7.36	2.052	1.792	8.08	2.138	1.889	8.80	2.220	1.982
6.66	1.966	1.692	7.38	2.054	1.795	8.10	2.140	1.892	8.82	2.222	1.984
6.68	1.969	1.696	7.40	2.057	1.798	8.12	2.142	1.894	8.84	2.225	1.987
6.70	1.971	1.698	7.42	2.059	1.800	8.14	2.144	1.897	8.86	2.227	1.989
6.72	1.974	1.702	7.44	2.062	1.803	8.16	2.147	1.900	8.88	2.229	1.992
6.74	1.976	1.705	7.46	2.064	1.805	8.18	2.149	1.902	8.90	2.231	1.994

h	$f(h)$	$g(h)$	h	$f(h)$	$g(h)$	h	$f(h)$	$g(h)$	h	$f(h)$	$g(h)$
8.90	2.231	1.994	9.7	2.319	2.092	13.3	2.678	2.484	16.9	2.995	2.823
8.92	2.233	1.997	9.8	2.329	2.104	13.4	2.687	2.494	17.0	3.003	2.832
8.94	2.236	2.000	9.9	2.340	2.116	13.5	2.696	2.503	17.1	3.011	2.841
8.96	2.238	2.002	10.0	2.351	2.128	13.6	2.705	2.513	17.2	3.019	2.850
8.98	2.240	2.005	10.1	2.362	2.140	13.7	2.714	2.523	17.3	3.027	2.859
9.00	2.242	2.007	10.2	2.372	2.151	13.8	2.723	2.533	17.4	3.035	2.868
9.02	2.244	2.009	10.3	2.382	2.162	13.9	2.732	2.543	17.5	3.044	2.876
9.04	2.247	2.012	10.4	2.392	2.173	14.0	2.741	2.553	17.6	3.052	2.885
9.06	2.249	2.014	10.5	2.403	2.185	14.1	2.750	2.563	17.7	3.060	2.894
9.08	2.251	2.016	10.6	2.413	2.196	14.2	2.759	2.573	17.8	3.068	2.903
9.10	2.253	2.019	10.7	2.423	2.207	14.3	2.768	2.582	17.9	3.076	2.910
9.12	2.255	2.021	10.8	2.433	2.218	14.4	2.777	2.592	18.0	3.084	2.919
9.14	2.258	2.024	10.9	2.443	2.221	14.5	2.787	2.602	18.1	3.093	2.928
9.16	2.260	2.026	11.0	2.454	2.242	14.6	2.796	2.612	18.2	3.101	2.937
9.18	2.262	2.029	11.1	2.465	2.253	14.7	2.805	2.621	18.3	3.109	2.945
9.20	2.264	2.031	11.2	2.475	2.264	14.8	2.814	2.631	18.4	3.117	2.952
9.22	2.266	2.034	11.3	2.484	2.274	14.9	2.823	2.640	18.6	3.133	2.969
9.24	2.269	2.036	11.4	2.494	2.285	15.0	2.832	2.649	18.8	3.148	2.986
9.26	2.271	2.039	11.5	2.504	2.296	15.1	2.841	2.659	19.0	3.164	3.002
9.28	2.273	2.042	11.6	2.514	2.307	15.2	2.850	2.669	19.2	3.180	3.010
9.30	2.275	2.044	11.7	2.524	2.318	15.3	2.859	2.678	19.4	3.196	3.035
9.32	2.277	2.046	11.8	2.534	2.329	15.4	2.867	2.688	19.6	3.227	3.052
9.34	2.280	2.049	11.9	2.544	2.340	15.5	2.876	2.697	19.8	3.231	3.068
9.36	2.282	2.051	12.0	2.554	2.350	15.6	2.885	2.706	20.0	3.243	3.085
9.38	2.284	2.053	12.1	2.563	2.360	15.7	2.894	2.715	20.2	3.258	3.101
9.40	2.286	2.056	12.2	2.573	2.370	15.8	2.903	2.724	20.4	3.273	3.117
9.42	2.288	2.058	12.3	2.582	2.381	15.9	2.911	2.733	20.6	3.288	3.132
9.44	2.291	2.061	12.4	2.592	2.392	16.0	2.919	2.742	20.8	3.303	3.148
9.46	2.293	2.063	12.5	2.601	2.401	16.1	2.928	2.751	21.0	3.318	3.164
9.48	2.295	2.065	12.6	2.610	2.411	16.2	2.936	2.760	21.2	3.333	3.179
9.50	2.297	2.067	12.7	2.620	2.421	16.3	2.945	2.769	21.4	3.348	3.195
9.52	2.299	2.069	12.8	2.630	2.432	16.4	2.953	2.778	21.6	3.362	3.210
9.54	2.302	2.072	12.9	2.640	2.443	16.5	2.961	2.787	21.8	3.377	3.226
9.56	2.304	2.075	13.0	2.650	2.454	16.6	2.970	2.796	22.0	3.392	3.241
9.58	2.306	2.077	13.1	2.660	2.465	16.7	2.979	2.805	22.2	3.401	3.256
9.60	2.308	2.080	13.2	2.669	2.475	16.8	2.987	2.814	22.4	3.421	3.272

h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)
22.4	3.421	3.272	29.6	3.912	3.782	36.8	4.348	4.232	72	6.042	5.959
22.6	3.436	3.287	29.8	3.924	3.795	37	4.359	4.243	73	6.084	6.001
22.8	3.450	3.303	30.0	3.937	3.808	38	4.417	4.302	74	6.123	6.041
23.0	3.465	3.318	30.2	3.950	3.821	39	4.473	4.360	75	6.166	6.084
23.2	3.480	3.333	30.4	3.963	3.835	40	4.528	4.416	76	6.205	6.124
23.4	3.494	3.348	30.6	3.976	3.848	41	4.584	4.473	77	6.245	6.168
23.6	3.509	3.363	30.8	3.989	3.862	42	4.638	4.528	78	6.285	6.205
23.8	3.523	3.378	31.0	3.002	3.875	43	4.692	4.584	79	6.326	6.246
24.0	3.538	3.393	31.2	4.014	3.888	44	4.743	4.636	80	6.364	6.284
24.2	3.551	3.407	31.4	4.026	3.900	45	4.797	4.691	81	6.405	6.326
24.4	3.563	3.423	31.6	4.039	3.913	46	4.848	4.744	82	6.443	6.364
24.6	3.577	3.437	31.8	4.051	3.925	47	4.899	4.795	83	6.481	6.403
24.8	3.590	3.452	32.0	4.063	3.938	48	4.952	4.850	84	6.519	6.442
25.0	3.603	3.467	32.2	4.075	3.951	49	5.000	4.898	85	6.558	6.481·
25.2	3.618	3.481	32.4	4.087	3.963	50	5.050	4.950	86	6.597	6.521
25.4	3.632	3.495	32.6	4.100	3.976	51	5.099	5.000	87	6.633	6.557
25.6	3.647	3.509	32.8	4.112	3.988	52	5.147	5.049	88	6.669	6.593
25.8	3.661	3.524	33.0	4.124	4.001	53	5.198	5.101	89	6.706	6.637
26.0	3.676	3.538	33.2	4.136	4.014	54	5.244	5.148	90	6.746	6.671
26.2	3.689	3.552	33.4	4.148	4.026	55	5.293	5.198	91	6.782	6.708
26.4	3.702	3.565	33.6	4.161	4.039	56	5.341	5.246	92	6.819	6.744
26.6	3.716	3.579	33.8	4.173	4.051	57	5.387	5.294	93	6.857	6.783
26.8	3.729	3.592	34.0	4.185	4.064	58	5.433	5.340	94	6.894	6.821
27.0	3.742	3.606	34.2	4.197	4.076	59	5.478	5.386	95	6.928	6.859
27.2	3.756	3.620	34.4	4.209	4.088	60	5.524	5.433	96	6.966	6.894
27.4	3.769	3.634	34.6	4.220	4.100	61	5.568	5.477	97	7.001	6.929
27.6	3.783	3.648	34.8	4.232	4.112	62	5.612	5.522	98	7.035	6.964
27.8	3.796	3.662	35.0	4.244	4.124	63	5.656	5.567	99	7.056	6.985
28.0	3.810	3.676	35.2	4.256	4.136	64	5.701	5.613	100	7.105	7.035
28.2	3.823	3.689	35.4	4.267	4.148	65	5.746	5.658	101	7.141	7.070
28.4	3.836	3.703	35.6	4.279	4.160	66	5.789	5.712	102	7.176	7.106
28.6	3.848	3.716	35.8	4.290	4.172	67	5.832	5.745	103	7.212	7.142
28.8	3.861	3.730	36.0	4.302	4.184	68	5.875	5.788	104	7.247	7.178
29.0	3.874	3.743	36.2	4.314	4.196	69	5.916	5.830	105	7.279	7.211
29.2	3.887	3.756	36.4	4.325	4.208	70	5.959	5.875	106	7.321	7.244
29.4	3.899	3.760	36.6	4.336	4.219	71	6.000	5.917	107	7.340	7.280

h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)	h	f(h)	g(h)
107	7.349	7.280	143	8.459	8.453	179	9.466	9.458	215	10.371	10.366
108	7.381	7.313	144	8.488	8.482	180	9.489	9.482	216	10.395	10.390
109	7.414	7.347	145	8.518	8.512	181	9.516	9.511	217	10.419	10.414
110	7.440	7.380	146	8.547	8.541	182	9.542	9.537	218	10.443	10.438
111	7.485	7.417	147	8.576	8.580	183	9.568	9.563	219	10.467	10.462
112	7.518	7.451	148	8.605	8.599	184	9.594	9.589	220	10.491	10.486
113	7.552	7.485	149	8.634	8.628	185	9.620	9.615	221	10.514	10.510
114	7.582	7.515	150	8.663	8.657	186	9.646	9.641	222	10.538	10.533
115	7.616	7.550	151	8.692	8.686	187	9.672	9.667	223	10.562	10.557
116	7.650	7.584	152	8.721	8.715	188	9.698	9.693	224	10.585	10.581
117	7.680	7.615	153	8.749	8.744	189	9.724	9.719	225	10.609	10.604
118	7.714	7.649	154	8.778	8.772	190	9.749	9.744	226	10.633	10.628
119	7.745	7.680	155	8.806	8.804	191	9.775	9.770	227	10.656	10.651
120	7.780	7.715	156	8.835	8.829	192	9.800	9.796	228	10.679	10.675
121	7.811	7.746	157	8.863	8.857	193	9.826	9.821	229	10.703	10.698
122	7.842	7.778	158	8.891	8.885	194	9.851	9.846	230	10.726	10.722
123	7.872	7.809	159	8.919	8.914	195	9.877	9.872	231	10.749	10.745
124	7.908	7.844	160	8.947	8.942	196	9.902	9.897	232	10.773	10.768
125	7.939	7.876	161	8.975	8.969	197	9.927	9.922	233	10.796	10.791
126	7.961	7.934	162	9.003	8.997	198	9.952	9.947	234	10.819	10.814
127	7.972	7.966	163	9.031	9.025	199	9.978	9.973	235	10.842	10.838
128	8.003	7.997	164	9.058	9.053	200	10.003	9.993	236	10.865	10.861
129	8.034	8.028	165	9.086	9.080	201	10.028	10.023	237	10.888	10.884
130	8.065	8.059	166	9.113	9.108	202	10.052	10.047	238	10.911	10.906
131	8.096	8.090	167	9.141	9.135	203	10.077	10.072	239	10.934	10.929
132	8.127	8.121	168	9.168	9.163	204	10.102	10.097	240	10.957	10.953
133	8.158	8.152	169	9.195	9.190	205	10.127	10.122	241	10.980	10.975
134	8.189	8.182	170	9.222	9.217	206	10.151	10.146	242	11.002	10.998
135	8.219	8.213	171	9.249	9.244	207	10.176	10.171	243	11.025	11.020
136	8.249	8.243	172	9.276	9.271	208	10.201	10.196	244	11.048	11.043
137	8.280	8.274	173	9.303	9.298	209	10.225	10.220	245	11.070	11.066
138	8.310	8.304	174	9.330	9.325	210	10.249	10.245	246	11.093	11.088
139	8.340	8.334	175	9.357	9.351	211	10.274	10.269	247	11.115	11.111
140	8.370	8.364	176	9.384	9.389	212	10.298	10.293	248	11.136	11.133
141	8.399	8.393	177	9.410	9.405	213	10.322	10.308	249	11.158	11.156
142	8.429	8.423	178	9.437	9.431	214	10.347	10.342	250	11.180	11.178

APPENDIX III

TABLES OF BESSEL FUNCTIONS*

TABLE 1. $J_0(y)$

y	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1.0000	0.9975	0.9900	0.9776	0.9604	0.9885	0.9120	0.8812	0.8463	0.8075
1	0.7652	0.7196	0.6711	0.6201	0.5669	0.5118	0.4554	0.3980	0.3400	0.2818
2	0.2239	0.1666	0.1104	0.0555	0.0025	-0.0484	-0.0968	-0.1424	-0.1850	-0.2243
3	-0.2601	-0.2921	-0.3202	-0.3443	-0.3643	-0.3801	-0.3918	-0.3992	-0.4026	-0.4018
4	-0.3971	-0.3887	-0.3766	-0.3610	-0.3423	-0.3205	-0.2961	-0.2693	-0.2404	-0.2097
5	-0.1776	-0.1443	-0.1103	-0.0758	-0.0412	-0.0068	0.0270	0.0599	0.0917	0.1220
6	0.1506	0.1773	0.2017	0.2238	0.2433	0.2601	0.2740	0.2851	0.2931	0.2981
7	0.3001	0.2991	0.2951	0.2882	0.2786	0.2663	0.2516	0.2346	0.2154	0.1944
8	0.1717	0.1475	0.1222	0.0960	0.0692	0.0419	0.0146	-0.0125	-0.0392	-0.0653
9	-0.0903	-0.1142	-0.1367	-0.1577	-0.1768	-0.1939	-0.2090	-0.2218	-0.2323	-0.2403
10	-0.2459	-0.2490	-0.2496	-0.2477	-0.2434	-0.2366	-0.2276	-0.2164	-0.2032	-0.1881
11	-0.1712	-0.1528	-0.1330	-0.1121	-0.0902	-0.0677	-0.0446	-0.0213	0.0020	0.0250
12	0.0477	0.0697	0.0908	0.1108	0.1296	0.1469	0.1626	0.1766	0.1887	0.1988
13	0.2069	0.2129	0.2167	0.2183	0.2177	0.2150	0.2101	0.2032	0.1943	0.1836
14	0.1711	0.1570	0.1414	0.1245	0.1065	0.0875	0.0679	0.0476	0.0271	0.0064
15	-0.0142	-0.0346	-0.0544	-0.0736	-0.0919	-0.1092	-0.1253	-0.1401	-0.1533	-0.1650

For $y \geqq 16$, an approximate formula is

$$J_0(y) \doteq \sqrt{\frac{2}{\pi y}} \left[\sin\left(y + \frac{\pi}{4}\right) + \frac{1}{8y} \sin\left(y - \frac{\pi}{4}\right) \right]$$

*By permission of the author and the publisher the numerical tables are taken from N. W. McLachlan, "Bessel Functions for Engineers," (Oxford, Clarendon Press, 1934).

TABLE 2. $J_1(y)$

y	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0.0000	0.0499	0.0995	0.1483	0.1960	0.2423	0.2867	0.3290	0.3688	0.4059
1	0.4401	0.4709	0.4983	0.5220	0.5419	0.5579	0.5699	0.5778	0.5815	0.5812
2	0.5767	0.5683	0.5560	0.5399	0.5202	0.4971	0.4708	0.4416	0.4097	0.3754
3	0.3391	0.3009	0.2613	0.2207	0.1792	0.1374	0.0955	0.0538	0.0128	-0.0272
4	-0.0660	-0.1033	-0.1386	-0.1719	-0.2028	-0.2311	-0.2566	-0.2791	-0.2985	-0.3147
5	-0.3276	-0.3371	-0.3432	-0.3460	-0.3453	-0.3414	-0.3343	-0.3241	-0.3110	-0.2951
6	-0.2767	-0.2559	-0.2329	-0.2081	-0.1816	-0.1538	-0.1250	-0.0953	-0.0652	-0.0349
7	-0.0047	0.0252	0.0543	0.0826	0.1096	0.1352	0.1592	0.1813	0.2014	0.2192
8	0.2346	0.2476	0.2580	0.2657	0.2708	0.2731	0.2728	0.2697	0.2641	0.2559
9	0.2453	0.2324	0.2174	0.2004	0.1816	0.1613	0.1395	0.1166	0.0928	0.0684
10	0.0435	0.0184	-0.0066	-0.0313	-0.0555	-0.0789	-0.1012	-0.1224	-0.1422	-0.1603
11	-0.1768	-0.1913	-0.2039	-0.2143	-0.2225	-0.2284	-0.2320	-0.2333	-0.2323	-0.2290
12	-0.2234	-0.2157	-0.2060	-0.1943	-0.1807	-0.1655	-0.1487	-0.1307	-0.1114	-0.0912
13	-0.0703	-0.0489	-0.0271	-0.0052	0.0166	0.0380	0.0590	0.0791	0.0984	0.1165
14	0.1334	0.1488	0.1626	0.1747	0.1850	0.1934	0.1999	0.2043	0.2066	0.2069
15	0.2051	0.2013	0.1955	0.1879	0.1784	0.1672	0.1544	0.1402	0.1247	0.1080

For $y \geqq 16$, an approximate formula is

$$J_1(y) \doteq \sqrt{\frac{2}{\pi y}} \left[\sin\left(y - \frac{\pi}{4}\right) + \frac{3}{8y} \sin\left(y + \frac{\pi}{4}\right) \right]$$

TABLE 3. $N_0(y)$

y	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	$-\infty$	-1.534	-1.081	-0.8073	-0.6060	-0.4445	-0.3085	-0.1907	-0.0868	0.0056
1	0.0883	0.1622	0.2281	0.2865	0.3379	0.3824	0.4204	0.4520	0.4774	0.4968
2	0.5104	0.5183	0.5208	0.5181	0.5104	0.4981	0.4813	0.4605	0.4359	0.4079
3	0.3769	0.3431	0.3071	0.2691	0.2296	0.1890	0.1477	0.1061	0.0645	0.0234
4	-0.0169	-0.0561	-0.0938	-0.1296	-0.1633	-0.1947	-0.2235	-0.2494	-0.2723	-0.2921
5	-0.3085	-0.3216	-0.3313	-0.3374	-0.3402	-0.3395	-0.3354	-0.3282	-0.3177	-0.3044
6	-0.2882	-0.2694	-0.2483	-0.2251	-0.1999	-0.1732	-0.1452	-0.1162	-0.0864	-0.0563
7	-0.0259	0.0042	0.0339	0.0628	0.0907	0.1173	0.1424	0.1658	0.1872	0.2065
8	0.2235	0.2381	0.2501	0.2595	0.2662	0.2702	0.2715	0.2700	0.2659	0.2592
9	0.2499	0.2383	0.2245	0.2086	0.1907	0.1712	0.1502	0.1279	0.1045	0.0804
10	0.0557	0.0307	0.0056	-0.0193	-0.0437	-0.0675	-0.0904	-0.1122	-0.1326	-0.1516
11	-0.1688	-0.1843	-0.1977	-0.2091	-0.2183	-0.2252	-0.2299	-0.2322	-0.2322	-0.2298
12	-0.2252	-0.2184	-0.2095	-0.1986	-0.1858	-0.1712	-0.1551	-0.1375	-0.1187	-0.0989
13	-0.0782	-0.0569	-0.0352	-0.0134	0.0085	0.0301	0.0512	0.0717	0.0913	0.1099
14	0.1272	0.1431	0.1575	0.1703	0.1812	0.1903	0.1974	0.2025	0.2056	0.2065
15	0.2055	0.2023	0.1972	0.1902	0.1813	0.1706	0.1584	0.1446	0.1295	0.1132

For $y \geqq 16$, an approximate formula is

$$N_0(y) \doteq \sqrt{\frac{2}{\pi y}} \left[\sin\left(y - \frac{\pi}{4}\right) - \frac{1}{8y} \sin\left(y + \frac{\pi}{4}\right) \right]$$

TABLE 4. $N_1(y)$

y	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	$-\infty$	-6.459	-3.324	-2.293	-1.781	-1.471	-1.260	-1.103	-0.9781	-0.8731
1	-0.7812	-0.6981	-0.6211	-0.5485	-0.4791	-0.4123	-0.3476	-0.2847	-0.2237	-0.1644
2	-0.1070	-0.0517	0.0015	0.0523	0.1005	0.1459	0.1884	0.2276	0.2635	0.2959
3	0.3247	0.3496	0.3707	0.3879	0.4010	0.4102	0.4154	0.4167	0.4141	0.4078
4	0.3979	0.3846	0.3680	0.3484	0.3260	0.3010	0.2737	0.2445	0.2136	0.1812
5	0.1479	0.1137	0.0792	0.0445	0.0101	-0.0238	-0.0568	-0.0887	-0.1192	-0.1481
6	-0.1750	-0.1998	-0.2223	-0.2422	-0.2596	-0.2741	-0.2857	-0.2945	-0.3002	-0.3029
7	-0.3027	-0.2995	-0.2934	-0.2846	-0.2731	-0.2591	-0.2428	-0.2243	-0.2039	-0.1817
8	-0.1581	-0.1331	-0.1072	-0.0806	-0.0535	-0.0262	0.0011	0.0280	0.0544	0.0799
9	0.1043	0.1275	0.1491	0.1691	0.1871	0.2032	0.2171	0.2287	0.2379	0.2447
10	0.2490	0.2508	0.2502	0.2471	0.2416	0.2337	0.2236	0.2114	0.1973	0.1813
11	0.1637	0.1446	0.1243	0.1029	0.0807	0.0579	0.0348	0.0114	-0.0118	-0.0347
12	-0.0571	-0.0787	-0.0994	-0.1189	-0.1371	-0.1538	-0.1689	-0.1821	-0.1935	-0.2028
13	-0.2101	-0.2152	-0.2182	-0.2190	-0.2176	-0.2140	-0.2084	-0.2007	-0.1912	-0.1798
14	-0.1666	-0.1520	-0.1359	-0.1186	-0.1003	-0.0810	-0.0612	-0.0408	-0.0202	0.0005
15	0.0211	0.0413	0.0609	0.0799	0.0979	0.1148	0.1305	0.1447	0.1575	0.1686

For $y \geqq 16$, an approximate formula is

$$N_1(y) \doteq \sqrt{\frac{2}{\pi y}} \left[\sin\left(y - \frac{3\pi}{4}\right) + \frac{3}{8y} \sin\left(y - \frac{\pi}{4}\right) \right]$$

TABLE 5. $J_0(yj^{-1/2}) = J_0(yj^{3/2}) = M_0(y)e^{j\theta_0(y)}$

y	$M_0(y)$	$\log_{10}\sqrt{y}\,M_0(y)$	$\theta_0(y)$	y	$M_0(y)$	$\log_{10}\sqrt{y}\,M_0(y)$	$\theta_0(y)$
0.00	1.000	.....	0.00°	2.0	1.229	0.2401	52.29°
0.05	1.000	$\bar{1}$.3995	0.04	2.1	1.274	0.2663	56.74
0.10	1.000	$\bar{1}$.5000	0.14	2.2	1.325	0.2933	61.22
0.15	1.000	$\bar{1}$.5880	0.32	2.3	1.381	0.3210	65.71
0.20	1.000	$\bar{1}$.6505	0.57	2.4	1.443	0.3493	70.19
0.25	1.000	$\bar{1}$.6990	0.90	2.5	1.511	0.3783	74.65
0.30	1.000	$\bar{1}$.7386	1.29	2.6	1.586	0.4077	79.09
0.35	1.000	$\bar{1}$.7721	1.75	2.7	1.666	0.4375	83.50
0.40	1.000	$\bar{1}$.8012	2.20	2.8	1.754	0.4676	87.87
0.45	1.001	$\bar{1}$.8260	2.90	2.9	1.849	0.4980	92.21
0.50	1.001	$\bar{1}$.8499	3.58	3.0	1.950	0.5286	96.52
0.55	1.001	$\bar{1}$.8708	4.33	3.1	2.059	0.5594	100.79
0.60	1.002	$\bar{1}$.8900	5.15	3.2	2.176	0.5902	105.03
0.65	1.003	$\bar{1}$.9077	6.04	3.3	2.301	0.6212	109.25
0.70	1.004	$\bar{1}$.9242	7.01	3.4	2.434	0.6521	113.43
0.75	1.005	$\bar{1}$.9397	8.04	3.5	2.576	0.6830	117.60
0.80	1.006	$\bar{1}$.9543	9.14	3.6	2.728	0.7140	121.75
0.85	1.008	$\bar{1}$.9682	10.31	3.7	2.889	0.7449	125.87
0.90	1.010	$\bar{1}$.9815	11.55	3.8	3.061	0.7758	129.99
0.95	1.013	$\bar{1}$.9943	12.86	3.9	3.244	0.8067	134.10
1.00	1.016	0.0067	14.23	4.0	3.439	0.8375	138.19
1.05	1.019	0.0187	15.66	4.5	4.618	0.9910	158.59
1.10	1.023	0.0304	17.16	5.0	6.231	1.1441	178.93
1.15	1.027	0.0419	18.72	5.5	8.447	1.2969	199.28
1.20	1.032	0.0533	20.34	6.0	11.50	1.4498	219.62
1.25	1.038	0.0645	22.02	7.0	21.55	1.7560	260.29
1.30	1.044	0.0756	23.75	8.0	40.82	2.0624	300.92
1.35	1.051	0.0867	25.54	9.0	77.96	2.3690	341.52
1.40	1.059	0.0978	27.37	10.0	149.8	2.6756	382.10
1.45	1.067	0.1089	29.26	11.0	289.5	2.9824	422.66
1.50	1.077	0.1201	31.19	12.0	561.8	3.2892	463.22
1.55	1.087	0.1314	33.16	14.0	2,137	3.9029	544.32
1.60	1.098	0.1428	35.17	16.0	8,217	4.5168	625.40
1.65	1.111	0.1544	37.22	18.0	3,185$_1$	5.1307	706.46
1.70	1.124	0.1661	39.30	20.0	1,242$_2$	5.7447	787.52
1.75	1.139	0.1779	41.41	25.0	3,809$_3$	7.2708	990.15
1.80	1.154	0.1900	43.51	30.0	1,192$_5$	8.8150	1,192.75
1.85	1.171	0.2022	45.70	35.0	3,786$_6$	10.3502	1,395.35
1.90	1.189	0.2146	47.88	40.0	1,215$_8$	11.8850	1,597.94
1.95	1.208	0.2273	50.08	45.0	3,920$_9$	13.4209	1,800.53

(1,215$_8$ represents 1,215 × 10⁸.)

For $y > 45$, approximate formulas are

$$\log_{10} M_0(y) = 0.3071y + \frac{0.0381}{y} - 0.3001 - \tfrac{1}{2}\log_{10} y$$

$$\theta_0(y) = 40.°514y - \frac{5.°709}{y} - 22.°5$$

If interpolation is required when $y \geqq 2.7$, the function $\ln[\sqrt{y}\,M_0(y)]$ should be used and $M_0(y)$ computed from it.

TABLE 6. $J_1(yj^{-\frac{1}{2}})e^{j180°} = J_1(yj^{\frac{3}{2}}) = M_1(y)e^{j\theta_1(y)}$

y	$M_1(y)$	$\log_{10}\sqrt{y}M_1(y)$	$\theta_1(y)$	y	$M_1(y)$	$\log_{10}\sqrt{y}M_1(y)$	$\theta_1(y)$
0.00	0.0000	......	135.00°	2.25	1.199	0.2548	170.50°
0.05	0.0250	$\bar{3}.7474$	135.02	2.30	1.232	0.2715	172.03
0.10	0.0500	$\bar{2}.1990$	135.07	2.35	1.266	0.2881	173.58
0.15	0.0750	$\bar{2}.4631$	135.16	2.40	1.301	0.3045	175.16
0.20	0.1000	$\bar{2}.6505$	135.29	2.45	1.337	0.3207	176.76
0.25	0.1250	$\bar{2}.7959$	135.45	2.50	1.374	0.3368	178.39
0.30	0.1500	$\bar{2}.9147$	135.64	2.55	1.411	0.3529	180.03
0.35	0.1750	$\bar{1}.0151$	135.88	2.60	1.450	0.3688	181.70
0.40	0.2000	$\bar{1}.1021$	136.15	2.65	1.489	0.3846	183.39
0.45	0.2250	$\bar{1}.1788$	136.45	2.70	1.530	0.4004	185.10
0.50	0.2500	$\bar{1}.2475$	136.79	2.80	1.615	0.4317	188.57
0.55	0.2751	$\bar{1}.3096$	137.17	2.90	1.705	0.4628	192.11
0.60	0.3001	$\bar{1}.3663$	137.58	3.00	1.800	0.4938	195.71
0.65	0.3252	$\bar{1}.4185$	138.03	3.10	1.901	0.5247	199.37
0.70	0.3502	$\bar{1}.4669$	138.51	3.20	2.009	0.5555	203.08
0.75	0.3753	$\bar{1}.5119$	139.03	3.30	2.124	0.5863	206.83
0.80	0.4004	$\bar{1}.5541$	139.58	3.40	2.240	0.6171	210.62
0.85	0.4256	1.5937	140.17	3.50	2.376	0.6479	214.44
0.90	0.4508	$\bar{1}.6311$	140.80	3.60	2.515	0.6788	218.30
0.95	0.4260	$\bar{1}.6665$	141.46	3.70	2.664	0.7096	222.17
1.00	0.5013	$\bar{1}.7001$	142.16	3.80	2.823	0.7405	226.07
1.05	0.5267	$\bar{1}.7321$	142.89	4.00	3.173	0.8025	233.90
1.10	0.5521	$\bar{1}.7627$	143.66	4.25	3.681	0.8801	243.77
1.15	0.5776	$\bar{1}.7920$	144.46	4.50	4.278	0.9579	253.67
1.20	0.6032	$\bar{1}.8201$	145.29	5.00	5.809	1.1136	273.55
1.25	0.6290	$\bar{1}.8471$	146.17	5.5	7.925	1.2692	293.48
1.30	0.6548	$\bar{1}.8731$	147.07	6.0	10.85	1.4245	313.45
1.35	0.6808	$\bar{1}.8982$	148.02	6.5	14.00	1.5795	333.46
1.40	0.7070	$\bar{1}.9225$	148.99	7.0	20.50	1.7343	353.51
1.45	0.7333	$\bar{1}.9460$	150.00	7.5	28.27	1.8889	373.59
1.50	0.7598	$\bar{1}.9688$	151.04	8.0	39.07	2.0434	393.69
1.55	0.7866	$\bar{1}.9909$	152.12	9.0	74.97	2.3520	433.96
1.60	0.8136	0.0125	153.23	10.0	144.7	2.6604	474.28
1.65	0.8408	0.0335	154.38	11.0	280.4	2.9685	514.63
1.70	0.8684	0.0539	155.55	12.0	545.0	3.2765	555.02
1.75	0.8962	0.0739	156.76	14.0	2,084	3.8920	635.84
1.80	0.9244	0.0935	158.00	16.0	8,038	4.5072	716.72
1.85	0.9530	0.1127	159.27	18.0	$3{,}123_1$	5.1222	797.63
1.90	0.9819	0.1315	160.57	20.0	$1{,}220_2$	5.7370	878.57
1.95	1.011	0.1499	161.90	25.0	$3{,}755_2$	7.2736	1,080.98
2.00	1.041	0.1680	163.27	30.0	$1{,}178_3$	8.8000	1,283.45
2.05	1.072	0.1859	164.66	35.0	$3{,}748_6$	10.3459	1,485.94
2.10	1.102	0.2035	166.08	40.0	$1{,}204_8$	11.8817	1,688.46
2.15	1.134	0.2208	167.53	45.0	$3{,}800_9$	13.4175	1,890.98
2.20	1.166	0.2379	169.00	50.0	$1{,}270_{11}$	14.9532	2,093.52

($1{,}178_3$ represents $1,718 \times 10^5$.)

For $y > 50$, approximate formulas are

$$\log_{10} M_1(y) \doteq 0.3071y - \frac{0.1152}{y} - 0.3091 - \frac{1}{2}\log_{10} y$$

$$\theta_1(y) \doteq 40.°514y + \frac{15.°19}{y} + 67.°5$$

If interpolation is required when $y \geqq 3.3$, the function $\ln [\sqrt{y}\, M_1(y)]$ should be used and $M_0(y)$ computed from it.

APPENDIX IV

MATERIAL CONSTANTS

CONSTANTS OF CONDUCTORS AND SEMICONDUCTORS

Material in order of decreasing conductivity	Conductivity, σ in $\dfrac{1}{\text{ohm-m}}$ at 20°C. constant	Relative dielectric constant ϵ_r at 20°C.
Silver...................................	6.14×10^7	1
Copper, annealed........................	5.80×10^7	1
Copper, hard drawn......................	5.65×10^7	1
Aluminum...............................	3.54×10^7	1
Tungsten................................	1.81×10^7	1
Zinc....................................	1.74×10^7	1
Brass (30 % Zinc).......................	$1.2 - 1.5 \times 10^7$	1
Nickel..................................	1.28×10^7	1
Iron, pure..............................	1.00×10^7	1
Steel...................................	$0.5 - 1 \times 10^7$	1
Tin.....................................	0.87×10^7	1
Manganin (84 % Cu, 12 % Mn, 4 % Ni)....	0.23×10^7	1
Constantin (60 % Cu, 40 % Ni)..........	0.20×10^7	1
Nichrome................................	0.10×10^7	1
Salt water..............................	$3 - 5$	80
Wet earth...............................	$10^{-2} - 10^{-3}$	$5 - 15$
Lake water..............................	$10^{-2} - 10^{-3}$	80
Distilled water.........................	2×10^{-4}	81
Dry earth...............................	$10^{-4} - 10^{-5}$	$2 - 6$

CONSTANTS OF DIELECTRICS AT LOW FREQUENCIES

Material in order of decreasing conductivity	Conductivity, σ in $\dfrac{1}{\text{ohm-m}}$ at 20°C.	Relative dielectric constant ϵ_r at 20°C.
Slate...........................	10^{-6}	$6.6 - 7.4$
Bakelite........................	$10^{-8} - 10^{-10}$	5.5
Wood, paraffined................	$10^{-8} - 10^{-11}$	$2 - 7$
Mica...........................	$10^{-11} - 10^{-15}$	2.1
Glass..........................	10^{-12}	$6 - 8$
Shellac........................	10^{-14}	$2.7 - 3.7$
Petroleum......................	10^{-14}	$2.0 - 3.2$
Paraffin.......................	$10^{-14} - 10^{-16}$	$1.9 - 2.3$
Rubber, hard...................	$10^{-14} - 10^{-16}$	$2.0 - 3.2$

CONSTANTS OF DIELECTRICS AT HIGH FREQUENCIES*

Material	Relative dielectric constant ϵ_{er}	Power factor $\sigma_e/\omega\epsilon_r = \epsilon''/\epsilon'$	Frequency, mega-hertz
Bakelite (sheet)...........	3.57	0.080	$3,000$
Glass.....................	$3.8 - 8.7$	$0.001 - 0.01$	$1,200$
Lucite....................	$\begin{cases} 2.58 \\ 2.56 \end{cases}$	0.0090 0.0087	$1,200$ $3,000$
Mycalex, red..............	5.91	0.0030	$1,200$
Mycalex, white............	5.74	0.0033	$1,200$
Paraffin..................	2.17	0.00019	$1,200$
Polyethylene..............	2.26	$0.00031 - 0.0023$	$1,200$
Polystyrene...............	2.45	$0.00028 - 0.00090$	$1,200$
Rubber, hard, black.......	2.69	0.00059	$3,000$
Rubber, soft, black.......	3.15	0.0058	$1,200$

* C. R. ENGLUND, *Bell System Tech. Jour.*, Vol. 23, p. 125 (January, 1944).

APPENDIX V

DIMENSIONS AND UNITS

V-1. Theoretical Analogues and Experimental Pointer Readings.—The primary purpose of the mathematical model of electromagnetism is to calculate theoretical analogues of directly observable pointer readings—these to be determined by means of suitably arranged experiments. Such an aim evidently can be achieved only if the equations constituting the mathematical structure have been so devised, and the symbols appearing in them have been so defined, that each equation or set of equations may be reduced to a relation between direct theoretical analogues of pointer readings. This does not mean that each, or even a single one, of the symbols appearing in a given equation has a direct experimental analogue. In fact, this usually is not the case. It does mean, however, that every symbol in every equation forming a part of the mathematical model must somewhere and somehow be functionally related to other symbols that do have such analogues. For example, the electric and magnetic vectors E and B have no experimental analogues as written in the field equations. But through the force equation they are functionally related to deflection variables of the mathematical-mechanical model. And these may be chosen to have direct experimental analogues.

All experimental pointer readings express how many times an arbitrary scale unit is contained in the observed scale interval that constitutes a measurement. The existence of basic scale units or standards is thus essential to all quantitative determinations. The establishment of standards may be described in terms of two interrelated parts. The first of these involves the selection of a minimum number of so-called fundamental dimensions from the symbols of the mathematical model. Such dimensions may themselves have direct experimental analogues, or they may be functionally related in a more or less complicated way to quantities that have such analogues. The second part is concerned with the experimental determination of the pointer readings in terms of which the unit dimension is to be defined.

Both the selection of fundamental dimensions and the experimental technique used to specify the units offer a wide variety of possibilities.[1] In every case, however, the basic criteria should be those of simplicity, convenience, and accuracy. In particular, the functional relationship between a proposed fundamental dimension and a quantity for which direct experimental analogues can be provided should be a simple one. Where possible, it is desirable to choose dimensions which themselves have direct experimental analogues. Furthermore, the experimental technique involved in obtaining the pointer readings in terms of which the unit is to be specified must be simple and easily and accurately reproduced.

It is possible to distinguish among three different methods of establishing fundamental units for a given choice of fundamental dimension.[2] (1) An arbitrary standard is constructed once and for all, and permanent constancy and invariance are *assumed* for it. Examples of this method are the setting aside of carefully selected pieces of matter such as the standard meter and the standard kilogram. (2) Proportionality constants appearing in equations in the mathematical model that involve the quantity chosen as a fundamental dimension are given arbitrary numerical values. The unit mass might be defined in terms of Newton's law of universal gravitation

$$f = \frac{km_1 m_2}{r^2}$$

By arbitrarily setting the constant k equal to 1, the definition of unit mass would be that mass which when placed a unit distance from a like mass attracts this with a unit of force. As another example, the unit of temperature might be defined in a similar way in terms of the general gas formula, by first assigning an arbitrary numerical value to the gas constant. (3) Fundamental dimensions that are simply related to direct theoretical analogues of accurate pointer readings are assigned arbitrary numerical values.

Example a: The fundamental dimension of length is theoretically closely related to the wave length of a red cadmium line

[1] ERNST WEBER, A Proposal To Abolish the Absolute Electrical Units Systems, *Trans. A.I.E.E.*, Vol. 51, 1932.

[2] G. MIE, *cf.* p. 648 "Handbuch der Experimentalphysik," Vol. XI/1.

which, in turn, has a direct spectroscopic pointer-reading analogue. The unit of length, the meter, is defined by assigning the numerical value

$$\lambda_{\text{cad,red}} = \frac{1}{1,553,164.13} \text{ meters}$$

to the wave length of the cadmium line.

Example b: Mechanical force is itself a direct theoretical analogue of the equilibrium pointer reading of a balance. A unit of force may be defined by assigning the value 0.999972 gram to the weight of a cubic centimeter of water at 4°C.

Example c: The dimension of temperature is theoretically related to mechanical or electrical changes in various devices, such as the expansion of a gas when heated. The unit of temperature may be defined by assigning the value 100.00 degrees to the temperature interval between the melting and boiling points of water under standard barometric pressure as observed for the calibration of any convenient device.

A brief consideration of these three methods suggests the following criticism. The first method is simple and experimentally convenient. But its accuracy is hardly adequate for high-precision measurements, since it is based on the hope rather than on the knowledge that the sample chosen remains invariant in time. The only way to verify that, in spite of all precautions, it does not change is to provide a dependable way for measuring it from time to time. Such a method, as for example the spectroscopic measurement of the standard meter, then supersedes the standard sample. It is evident also that only a very limited number of dimensions can be assigned even a reasonably permanent unit in this way. The second method is mathematically simple, but it is most unfortunate from the experimental point of view. It actually does not specify any series of experimental operations for determining the pointer-reading analogues of the dimension chosen, it merely imagines an experiment. This is as likely as not inconvenient, inaccurate, and even impossible. The third method is the most satisfactory and the most common. By proper choice of dimension and a careful selection of an experiment which combines a high degree of precision with convenience, a unit may be specified accurately in a way that is both experimentally and theoretically convenient.

If fundamental dimensions are assigned units according to the third method, then the proportionality constant in important relations such as Newton's law of gravitation and the general gas formula are determined by the variables in the relation. By a suitably arranged experiment in which all these variables may be assigned numerical values by substituting pointer readings for theoretical analogues, the constant may be computed as a number with dimensions. This is done, for example, both in Newton's law and in the general gas formula.

V-2. The Dimensional Formulation of the Electromathematical Model.—In the mathematical description of the atomic model in terms of continuous functions, six densities were constructed to characterize the average condition of charge and of moving charge. In addition to using the dimensions of length and time, these were defined in terms of one specifically electrical dimension introduced to stand for the property of matter called charge. Thus, the mathematical analogue of the atomic model deals with the four dimensions symbolically represented by L, M, T, and Q. The unit names introduced for these dimensions are for length, the meter; for mass, the kilogram; for time, the second; and for charge, the coulomb.

In order to examine the dimensional character of the field and potential equations, these may be written in terms of the quantities involved insofar as these can be assigned. Quantities to which dimensions can be assigned in these equations are

$$\bar{\rho} \approx \frac{Q}{L^3} \tag{1}$$

$$\overline{\rho_m \mathsf{v}} \approx \frac{Q}{L^2 T}$$

From their definitions, it is easily verified that the vector operators which occur have the following dimensional equivalents:

$$\text{div} \approx \frac{1}{L}$$

$$\text{curl} \approx \frac{1}{L}$$

$$\text{grad} \approx \frac{1}{L} \tag{2}$$

$$\nabla^2 \approx \frac{1}{L^2}$$

Since the field equations and the potential equations are analytically equivalent, it is evidently not necessary to examine the dimensional characteristics of both. The relation between the field vectors and the potentials is defined by

$$E = -\operatorname{grad} \phi - \dot{A} \tag{3}$$
$$B = \operatorname{curl} A$$

Consequently, their dimensional interdependence is simply

$$E \approx \phi/L \tag{4}$$
$$B \approx A/L$$

Because of their symmetrical form the potential equations (III.4.19,20) are more convenient than the field equations. The three terms in each lead to the following quasi-dimensional forms:

$$\frac{\phi}{L^2} \approx \frac{\phi \epsilon_0/\nu_0}{T^2} \approx \frac{Q/L^3}{\epsilon_0} \tag{5a}$$

$$\frac{A}{L^2} \approx \frac{A \epsilon_0/\nu_0}{T^2} \approx \frac{Q/TL^2}{\nu_0} \tag{5b}$$

The potential equation of continuity (III.4.17) has the dimensional form

$$\frac{A}{L} \approx \frac{\phi \epsilon_0/\nu_0}{T} \tag{5c}$$

In (5a, b, c), ϕ, A, ϵ_0, and ν_0 as yet cannot be written dimensionally.

From these relations it is apparent, as was already evident from the potential equations themselves, that (ϵ_0/ν_0) has the dimensions

$$\frac{\epsilon_0}{\nu_0} \approx \frac{T^2}{L^2} \tag{6}$$

This is the square of a reciprocal velocity that has been assigned the symbol v_0. Thus,

$$v_0^2 \equiv \frac{\nu_0}{\epsilon_0} \approx \frac{L^2}{T^2} \tag{7}$$

Upon substituting this dimensional equivalent for (ϵ_0/ν_0) in (5) and rearranging terms, it follows that

$$\epsilon_0 \approx \frac{1}{\phi} \frac{Q}{L}$$

$$\nu_0 \approx \frac{1}{A} \frac{Q}{T} \approx \frac{1}{\phi} \frac{QL}{T^2} \tag{8}$$

$$A \approx \phi \frac{T}{L}$$

This is a set of three relations involving four quantities as yet without dimensional equivalents. Since three relations are not sufficient to define four quantities, it is convenient for the present to assign an independent dimension to the scalar potential ϕ. Let its dimensional symbol be V, and its unit name the volt. All purely electrical quantities can now be expressed in terms of combinations of the four dimensions L, T, Q, V and the corresponding units the meter, the second, the coulomb, and the volt. In most electrical problems, the use of the fundamental dimensions L, T, and Q and the auxiliary dimension V leads to simple and useful dimensional forms.

Proceeding to a consideration of the force equation (II.12.5), it follows from (4) that

$$E \approx \frac{V}{L}; \qquad B \approx \frac{VT}{L^2} \tag{9}$$

Accordingly, the electromagnetic force F has the following dimensions and units:

$$F \approx \frac{QV}{L} \frac{\text{coulomb-volts}}{\text{meter}} \tag{10}$$

The mechanical equivalent of electricity A_e is defined by the force equation. Dimensionally, it must be

$$A_e = -\frac{F_m}{F} \approx \frac{MLT^{-2}}{QVL^{-1}} \frac{\text{newtons}}{\text{coulomb-volts per meter}^1} \tag{11}$$

or

$$A_e \approx \frac{ML^2T^{-2}}{QV} \frac{\text{mechanical joules}}{\text{coulomb-volts}} \tag{12}$$

Here the numerator has the dimension of mechanical energy, the denominator that of electromagnetic energy.

If in the relations (11) and (12), V is treated as a fifth independent dimension, A_e is a dimensional constant. Its numerical value is the ratio between mechanical and electrical force or mechanical and electrical energy functions; it must be determined experimentally and treated as a fundamental constant.[2]

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[1] 1 newton = 1 mechanical joule/meter = force to accelerate 1 kilogram one meter/second/second.

[2] This is the system adopted by Mie, "Elektrodynamik, Handbuch der Experimentalphysik," Vol. XI/1.

On the other hand, if V is used as an auxiliary dimension it may be expressed in terms of the four fundamental dimensions L, T, M, and Q. This is readily done (following method 2 described in section 1 of this Appendix) by assigning a dimensionless numerical value of unity to the constant A_e.* In this case the fundamental dimensions of V are given by

$$V \approx \frac{ML^2}{QT^2} \qquad (13)$$

and the unit of V, the volt, may be defined directly or indirectly in terms of the kilogram, meter, second, and coulomb.

UNIT NOMENCLATURE

Coulomb/second	is equivalent to	Ampere
Coulomb-volt		joule†
Coulomb-volt/second Ampere-volt Joule/second }		watt†
Coulomb/volt		farad
Volt-second/coulomb Volt/ampere }		ohm
Coulomb/volt-second Ampere/volt 1/ohm }		mho or siemens
Volt-second/square meter Pramaxwell/square meter }		{ pragauss‡ of { weber/m² ‡
Volt-second/coulomb/second Volt-second/ampere Pramaxwell/ampere Weber/ampere }		henry

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* This is done in the rationalized practical system used in this text.

† Note that the unit names joule and watt may be taken to be *purely electrical* in the same way as the calorie and the calorie/second are purely thermal. The relation of the electrical joule to the mechanical joule, or erg × 10^7, is then contained in the mechanical equivalent of electricity A_e. If, as in the practical system used in this text, A_e is set equal to 1, mechanical and electrical joules are identical.

‡ pramaxwell }
  weber   } = 10^8 maxwell; pragauss = 10^4 gauss.

TABLE OF DIMENSIONS AND UNITS

Electric Quantities

ρ	volume density of charge	$\{\frac{Q}{L^3}\}$	$\frac{coulomb}{meter^3}$
ρ̄	essential volume characteristic of charge	$\{\frac{Q}{L^3}\}$	$\frac{coulomb}{meter^3}$
η	surface density of charge	$\{\frac{Q}{L^2}\}$	$\frac{coulomb}{meter^2}$
η̄	essential surface characteristic of charge	$\{\frac{Q}{L^2}\}$	$\frac{coulomb}{meter^2}$
P	volume density of polarization	$\{\frac{Q}{L^2}\}$	$\frac{coulomb}{meter^2}$
D	auxiliary electric vector		
φ	scalar potential	$\{V\}$	volt
Z	polarization potential	$\{VL\}$	volt-meter
E	electric vector	$\{\frac{V}{L}\}$	$\frac{volt}{meter}$
U_e	electric energy function	$\{QV\}$	joule
$ε_0$	universal electric constant	$\{\frac{Q}{VL}\}$	$\frac{farad}{meter}$
$χ_e$	electric susceptibility	numeric	
$ε_r$	relative dielectric constant	numeric	

Magnetic Quantities

i	volume density of moving charge	$\{\frac{Q}{L^2T}\}$	$\frac{ampere}{meter^2}$
$\barρ_m v$	essential volume characteristic of moving charge	$\{\frac{Q}{L^2T}\}$	$\frac{ampere}{meter^2}$
I	surface density of moving charge	$\{\frac{Q}{LT}\}$	$\frac{ampere}{meter}$
$\barη_m v$	essential surface characteristic of moving charge	$\{\frac{Q}{LT}\}$	$\frac{ampere}{meter}$
M	volume density of magnetization	$\{\frac{Q}{LT}\}$	$\frac{ampere}{meter}$
H	auxiliary magnetic vector		
A	vector potential	$\{\frac{VT}{L}\}$	$\frac{volt-second}{meter}$
Y	magnetization potential	$\{VT\}$	volt-second
B	magnetic vector	$\{\frac{VT}{L^2}\}$	$\frac{volt-second}{meter^2}$
U_m	magnetic energy function	$\{QV\}$	joule
$ν_0$	universal magnetic constant $\left(\frac{1}{μ_0}\right)$	$\{\frac{QL}{VT^2}\}$	$\frac{meter}{henry}$
$χ_m$	magnetic susceptibility	numeric	
$ν_r$	relative reluctivity $\left(\frac{1}{μ_0}\right)$	numeric	

TABLE OF DIMENSIONS AND UNITS.—*(Continued)*

Electromagnetic Quantities

F	electromagnetic force function	$\left\{\dfrac{QV}{L}\right\}$ $\left\{\dfrac{ML}{T^2}\right\}$	$\dfrac{\text{joule}}{\text{meter}}$ newton when $A_e=1$ as in this book	S	Poynting-vector	$\left\{\dfrac{QV}{L^2T}\right\}$ $\dfrac{\text{watt}}{\text{meter}^2}$
T	electromagnetic torque	$\left\{\dfrac{ML^2}{T^2}\right\}$	newton-meter when $A_e=1$	v_0	characteristic velocity of propagation	$\left\{\dfrac{L}{T}\right\}$ $\dfrac{\text{meter}}{\text{second}}$
σ	conductivity	$\left\{\dfrac{Q}{VTL}\right\}$	$\dfrac{1}{\text{ohm–meter}}$	ζ_0	characteristic resistance	$\left\{\dfrac{VT}{Q}\right\}$ ohm
T	electromagnetic energy transfer function	$\left\{\dfrac{QV}{T}\right\}$	watt	A_e	mechanical equivalent of electricity	$\left\{\dfrac{ML^2T^{-2}}{QV}\right\}$ $\dfrac{\text{mechanical joule}}{\text{electrical joule}}=1$ in the practical system used in this book

The dimensions of an electric quantity are changed to those of the analogous magnetic quantity by multiplying Q by L/T and V by T/L. The dimensions of a magnetic quantity are changed to those of the analogous electric quantity by multiplying Q by T/L and V by L/T.

V-3. Electromagnetic Units and Constants.—The dimensional formulation of the electromagnetic model outlined in the preceding two sections involves the selection of a fundamental electrical dimension Q for electric charge and an auxiliary electrical dimension V for the scalar potential. The practical unit of charge the coulomb and the practical unit of potential the volt must be specified in terms of one of the three methods outlined in V-1. If both Q and V are treated as independent dimensions (Mie), the coulomb and the volt may be defined directly in terms of pointer-reading analogues of quantities related to Q and V in specially devised and easily and accurately reproduced experiments. The most convenient method from the experimental point of view involves the definition of the coulomb (*e.g.*, in terms of the electrolysis of silver) and the definition of the volt-second per coulomb or ohm in terms of a standard resistance (such as a suitable column of mercury). The units so defined are the international coulomb and the international volt. With Q and V thus defined, further experiments must be performed to determine numerical values for the fundamental constants ν_0 and ϵ_0 (and for the mechanical equivalent of electricity A_e if this is not set equal to one).

In the practical system of units used in this text, the coulomb but not the volt is treated as an independent unit. In this system, the dimensionless value of unity is assigned to the mechanical equivalent of electricity A_e, so that any experiment performed to determine A_e may be used to define the product QV in mechanical joules (which are here identical with electrical joules). Similarly, the experiment defining VT/Q in terms of the standard ohm may be used to define the ratio V/Q. With the coulomb-volt and the volt per coulomb uniquely specified, two equations in two unknowns are available to define the coulomb and the volt in terms of the ohm and the mechanical joule and, hence, in terms of the meter, kilogram, ohm, and second. The fundamental constants ν_0 and ϵ_0 are determined experimentally, and from them the characteristic velocity $v_0 = \sqrt{\nu_0/\epsilon_0}$ and the characteristic resistance $\zeta_0 = 1/\sqrt{\nu_0\epsilon_0}$ may be calculated. Or, v_0 and one of ν_0 and ϵ_0 may be measured experimentally.

Based upon the definition of the international coulomb and volt, Mie[1] has calculated the following as the best numerical values in the rationalized system:

[1] "Electrodynamik," Handbuch der Experimentalphysik, Vol. XI/1, p. 484.

$\epsilon_0 = 0.8859 \times 10^{-11}$ farad/meter

$\nu_0 = 1/(1.25598 \times 10^{-6})$ meter/henry

$v_0 = \sqrt{\nu_0/\epsilon_0} = 2.9979 \times 10^8$ meters/second

$\doteq 3 \times 10^8$ meters/second

$A_e = 1.00043$ mechanical joules/electrical joule

$A_q/A_e = 4.1842$ electrical joules/calorie

In the rationalized practical system, slightly different values obtain because A_e is by definition unity so that the product QV is slightly changed. The values are

$\epsilon_0 = 10^7/4\pi v_0^2 = 0.8854 \times 10^{-11}$ farad/meter

$\nu_0 = 1/(4\pi \times 10^{-7}) = 1/(1.257 \times 10^{-6})$ meter/henry

$v_0 = \sqrt{\nu_0/\epsilon_0} = 2.9979 \times 10^8$ meters/second

$\doteq 3 \times 10^8$ meters/second

$\zeta_0 = 1/\sqrt{\nu_0\epsilon_0} = 376.7$ ohms

$A_e = 1$

$A_q/A_e = A_q = 4.186$ joules/calorie

Some writers[1] using the practical system feel that, if the ohm is established as a fundamental unit, resistance should be introduced with a symbol R in all dimensional formulas instead of Q. However, resistance is not a fundamental concept as is electric charge, and its use along with M, L, and T leads to intricate dimensional formulas such as

$$Q \approx \{M^{1/2}LR^{-1/2}T^{-1/2}\}, \quad V \approx \{M^{1/2}LR^{1/2}T^{-3/2}\}$$

Since dimensional analysis in no way requires that the dimensions of experimentally convenient fundamental units be used in preference to theoretically more fundamental quantities, it is in any case desirable to retain Q rather than R as a fourth fundamental dimension and V as an often convenient auxiliary dimension. Fractional powers do not appear in the simple dimensional formulas using Q or Q and V. The accompanying table of dimensions is written in terms of Q and V. Whenever required, the relation (V-2.13) that expresses V in terms of M, L, T, and Q may be introduced.

The practical system of units described above and used throughout this text is not the only system in common use. Although other systems may have certain advantages in exclu-

[1] G. E. M. JAUNCEY and A. S. LANGSDORF, "M.K.S. Units and Dimensions."

sively theoretical work, especially for those who are accustomed to them, only the practical system has the *unique and compelling characteristic that it alone is entirely adequate for both theoretical and experimenal work*, so that confusing conversion tables are not required. *One system is always simpler than two* when, as in applied physics and engineering, theory and experiment must work in close cooperation. It is the responsibility of writers using systems other than the practical to convert their results to the practical system or at least to provide adequate conversion tables.

List of Books

A selected list of books with brief comments is given to help the student to select material for supplementary reading and for more elementary as well as more advanced study.

Electromagnetic Theory

ABRAHAM-BECKER: "The Classical Theory of Electricity and Magnetism," Blackie & Son, Ltd., Glasgow, 1932.

——: "Theorie der Elektrizität," Vol. I, B. G. Teubner, Leipzig, 1930. An excellent introduction to Maxwell's theory following classical lines.

BECKER, R.: "Theorie der Elektrizität," Vol. II, B. G. Teubner, Leipzig, 1933. Classical electron theory.

BENNETT, E., and H. M. CROTHERS: "Introductory Electrodynamics for Engineers," McGraw-Hill Book Company, Inc., New York, 1926. One of the best introductory texts.

BREISIG, F.: "Theoretische Telegraphie," Friederich Vieweg & Sohn, Brunswick, Germany, 1924.

BRILLOUIN, M.: "Propagation de L'Électricité," Hermann & Cie, Paris, 1904. An old but excellent historical and mathematical treatise.

CULWICK, E. G.: "Fundamentals of Electro-Magnetism," The Macmillan Company, New York, 1939. An introductory text with a critical point of view. M.K.S. units.

FRANK, N. H.: "Introduction to Electricity and Optics," McGraw-Hill Book Company, Inc., New York, 1940. An introductory text. M.K.S. units.

FRANK P., and R. VON MISES: "Differentialgleichungen der Physik," Vol. II, Friederich Vieweg & Sohn, Brunswick, Germany, 1935; Mary S. Rosenberg, New York, 1943. Part IV by Noether on the stationary and quasi-stationary electromagnetic field and Part V by Sommerfeld on electric oscillations are advanced but clearly written and authoritative.

FRENKEL, J.: "Elektrodynamik," Vols. I and II. Verlag Julius Springer, Berlin, 1926, 1928. Comprehensive, advanced, difficult.

GEIGER-SCHEEL: "Handbuch der Physik," Vols. XV and XII, Verlag Julius Springer, Berlin, 1927.

HARNWELL, G. P.: "Principles of Electricity and Electromagnetism," McGraw-Hill Book Company, Inc., New York, 1938. An excellent intermediate treatment of both theoretical and experimental electromagnetism and its applications. M.K.S. units.

JEANS, J. H.: "The Mathematical Theory of Electricity and Magnetism," Cambridge University Press, London, 1927. An old classic by a master calculator.

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KÜPFMÜLLER, K.: "Einführung in die theoretische Elektrotechnik," Verlag Julius Springer, Berlin, 1932. One of the best books on technical electromagnetism; clear and to the point. M.K.S. units.

LINDSAY, R. B., and H. MARGENAU: "Foundations of Physics," John Wiley & Sons, Inc., New York, 1936. Chapter VI is a critical and illuminating discussion of electromagnetic theory.

MASON, M., and W. WEAVER: "The Electromagnetic Field," University of Chicago Press, Chicago, 1929. A· critical mathematical study of the foundations of electromagnetism that every mathematically minded student in the field should read.

MAXWELL, J. C.: "Electricity and Magnetism," Vol. II, Oxford University Press, New York, 1892.

OLLENDORF, F.: "Potentialfelder der Elektrotechnik," Verlag Julius Springer, Berlin, 1932. A brilliant application of electromagnetic theory to the solution of practical problems. M.K.S. units.

PAGE, L., and N. I. ADAMS: "Principles of Electricity," D. Van Nostrand Company, Inc., 1931. A standard intermediate text.

————: "Electrodynamics," D. Van Nostrand Company, Inc., New York, 1940. An advanced treatise.

PIERCE, G. W.: "Electric Oscillations and Electric Waves," McGraw-Hill Company, Inc., New York, 1920. Book II contains a concise study of the field equations with application to plane waves.

RAMO, S., and J. R. WHINNERY: "Fields and Waves in Modern Radio," John Wiley & Sons, Inc., New York, 1944. A good and concise treatment of mathematical methods in applied electromagnetism. M.K.S. units.

SCHAEFER, C.: "Einführung in die theoretische Physik," Bd. III, 1. Teil, "Elektrodynamik und Optik," Walter de Gruyter & Company, Berlin, 1932. A detailed and comprehensive treatment clearly and simply written.

SCHELKUNOFF, S.: "Electromagnetic Waves," D. Van Nostrand Company, Inc., New York, 1943. An advanced treatise stressing the impedance concept and the point of view of transmission-line theory. M.K.S. units.

SKILLING, H. H.: "Fundamentals of Electric Waves," John Wiley & Sons, Inc., New York, 1942. A good introductory survey.

SMYTHE, W. R.: "Static and Dynamic Electricity," McGraw-Hill Book Company, Inc., New York, 1939. An advanced mathematical treatise.

STRATTON, J. A.: "Electromagnetic Theory," McGraw-Hill Book Company, Inc., New York, 1941. A comprehensive mathematical treatise and reference book. M.K.S. units.

WIEN-HARMS: "Handbuch der Experimentalphysik," Vol. XI/1. Akademische Verlagsgesellschaft, m.b.H., Leipzig, 1932. This volume entitled Elektrodynamik and written by Mie is an excellent critical treatment from the experimental point of view.

Engineering Mathematics

BURINGTON, R. S., and C. C. TORRANCE: "Higher Mathematics," McGraw-Hill Book Company, Inc., New York, 1939.

DOHERTY, R. E., and E. G. KELLER: "Mathematics for Modern Engineering," John Wiley & Sons, Inc., New York, 1936.

MARGENAU, H., and G. M. MURPHY: "The Mathematics of Physics and Chemistry," D. Van Nostrand Company, Inc., New York, 1943.

SOKOLNIKOFF, I. S., and E. S. SOKOLNIKOFF: "Higher Mathematics for Engineers and Physicists," McGraw-Hill Book Company, Inc., New York, 1934.

WILSON, E. B.: "Advanced Calculus," Ginn and Company, Boston, 1912.

Vector Analysis

GANS, R.: "Vector Analysis with Applications to Physics," Blackie & Son, Ltd., Glasgow, 1932.

————: "Einführung in die Vektoranalysis," B. G. Teubner, Leipzig, 1923.

PHILLIPS, H. B.: "Vector Analysis," John Wiley & Sons, Inc., New York, 1933.

SHORTER, L. R.: "Introduction to Vector Analysis," The Macmillan Company, New York, 1931.

See also chapters in all the books listed under Engineering Mathematics, in the Appendix in Mason and Weaver, "The Electromagnetic Field," and in introductory chapters in many of the books on electromagnetic theory.

Bessel Functions

MCLACHLAN, N. W.: "Bessel Functions for Engineers," Oxford University Press, New York, 1934.

Differential and Integral Equations

FRANK, P., and R. VON MISES: "Differentialgleichungen der Physik," Vol. I, Friederich Vieweg & Sohn, Brunswick, Germany, 1935; Mary S. Rosenberg, New York, 1943.

WEBSTER, A. G.: "Partial Differential Equations of Mathematical Physics," G. E. Stechert & Company, New York, 1933.

Tables

DWIGHT, H. B.: "Mathematical Tables," McGraw-Hill Book Company, Inc., New York, 1941.

————: "Tables of Integrals and Other Mathematical Data," The Macmillan Company, New York, 1934.

JAHNKE-EMDE: "Tables of Functions," B. G. Teubner, Leipzig, 1938; Dover Publications, New York, 1943.

PIERCE, B. O.: "Tables of Integrals," Ginn and Company, Boston, 1929.

History of Electromagnetism

WHITTAKER, E. T.: "History of the Theories of the Aether and Electricity," Dublin University Press, Longmans, Green & Company, London, 1910.

Küpfmüller, K.: "Einführung in die theoretische Elektrotechnik," Verlag Julius Springer, Berlin, 1932. One of the best books on technical electromagnetism; clear and to the point. M.K.S. units.

Lindsay, R. B., and H. Margenau: "Foundations of Physics," John Wiley & Sons, Inc., New York, 1936. Chapter VI is a critical and illuminating discussion of electromagnetic theory.

Mason, M., and W. Weaver: "The Electromagnetic Field," University of Chicago Press, Chicago, 1929. A· critical mathematical study of the foundations of electromagnetism that every mathematically minded student in the field should read.

Maxwell, J. C.: "Electricity and Magnetism," Vol. II, Oxford University Press, New York, 1892.

Ollendorf, F.: "Potentialfelder der Elektrotechnik," Verlag Julius Springer, Berlin, 1932. A brilliant application of electromagnetic theory to the solution of practical problems. M.K.S. units.

Page, L., and N. I. Adams: "Principles of Electricity," D. Van Nostrand Company, Inc., 1931. A standard intermediate text.

————: "Electrodynamics," D. Van Nostrand Company, Inc., New York, 1940. An advanced treatise.

Pierce, G. W.: "Electric Oscillations and Electric Waves," McGraw-Hill Company, Inc., New York, 1920. Book II contains a concise study of the field equations with application to plane waves.

Ramo, S., and J. R. Whinnery: "Fields and Waves in Modern Radio," John Wiley & Sons, Inc., New York, 1944. A good and concise treatment of mathematical methods in applied electromagnetism. M.K.S. units.

Schaefer, C.: "Einführung in die theoretische Physik," Bd. III, 1. Teil, "Elektrodynamik und Optik," Walter de Gruyter & Company, Berlin, 1932. A detailed and comprehensive treatment clearly and simply written.

Schelkunoff, S.: "Electromagnetic Waves," D. Van Nostrand Company, Inc., New York, 1943. An advanced treatise stressing the impedance concept and the point of view of transmission-line theory. M.K.S. units.

Skilling, H. H.: "Fundamentals of Electric Waves," John Wiley & Sons, Inc., New York, 1942. A good introductory survey.

Smythe, W. R.: "Static and Dynamic Electricity," McGraw-Hill Book Company, Inc., New York, 1939. An advanced mathematical treatise.

Stratton, J. A.: "Electromagnetic Theory," McGraw-Hill Book Company, Inc., New York, 1941. A comprehensive mathematical treatise and reference book. M.K.S. units.

Wien-Harms: "Handbuch der Experimentalphysik," Vol. XI/1. Akademische Verlagsgesellschaft, m.b.H., Leipzig, 1932. This volume entitled Elektrodynamik and written by Mie is an excellent critical treatment from the experimental point of view.

Engineering Mathematics

Burington, R. S., and C. C. Torrance: "Higher Mathematics," McGraw-Hill Book Company, Inc., New York, 1939.

oherty, R. E., and E. G. Keller: "Mathematics for Modern Engineering," John Wiley & Sons, Inc., New York, 1936.

argenau, H., and G. M. Murphy: "The Mathematics of Physics and Chemistry," D. Van Nostrand Company, Inc., New York, 1943.

okolnikoff, I. S., and E. S. Sokolnikoff: "Higher Mathematics for Engineers and Physicists," McGraw-Hill Book Company, Inc., New York, 1934.

ilson, E. B.: "Advanced Calculus," Ginn and Company, Boston, 1912.

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ans, R.: "Vector Analysis with Applications to Physics," Blackie & Son, Ltd., Glasgow, 1932.

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orter, L. R.: "Introduction to Vector Analysis," The Macmillan Company, New York, 1931.

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Bessel Functions

cLachlan, N. W.: "Bessel Functions for Engineers," Oxford University Press, New York, 1934.

Differential and Integral Equations

ank, P., and R. von Mises: "Differentialgleichungen der Physik," Vol. I, Friederich Vieweg & Sohn, Brunswick, Germany, 1935; Mary S. Rosenberg, New York, 1943.

ebster, A. G.: "Partial Differential Equations of Mathematical Physics," G. E. Stechert & Company, New York, 1933.

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——: "Tables of Integrals and Other Mathematical Data," The Macmillan Company, New York, 1934.

unke-Emde: "Tables of Functions," B. G. Teubner, Leipzig, 1938; Dover Publications, New York, 1943.

irce, B. O.: "Tables of Integrals," Ginn and Company, Boston, 1929.

History of Electromagnetism

hittaker, E. T.: "History of the Theories of the Aether and Electricity," Dublin University Press, Longmans, Green & Company, London, 1910.

Index of Symbols[1]

The following representation applies in general. Real scalars are in italic or Greek type; complex scalars in bold-faced italic or bold-faced Greek type. Real vectors are in gothic; complex vectors in bold-faced gothic. For example, if a real vector is fixed in direction but varies periodically in amplitude it is written

$$\mathsf{A}_{\text{Inst}} = \mathsf{A}\cos(\omega t + \theta) = \text{real part } \{\mathbf{A}_{\text{Inst}} = \mathbf{A}e^{j(\omega t+\theta)} = \mathbf{A}e^{j\omega t}\}$$

A unit vector is written $\hat{\mathsf{A}}$, so that $\mathsf{A} = \hat{\mathsf{A}}A$. If the vector is complex, $\mathbf{A} = \hat{\mathsf{A}}\mathbf{A} = \hat{\mathsf{A}}Ae^{j\theta}$. If a real scalar varies periodically in amplitude,

$$\phi_{\text{Inst}} = \phi\cos(\omega t + \theta) = \text{real part } \{\boldsymbol{\phi}_{\text{Inst}} = \boldsymbol{\phi}e^{j(\omega t+\theta)} = \boldsymbol{\phi}e^{j\omega t}\}$$

In listing vector quantities that may be constant or vary periodically in time, the four symbols $\mathsf{A}, A, \mathbf{A}, \mathbf{A}$ are given in this order. A is a constant vector or the directed real amplitude of a vector that is fixed in direction but varying periodically in time; A is the scalar magnitude of A. $\mathbf{A}$ is the directed complex amplitude of a vector that is fixed in direction but varying periodically in time; $\mathbf{A}$ is the complex scalar magnitude of $\mathbf{A}$; A is the real scalar magnitude of $\mathbf{A}$. Bold-faced Greek letters *with accent* are used as unit vectors in cylindrical and polar coordinates, *e.g.*, $\hat{\boldsymbol{\Theta}}$. No confusion can arise with complex scalars since a unit vector is necessarily real and only unit vectors are accented.

[1] Vector operators that are summarized in Appendix I are not included. A few symbols used for special purposes in a limited way are omitted.

INDEX OF SYMBOLS

Symbol	Name	Units	Page where first defined
$A, A, \mathbf{A}, A$	Vector potential	$\dfrac{\text{volt-second}}{\text{meter}}$	164
A_e	Mechanical equivalent of electricity	$\dfrac{\text{mechanical joule}}{\text{electrical joule}}$ $= 1$ (dimensionless)	126
A_q	Mechanical equivalent of heat	$\dfrac{\text{joule}}{\text{calorie}}$	181
a	Radius of cylindrical conductor	meter	81
$\mathring{a}$	Unit vector		161
$B, B, \mathbf{B}, B$	Fundamental magnetic vector	$\dfrac{\text{volt-second}}{\text{square meter}}$ $\dfrac{\text{weber}}{\text{square meter}}$	96
b	Radius of circular condenser; inner radius of tubular conductor; etc.	meter	351; 368
$C, C, \mathbf{C}, C$	Typical vector		163
C_0	Static capacitance of a condenser	farad	383
c	Capacitance per unit length of line	$\dfrac{\text{farad}}{\text{meter}}$	476
c_0	Static capacitance of condenser per unit thickness of dielectric	$\dfrac{\text{farad}}{\text{meter}}$	382
$D, D, \mathbf{D}, D$	Auxiliary electric vector	$\dfrac{\text{coulomb}}{\text{square meter}}$	109
D	Directivity	dimensionless	298
d_s	Skin depth	meter	210

Symbol	Name	Units	Page where first defined
$\mathsf{E}, E, \mathbf{E}, E$	Fundamental electric vector	$\dfrac{\text{volt}}{\text{meter}}$	96
$\mathsf{E}^{\bullet}, E^{\bullet}, \mathbf{E}^{\bullet}, E^{\bullet}$	Externally maintained electric field	$\dfrac{\text{volt}}{\text{meter}}$	154
$E\left(\dfrac{\pi}{2}, k\right)$ or E	Elliptic integral of second kind		445
e	Electric charge	coulomb	10
e	Base of natural logarithm	dimensionless	146
$\mathsf{F}, F, \mathbf{F}, F$	Electromagnetic force	newton	126
$\mathsf{F}_M, F_M, \mathbf{F}_M, F_M$	Mechanical force	newton	126
f	Frequency	hertz $\dfrac{\text{cycle}}{\text{second}}$	
$f(h)$	Function	dimensionless	208
g	Leakage conductance per unit length of line	$\dfrac{\text{mho}}{\text{meter}}$	476
$g(h)$	Function	dimensionless	208
$\mathsf{H}, H, \mathbf{H}, H$	Auxiliary magnetic vector	$\dfrac{\text{ampere}}{\text{meter}}$	109
$H_n^{(1)}(x), H_n^{(2)}(x)$	Hankel function; or Bessel function of third kind and order n		329
h	Half-length of antenna	meter	81
$h_e = \sigma_e/\omega\epsilon_e$	Fundamental ratio in simple media (power factor in dielectric)	dimensionless	207

INDEX OF SYMBOLS.—(*Continued*)

Symbol	Name	Units	Page where first defined
$h_m = \nu''/\nu'$	Fundamental ratio in simple media with time lags in magnetization	dimensionless	207
I,I	Total current in a conductor	ampere	83
i,i,l,i	Volume density of current	$\dfrac{\text{ampere}}{\text{square meter}}$	43
i_f,i_f,l_f,i_f	Volume density of convection current or of moving free charge	$\dfrac{\text{ampere}}{\text{square meter}}$	64
i_m,i_m,l_m,i_m	Volume density of magnetization current	$\dfrac{\text{ampere}}{\text{square meter}}$	62
i_p,i_p,l_p,i_p	Volume density of polarization current	$\dfrac{\text{ampere}}{\text{square meter}}$	77
i	Index of summation		11
$J_n(x)$	Bessel function of first kind and order n		328
$j = \sqrt{-1}$	Imaginary unit		195
j	Index of summation		10
K	General conduction parameter	$\dfrac{1}{\text{ohm}}$ mho siemens	210
K	A constant		
$K\left(\dfrac{\pi}{2},k\right)$ or K	Elliptic integral of first kind		445
k	Surface density of polarization	$\dfrac{\text{coulomb}}{\text{meter}}$	37

INDEX OF SYMBOLS.—*(Continued)*

Symbol	Name	Units	Page where first defined
k	Index of summation		42
k_{12}, k_{12}	Coefficient of coupling		410
k_E, k_M, k	Complex factor	$\dfrac{1}{\text{meter}}$	215, 323
L	Dimensional symbol for length		5
L_1^e	External self-inductance of circuit 1	henry	418
L_1^i	Internal self-inductance of circuit 1	henry	429
L_{12}	Mutual inductance of circuit 1 due to circuit 2	henry	419
l, l, l, l	Surface density of current	$\dfrac{\text{ampere}}{\text{meter}}$	45
l_f, l_f, l_f, l_f	Surface density of current (free charge)	$\dfrac{\text{ampere}}{\text{meter}}$	89
l	Inductance per unit length of transmission line	$\dfrac{\text{henry}}{\text{meter}}$	475
l_0^e	External inductance per unit length of transmission line	$\dfrac{\text{henry}}{\text{meter}}$	473
l^i	Internal inductance per unit length of transmission line	$\dfrac{\text{henry}}{\text{meter}}$	475
l_0^i	Internal inductance per unit length	$\dfrac{\text{henry}}{\text{meter}}$	347
M, M, M, M	Volume density of magnetization	$\dfrac{\text{ampere}}{\text{meter}}$	51

INDEX OF SYMBOLS.—(*Continued*)

Symbol	Name	Units	Page where first defined
M	Mass	kilogram	130
M	Dimensional symbol for mass		5
m,m,m,m	Magnetization of circulating charges; magnetic moment	ampere-meter^2	50
m	Mass	kilogram	10
N,N	Generalized index of refraction	dimensionless	207
$\hat{N}$	Unit normal		65
$N_n(x)$	Neumann function or Bessel function of second kind and order n		328
N_s	Real index of refraction	dimensionless	207
n	Integer		
$\hat{n}$	Unit external normal		25
P,P,P,P	Volume density of polarization	$\dfrac{\text{coulomb}}{\text{square meter}}$	21
P,p,P,p	Polarization of group of charges; electric moment	coulomb-meter	19
Q,Q	Total charge	coulomb	84
Q	Dimensional symbol for charge		5
Q	Thermal energy function	calorie	181
q,q	Charge per unit length or radial width	$\dfrac{\text{coulomb}}{\text{meter}}$	83

Symbol	Name	Units	Page where first defined
R	Vector from origin in spherical coordinates	meter	
R	Spherical coordinate	meter	
R	Distance from an element of volume to any point	meter	225
R_{11}	Total resistance of circuit 1	ohm	409
R_{11}	Distance between two points in circuit 1	meter	407
R_1^*	Total external resistance of circuit 1	ohm	409
R_1^i	Total internal resistance of circuit 1	ohm	409
R_{12}	Mutual resistance of circuit 1 due to circuit 2	ohm	409
R_{12}	Distance between a point in circuit 1 and a point in circuit 2	meter	407
R_p^*	External or radiation resistance referred to I_p	ohm	296
R_o^*	External or radiation resistance referred to I_0	ohm	296
R_0^i	D.c. resistance	ohm	384
R^*	Surface resistance	ohm	349

Symbol	Name	Units	Page where first de-fined
r	Vector from origin in cylindrical coordinates	meter	
r	Cylindrical coordinate	meter	
r_0	D.c. resistance per unit length	$\dfrac{\text{ohm}}{\text{meter}}$	347
r^i	Internal resistance per unit length	ohm	346
$\mathsf{S}, S, \mathsf{S}, S$	Poynting vector	$\dfrac{\text{watts}}{\text{square meter}}$	187
S	Surface, area	square meter	114
s	Space vector	meter	
s	Distance; contour	meter	115
$\mathsf{T}, T, \mathsf{T}, T$	Electromagnetic torque	newton-meters	126
T, T	Electromagnetic energy transfer function	watt	181
T	Dimensional symbol for time		5
T_q	Thermal energy transfer function	$\dfrac{\text{caloric}}{\text{square meter}}$	181
T_p	Period	second	218
T_E	Time of relaxation	second	147
t	Time	second	73
U	Electromagnetic energy function	joule	180
U_E	Electric energy function	joule	189
U_M	Magnetic energy function	joule	189

Symbol	Name	Units	Page where first de-fined
U,*U*,**U**,*U*	Modified polarization potential	ampere	216
u,*u*,**u**,*u*	Nonrandom velocity of free charges	$\dfrac{\text{meter}}{\text{second}}$	43
û	Unit vector in direction of *u* coordinate		
V	Dimensional symbol for scalar potential		97
V,*V*	Scalar potential difference	volt	468
V•,*V*•	Impressed driving potential difference	volt	406
*V*ⁱ,*V*ⁱ	Induced voltage	volt	411
V,*V*,V,*V*	Modified magnetization potential	volt	216
v,*v*,**v**,*v*	Velocity	$\dfrac{\text{meter}}{\text{second}}$	42
v̂	Unit vector in direction of *v* coordinate		
v•	Characteristic velocity in simple medium	$\dfrac{\text{meter}}{\text{second}}$	208
*v*₀	Characteristic velocity in free space	$\dfrac{\text{meter}}{\text{second}}$	112
v•	Real phase velocity in a simple medium	$\dfrac{\text{meter}}{\text{second}}$	207
W	Mechanical energy function	joule	181
W,*W*,**W**,*W*	Vector potential difference	$\dfrac{\text{volt-second}}{\text{meter}}$	468

INDEX OF SYMBOLS.—*(Continued)*

Symbol	Name	Units	Page where first defined
w	Width, thickness	meter	367
$\hat{w}$	Unit vector in direction of w coordinate		
X_{11}	Self-reactance of circuit 1	ohm	409
X_1^e	External self-reactance of circuit 1	ohm	409
X_1^i	Internal self-reactance of circuit 1	ohm	409
X_{12}	Mutual reactance of circuit 1 due to circuit 2	ohm	409
X^s	Surface reactance	ohm	349
$\mathbf{x}$	Vector in direction of positive x axis	meter	
x	Cartesian coordinate	meter	
x_i	Internal reactance per unit length	$\dfrac{\text{ohm}}{\text{meter}}$	346
$Y,Y,\mathsf{Y},\mathcal{Y}$	Magnetization potential	volt-second	174
$\mathbf{y}$	Vector in direction of positive y axis	meter	
y	Cartesian coordinate	meter	
y,y	Admittance per unit length	$\dfrac{\text{mho}}{\text{meter}}$	216
$Z,Z,\mathsf{Z},\mathcal{Z}$	Polarization potential or Hertzian vector	volt-meter	174
Z_c,Z_c	Characteristic impedance	ohm	217
Z_{11},Z_{11}	Self-impedance of circuit 1	ohm	408
Z_1^e,Z_1^e	External self-impedance of circuit 1	ohm	408

Symbol	Name	Units	Page where first defined
Z_1^i, Z_1^i	Internal self-impedance of circuit 1	ohm	408
Z_{12}, Z_{12}	Mutual impedance in circuit 1 due to circuit 2	ohm	408
Z^s, Z^s	Surface impedance	ohm	349
$\mathbf{z}$	Vector in direction of positive z axis	meter	
z	Cartesian coordinate	meter	
$z^i, z^s; z^o, z^o$	Internal, external impedance per unit length	$\dfrac{\text{ohm}}{\text{meter}}$	346, 473

Greek Symbols

Symbol	Name	Units	Page
α_s (alpha)	Attenuation constant of simple medium	$\dfrac{\text{neper}}{\text{meter}}$	207
β, β (beta)	Phase constant	$\dfrac{1}{\text{meter}}$	207
β_0	Phase constant of free space	$\dfrac{\text{radian}}{\text{meter}}$	197
β_s	Real phase constant of simple medium	$\dfrac{\text{radian}}{\text{meter}}$	207
γ, γ (gamma)	Propagation constant	$\dfrac{1}{\text{meter}}$	215
δ (delta)	Thickness of surface layer	meter	17
ϵ, ε (epsilon)	Absolute dielectric constant of simple medium	$\dfrac{\text{farad}}{\text{meter}}$	139, 200

INDEX OF SYMBOLS.—(Continued)

Symbol	Name	Units	Page where first defined
ϵ_e	Absolute effective dielectric constant	$\dfrac{\text{farad}}{\text{meter}}$	201
ϵ_0	Fundamental electric constant (dielectric constant of space)	$\dfrac{\text{farad}}{\text{meter}}$	98
$\epsilon_r, \mathbf{\epsilon}_r$	Relative dielectric constant of simple medium	dimensionless	139, 199
$\epsilon_r', \epsilon_r''$	Real and imaginary parts in $\varepsilon_r = \epsilon_r' - j\epsilon_r''$	dimensionless	199
$\zeta, \mathbf{\zeta}$ (zeta)	Characteristic impedance	ohm	204
ζ_e	Effective characteristic impedance in a simple medium	ohm	208
ζ_0	Characteristic resistance	ohm	112
$\eta, \mathbf{n}$ (eta)	Surface density of charge	$\dfrac{\text{coulomb}}{\text{square meter}}$	17
$\eta', \mathbf{n}'$	Surface density of charge engaged in nonrandom motion	$\dfrac{\text{coulomb}}{\text{square meter}}$	45
$\eta_f, \mathbf{n}_f$	Surface density of free charge	$\dfrac{\text{coulomb}}{\text{square meter}}$	35
$\bar{\eta}, \bar{\mathbf{n}}$	Essential surface characteristic of charge	$\dfrac{\text{coulomb}}{\text{square meter}}$	34
$\overline{\eta_m \mathbf{v}}, \overline{\eta_m v}, \overline{\mathbf{n}_m \mathbf{v}}, \overline{\mathbf{n}_m v}$	Essential surface characteristic of moving charge	$\dfrac{\text{coulomb}}{\text{square meter}}$	81
Θ (theta) $\hat{\mathbf{\Theta}}$	Polar coordinate Unit vector	radian, degree	

INDEX OF SYMBOLS.—*(Continued)*

Symbol	Name	Units	Page where first defined
θ	Cylindrical coordinate; phase angle	radian, degree	
$\hat{\theta}$	Unit vector		
λ_0 (lambda)	Wave length in space	meter	250
λ_s	Wave length in a simple medium	meter	256
$\mu, \mathbf{\mu}$ (mu)	Absolute permeability of simple medium	$\dfrac{\text{henry}}{\text{meter}}$	139, 203
μ_0	Absolute permeability of space	$\dfrac{\text{henry}}{\text{meter}}$	98
$\mu_r, \mathbf{\mu}_r$	Relative permeability of simple medium	dimensionless	139
$\nu, \mathbf{\nu}$ (nu)	Absolute reluctivity or absolute diamagnetic constant of simple medium	$\dfrac{\text{meter}}{\text{henry}}$	139, 200
ν_0	Fundamental magnetic constant (reluctivity of space, diamagnetic constant of space)	$\dfrac{\text{meter}}{\text{henry}}$	98
$\nu_r, \mathbf{\nu}_r$	Relative reluctivity, or relative diamagnetic constant of simple medium	dimensionless	139, 199
ν_r', ν_r''	Real and imaginary parts of $\mathbf{\nu}_r = \nu_r' - j\nu_r''$	dimensionless	199
$\xi, \boldsymbol{\xi}$ (xi)	Dielectric factor	$\dfrac{\text{farad}}{\text{meter}}$	203
$\rho, \boldsymbol{\rho}$ (rho)	Volume density of charge	$\dfrac{\text{coulomb}}{\text{cubic meter}}$	12

Symbol	Name	Units	Page where first defined
ρ',ϱ'	Volume density of charge engaged in nonrandom motion	$\dfrac{\text{coulomb}}{\text{cubic meter}}$	43
ρ_f,ϱ_f	Volume density of free charge	$\dfrac{\text{coulomb}}{\text{cubic meter}}$	28
$\bar{\rho},\bar{\varrho}$	Essential volume characteristic of charge	$\dfrac{\text{coulomb}}{\text{cubic meter}}$	28
$\rho_m v,\rho_m \upsilon,\varrho_m v,\varrho_m \upsilon$	Essential volume characteristic of moving charge	$\dfrac{\text{coulomb}}{\text{cubic meter}}$	76
Σ,σ (sigma)	Surface	square meter	25
σ,δ	Conductivity	mho	[136, 199]
σ_e	Real effective conductivity	mho	201
σ',σ''	Real and imaginary parts in $\delta = \sigma' - j\sigma''$	mho	199
τ (tau)	Volume	cubic meter	10
Φ (phi)	Polar coordinate	radian	
Φ'	Phase function for transmission line	meter	477
$\hat{\Phi}$	Unit vector		
ϕ,φ	Scalar potential	volt	163
ϕ^e,φ^e	Externally maintained driving potential	volt	405
X_e (chi)	Extinction coefficient	dimensionless	207
$\chi,\varkappa$	Electric susceptibility	dimensionless	136
$\chi_m,\varkappa_m$	Magnetic susceptibility	dimensionless	136
Ψ,Ψ (psi)	Scalar function	ampere	216
ψ,ψ	Scalar function	$\dfrac{\text{volt-second}}{\text{meter}}$	211
ψ	Angle	radian, degree	
ω (omega)	Angular velocity	$\dfrac{\text{radian}}{\text{second}}$	147

Problems

Problems preceded by an asterisk may be omitted by technical students not interested in more advanced mathematics. All other problems should be solved.

Chapter I

1. With the aid of schematic diagrams, describe the orientation and distribution of simple dipoles in the following bodies. It is assumed that only neutral bound-charge groups are present which may be represented by equivalent simple dipoles when they are distorted. In each case, a suitable external force is presupposed. The values of P assume a subdivision into volume cells only. a is a constant.

    a. A cube of side S placed with its center at the origin of coordinates and in which $P = P_x = ax; P_y = P_z = 0$.

    b. A cylinder of height h and radius b in which $P = ar$.

$$(r^2 = x^2 + y^2.)$$

    c. A sphere of radius R in which $P = aR$. $(R^2 = x^2 + y^2 + z^2.)$

    d. What are the values of ρ and η in each case?

2. Let the mode of subdivision in Problem 1 be changed into one using both volume and surface cells so chosen that $P = 0$. Calculate ρ and η for the three cases a, b, and c in Problem 1. *Use the fundamental definition of divergence.*

3. Assume the ionosphere to consist of a layer of electrons of uniform density which begins abruptly at a height h_1 and ends at a height h_2 above the earth. Show how such a layer can be represented mathematically in terms of a volume density of polarization and a surface density of charge. (The latter need be defined only at the upper boundary.)

4. Obtain the expression for div **A** given in Appendix I for cylindrical coordinates by direct evaluation from the fundamental definition.

5. Repeat Problem 4 for spherical coordinates.

6. Calculate div P *using the appropriate coordinate form of the divergence* for each of the three values of P given in Problem 1.

7. A long cylindrical antenna of radius a has, at a given instant, a charge distribution given by $\rho = 0$, $\eta = (q/2\pi a) \sin \beta z$. $P = 0$. Here q and β are independent of z. The ends of the antenna are at $\beta z = \pm \pi/2$. Express the charge distribution in terms of ρ and P alone. Show schematic diagrams for the two representations. (Neglect the small circular end surfaces.)

8. The electrostatic properties of a homogeneous sphere of radius a under the action of a symmetrical external force can be described by the following volume density functions using a subdivision into volume cells only. $P = 5R/R$; $\rho = 10/R$. Describe the same body in terms of ρ and η

alone using an appropriately changed mode of subdivision. Which representation is to be preferred on the basis of simplicity? Of physical plausibility?

*9. Derive the formula for at least one component of curl **A** in cylindrical coordinates using the fundamental definition of the curl.

*10. Repeat Problem 9 for spherical coordinates.

11. Derive the formula for at least one component of curl **A** in Cartesian, cylindrical, and spherical coordinates using the alternative definition of the curl in terms of its component normal to a surface.

12. A section of a long cylindrical region of radius r is characterized in the steady state by the following densities using a subdivision into volume cells only. $M = -40; i = 42/r$. Calculate the essential densities $\overline{\rho_m \mathbf{v}}$ and $\overline{\eta_m \mathbf{v}}$. Determine i and l for a subdivision into volume and surface cells such that $M = 0$. (Consider only the cylindrical surface in calculating l.) Show the directions of motion of the charges and the orientation of the density vectors for both representations by means of cross-sectional sketches. Which representation is to be preferred? Why?

13. A cylindrical bar of iron is completely characterized electrically by $M = M_z \hat{z}$. Here M_z is a constant and $\hat{z}$ is a unit vector along the axis of the bar. Describe an equivalent representation in which $M = 0$, using other density functions and a different mode of subdivision.

14. A hoop of flat copper lying in the r, θ plane has a small thickness d, a width w, and a mean radius R. It is characterized completely by a surface density of current l_θ cos ωt defined for both *inner and outer* surfaces of width w. Consider a representation entirely in terms of M_z. The equivalence in representation is to be valid only as observed from points *outside* the volume bounded by the hoop. That is, M_z may be defined at all points in this flat volume as well as in the copper. Discuss the characterization of a loop antenna of n turns each carrying a current I in terms of an equivalent magnetic shell, *i.e.*, in terms of M_z.

15. A long cylindrical antenna of radius a and with axis parallel to z is characterized approximately by $\bar{\rho} = 0$; $\bar{\eta} = (q/2\pi a)$ sin βz cos ωt; $\overline{\rho_m v_\theta} = 0$; $\overline{\rho_m v_z} = K$ cos βz sin ωt; $\overline{\eta_m v} = 0$. Here q, K, and β are independent of z and t. The ends of the antenna are at $\beta z = \pm \pi/2$. Determine $\overline{\rho_m v_r}$. Discuss a representation in terms of (a) η and i; (b) η, ρ, and P_z; (c) P alone; from the mathematical and physical points of view. (Neglect the small end surfaces.)

16. The total current in a cylindrical diode of great length h compared with the small radius b of the anode is given approximately by the constant radial current

$$I_r = 14.7 \times 10^{-6} \frac{h \bar{E}_p{}^{3/2}}{b} \text{ amp.}$$

Here $\bar{E}_p$ is the constant plate voltage. Obtain an expression for the essential density $\overline{\rho_m v}$ in the interelectrode space. Prove that $\bar{\rho}$ is constant in time. Discuss the interpretation of $\overline{\rho_m v}$ in terms of i, M, P.

Chapter II

1. Write the boundary conditions in shorthand and in expanded form for the E vector and the B vector at the boundaries between two regions.

a. Region 1 is a good conductor described in terms of η_f, I_f, and M; region 2 is space.

b. Regions 1 and 2 are both dielectrics described in terms of different values of P.

c. Regions 1 and 2 are both simply polarizing dielectrics characterized by different dielectric constants.

d. Region 1 is a perfect conductor described in terms of η_f and I_f; region 2 is a simply polarizing but imperfect dielectric described in terms of a dielectric constant η_f and i_f.

2. Prove that the field equations are consistent with the principle of conservation of electric charge as formulated in the equation of continuity.

3. A very long coaxial cable consists of a solid inner conductor of radius a and of an outer conductor of inner radius b_1 and outer radius b_2. The cable is parallel to the z axis. The space between the two conductors is evacuated. The inner conductor is characterized by a uniform density i_z; the outer conductor by a uniform density $-i'_z$ such that the total current in the inner and outer conductors are the same in magnitude. Determine the B field and the H field for all radii from zero to points completely outside the cable in terms of the total current in the inner conductor. Write the boundary conditions in *coordinate form* at each boundary.

4. Determine the effect on the B field and on the H field in Problem 3 if the medium between the conductors is characterized by a volume density of magnetization $(-M)$ that is proportional to B in magnitude and that is (a) in the same direction as B [diamagnetic medium; use $\nu_0(\nu_r - 1)$ with $\nu_r > 1$ as the proportionality constant]; (b) in the opposite direction [paramagnetic medium; use $\nu_0(\nu_r - 1)$ with $\nu_r < 1$ as the proportionality factor].

5. The inner conductor of a cable like that in Problem 3 is characterized by a uniform surface density of charge η; the inner surface of the outer conductor by a uniform surface density $-\eta'$ such that the total charge per unit length is the same in magnitude on the outer and inner conductors. Determine the E field and the D field for all radii from zero to points completely outside the cable in terms of the total charge per unit length on the inner conductor. Write the boundary conditions in *coordinate form* at each boundary.

6. Determine the effect on the E field and on the D field in Problem 5 if the medium between the two conductors is characterized by a volume density of polarization P that is proportional to E and in the same direction (dielectric medium). Use $\epsilon_0(\epsilon_r - 1)$ as the proportionality factor.

7. Obtain an expression giving direction and magnitude of the force per unit length on each of two infinitely long parallel conductors each with a total axial current I if the currents are

a. In the same direction.

b. In opposite directions.

The distance b between centers is large compared with the radii of the conductors. Use the Ampère-Maxwell theorem of circuitation and the appropriate form of the general expression for electromagnetic force.

8. An extremely long (assume infinitely long) transmission line consists of two parallel wires in air each of radius 1 mm. separated a distance of 5 cm.

between centers.   The two wires carry direct currents in opposite directions; the current in wire 1 is 2 amp., that in wire 2 is 1 amp.   Beginning with the integral form of the field equations and showing all steps, determine numerical values with correct practical units of the B vector at a distance of 3 cm. from wire 2 and 4 cm. from wire 1 in a cross-sectional plane.   Show a diagram with directions of currents and field vector indicated.
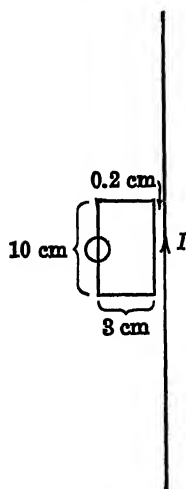
9. Beginning with the *general* definition for electromagnetic force and showing all steps, determine the force per unit length acting on one of the wires in Problem 8.   Assume the current concentrated at the axis for this purpose.   Show a diagram with directions of current, field, and force.

10. The magnitude of the quantity $\mathscr{J}(\mathbf{E}, \mathbf{ds})$ around the small rectangle of wire in the figure is determined by measurement to be 4 volts.   What is the magnitude of the alternating current I opposite the center of the loop?   The frequency is 10 megahertz (megacycles/second).

11. Two spherical pith balls are charged electrostatically so that one has a total surface charge Q_1, the other a total surface charge Q_2.   The balls are separated a distance R_0 between centers, and the charge is uniformly distributed over the surfaces, the interior is uncharged; assume the pith balls to be perfect nonconductors so that no redistribution of charge occurs.   Prove that Coulomb's law $F_R = Q_1 Q_2 / 4\pi \Delta R_0^2$ is true for the balls if distances are measured between centers.   Use Gauss's theorem and the general expression for electrostatic force.

12. The xy plane is the boundary between a good (but not perfect) conductor (region 1) and a perfect dielectric (region 2).   Each is a simple medium with ϵ_r, ν_r, and σ assumed known.

  a. Write the defining relation for E and B in each region and at the boundary *in appropriately specialized forms*.

  b. Assume that the E vector has only an x component, the B vector only a y component (i.e., $E_y = E_z = B_x = B_z = 0$).   Specialize *a* according to these conditions, and show that both E and B satisfy equations of the form

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \text{ in the dielectric}$$

$$\frac{\partial^2 f}{\partial z^2} = K\frac{\partial f}{\partial t} \text{ in the conductor}$$

Express v and K in terms of appropriate parameters and constants.   Obtain the specialized boundary conditions.

  c. Assuming E and B to be determined from the solutions of the equations subject to the boundary conditions in *b*, write down expressions for ρ, η, P; i, I, $-$M in terms of E and B.   If any are

0.2 cm

10 cm

I

3 cm

PROBLEM 10.

zero, state why. For those that are not zero, state what mode of subdivision is implicit.

13. Determine whether salt water $\sigma = 4$ (1/ohm-meter), $\epsilon_r = 80$ may be assumed to be a good conductor at $f = 60$ hertz, $f = 6$ kilohertz, $f = 6$ megahertz, $f = 6,000$ megahertz. Assume a good conductor to satisfy the condition $\sigma/\omega\epsilon \geq 100$.

14. Determine the upper frequency limit for which each of the following may be considered to be a good conductor as defined by $\sigma/\omega\epsilon \geq 100$: copper, distilled water, dry earth, moist earth.

Chapter III

*1. Show that the electromagnetic force defined by II.12.5 may be expanded into

$$\mathsf{F} = \int_\tau \{\rho\mathsf{E} + [\mathsf{i},\mathsf{B}]\}\,d\tau + \int_\Sigma \{\eta\mathsf{E} + [\mathsf{l},\mathsf{B}]\}\,d\sigma + \int_\tau (\mathsf{P},\nabla)\mathsf{E}\,d\tau + \int_\tau (\mathsf{M},\nabla)\mathsf{B}\,d\tau \\ + \int_\tau [\dot{\mathsf{P}},\mathsf{B}]\,d\tau$$

*2. Show that the electromagnetic torque defined by II.12.6 may be expanded into

$$\mathsf{T} = \int_\tau [\mathsf{r},\rho\mathsf{E}]\,d\tau + \int_\Sigma [\mathsf{r},\eta\mathsf{E}]\,d\sigma + \int_\tau [\mathsf{r},[\mathsf{i},\mathsf{B}]]\,\dot{d\tau} + \int_\Sigma [\mathsf{r},[\mathsf{l},\mathsf{B}]]\,d\sigma \\ + \int_\tau [\mathsf{r},(\mathsf{P},\nabla)\mathsf{E}]\,d\tau + \int_\tau [\mathsf{r},(\mathsf{M},\nabla)]\mathsf{B}\,d\tau + \int_\tau [\mathsf{r},[\dot{\mathsf{P}},\mathsf{B}]]\,d\tau + \int_\tau [\mathsf{P},\mathsf{E}]\,d\tau \\ + \int_\tau [\mathsf{M},\mathsf{B}]\,d\tau$$

Do this by showing that

$$\int_\tau [\mathsf{P},\mathsf{E}]\,d\tau + \int [\mathsf{r},(\mathsf{P},\nabla)\mathsf{E}]\,d\tau = -\int_\tau [\mathsf{r},\mathsf{E}]\,\operatorname{div}\mathsf{P}\,d\tau + \int_\Sigma [\mathsf{r},\mathsf{E}](\mathsf{n},\mathsf{P})\,d\sigma$$

$$\int_\tau [\mathsf{M},\mathsf{B}]\,d\tau + \int_\tau [\mathsf{r},(\mathsf{M},\nabla)\mathsf{B}]\,d\tau = \int_\tau [\mathsf{r},[\operatorname{curl}\mathsf{M},\mathsf{B}]]\,d\tau - \int_\Sigma [\mathsf{r},[[\mathsf{n},\mathsf{M}],\mathsf{B}]]\,d\sigma$$

This is most easily accomplished in rectangular coordinates, using the formulas for div $\phi\mathsf{C}$ and curl $\phi\mathsf{C}$ to permit application of the divergence theorem and the curl theorem to each component.

3. The analysis of the motion of electrons in a plane diode (assumed infinite in extent) begins with the relations

$$\frac{dJ_x}{dx} = 0; \qquad J_x = q_x u_x + \epsilon_0 \frac{\partial E_x}{\partial t}$$

$$q_x E_x = M_x \frac{du_x}{dt} = M_x a_x$$

The electrodes lie in the yz plane at $x = 0$ and $x = b$. q_x is the total charge in an infinite slice of unit thickness at x; M_x is the mass of this charge; u_x is its nonrandom velocity; a_x its nonrandom acceleration. Obtain the above relations from the field and force equations. Justify the statement that J_x

is not a function of distance; show that J_x is equal to the total conduction current I_e entering the cathode from the connected circuit; show that $I_e = I_p$.

4. Proceeding from the relations in Problem 3, show first that

$$J_x = \epsilon_0 \frac{dE_x}{dt}$$

where

$$\frac{dE_x}{dt} = \frac{\partial E_x}{\partial x}\frac{dx}{dt} + \frac{\partial E_x}{\partial t}$$

then show that

$$\frac{da_x}{dt} = \frac{eJ_x}{m\epsilon_0}$$

where e is the charge, m the mass of an electron. (Note that $\partial E_x/\partial t$ is the variation in time of E_x at a fixed value of x; dE_x/dt the variation in time of E_x experienced by the moving charge.)

5. Show that if it is assumed that

$$\frac{da_x}{dt} = C + \psi'''(t)$$

where C is a constant and $\psi'''(t)$ the third derivative with respect to the time of a function $\psi(t)$, the mean nonrandom acceleration a_x, velocity u_x, and displacement x of the slice of charge q_x are

$$a_x = C(t - t_c) + \phi''(t) - \phi''(t_c) + a_c$$
$$u_x = \tfrac{1}{2}C(t - t_c)^2 + \phi'(t) - \phi'(t_c) - (t - t_c)\phi''(t_c) + (t - t_c)a_c + u_c$$
$$x = \tfrac{1}{6}C(t - t_c)^3 + \phi(t) - \phi(t_c) - (t - t_c)\phi'(t_c) - \tfrac{1}{2}(t - t_c)^2\phi''(t_c)$$
$$+ \tfrac{1}{2}(t - t_c)^2 a_c + (t - t_c)u_c$$

The initial conditions are assumed to be $a_x = a_c$, $u_x = u_c$ at the cathode at $x = 0$ at time $t = t_c$. The subscript c refers to the cathode.

6. Specialize the general formulas in Problem 5 to the stationary state to obtain $(a_x)_0$, $(u_x)_0$, x_0. The subscript 0 here refers to the stationary state.

7. Verify the following expression for the potential difference between the electrodes of the diode at $x = 0$ and $x = b$ in the stationary state for the special case of such small emission from the cathode that effect of space charge is negligible so that terms in C and u_c are negligible.

$$V_0 = \phi_p - \phi_c = -\frac{m}{2e}(u_p)_0^2 = -\frac{m}{2e}(a_p)_0^2 T^2$$

where $T = t_p - t_c$, $(u_p)_0$ and $(a_p)_0$ are stationary-state values of u_x and a_x at the plate $(x = b)$, and $e/m = -1.76 \times 10^{11}$ coulombs/kilogram. (Note that in the integration of E_x with respect to x, t is a constant.) Show that this is equivalent to the dynamical equation for a positive charge q_x of mass M_x.

$$q_x(\phi_p - \phi_c) = \tfrac{1}{2}M_x(u_p^2 - u_c^2)$$

Show that the transit time $T = t_p - t_c = b\sqrt{-\dfrac{2m}{eV_0}}$

8. Derive Child's law for the stationary state in a plane diode by assuming "complete space charge" with zero initial velocity and acceleration ($a_o = 0$, $u_o = 0$) in the expressions obtained in Problem 6.

$$I_0 = \frac{8\epsilon_0 V_0^{3/2}}{9b^2} \sqrt{-\frac{e}{2m}}$$

Here I_0 is the total d.c. conduction current. Show that the transit time is $\frac{3}{2}$ that obtained in Problem 7 with no space charge.

*9. Show that the scalar potential in the stationary state, *viz.*,

$$\phi = \frac{1}{4\pi\epsilon_0} \left\{ \int_\tau \frac{\bar{\rho}'}{R} \, d\tau' + \int_\Sigma \frac{\bar{\eta}'}{R} \, d\sigma' \right\}$$

is equivalent to

$$\phi = \frac{1}{4\pi\epsilon_0} \left\{ \int_\tau \frac{\rho'}{R} \, d\tau' + \int_\Sigma \frac{\eta'}{R} \, d\sigma' + \int_\tau \left(\nabla' \frac{1}{R}, \mathsf{P}' \right) d\tau' \right\}$$

*10. Show that the vector potential in the stationary state, *viz.*,

$$\mathsf{A} = \frac{1}{4\pi\nu_0} \left\{ \int_\tau \frac{\bar{\mathsf{I}}'}{R} \, d\tau' + \int_\Sigma \frac{\bar{\mathsf{I}}'}{R} \, d\sigma' \right\}$$

is equivalent to

$$\mathsf{A} = \frac{1}{4\pi\nu_0} \left\{ \int_\tau \frac{\mathsf{I}'}{R} \, d\tau' + \int_\Sigma \frac{\mathsf{I}'}{R} \, d\sigma' + \int_\tau \left[\nabla' \frac{1}{R}, -\mathsf{M}' \right] d\tau' \right\}$$

11. Show why $h_e = \sigma_e/\omega\epsilon_e$ may be more nearly independent of frequency in a dielectric than σ_e.

12. Calculate the ratio $h_e = \sigma_e/\omega\epsilon_e$, the extinction coefficient, the attenuation constant, the real index of refraction, the real phase constant, and the real phase velocity in (a) copper, (b) salt water, (c) dry earth, assuming no time lags in conduction or polarization at the following frequencies: $f = 1,000$ hertz, 1 megahertz, 1,000 megahertz.

13. (a) Define a good conductor at a frequency of 600 megahertz. (b) Define a good dielectric at the same frequency.

14. Over what range of frequencies is moist earth (a) a good conductor; (b) a good dielectric. (Neglect time lags in polarization and conduction.)

15. The effective average relative dielectric constant ϵ_{er} for the interelectrode space of a cylindrical diode is 0.62.

    a. Determine the average volume density of charge of the electrons and the average number of electrons per unit volume at a frequency of 100 megahertz.

    b. If an equivalent model using a dielectric between the electrodes could be constructed to have a relative dielectric constant $\epsilon_{er} = 0.62$, would the "dipoles" have to arrange themselves to increase or decrease the electric field?

Chapter IV

1. Carry out the steps to obtain the scalar and vector potential due to a Hertzian dipole given by (IV.12.9) and (IV.12.13).

2. Proceeding from

$$\mathsf{E} = -\text{grad } \phi - j\omega\mathsf{A}$$
$$\mathsf{B} = \text{curl } \mathsf{A}$$

derive the general formulas (IV.12.17) and (IV.12.18) for the electromagnetic field of a Hertzian dipole.

3. Show that the electric field in the far zone of a Hertzian dipole is proportional to sin Θ. Plot the so-called "vertical" field pattern in a polar plot and in a rectangular plot.

4. The leading term in the distribution of current along a thin center-driven cylindrical antenna of half-length $h = \lambda/4$ and with center at $z = 0$ is

$$I_s = I_0 \cos \beta_0 z$$

a. Use this distribution function to determine the complex amplitudes **A**, **E**, and **B** in the far zone in free space.

b. Also obtain expressions for the instantaneous values of these vectors with all phases referred to the input current I_0.

c. Show that the dependence of **E** and **B** in the far zone upon the polar coordinate Θ measured from the axis of the antenna is the so-called vertical field factor given by

$$F(\Theta) = \frac{\cos \left(\dfrac{\pi}{2} \cos \Theta \right)}{\sin \Theta}$$

Plot this as a function of Θ, and compare it with the "vertical" pattern of the Hertzian dipole. Explain the practical value of the field pattern from the point of view of a receiving antenna (placed parallel to **E**) in which the induced voltage across the load is proportional to **E**.

5. The axial distribution of current in a center-driven antenna of half-length h that satisfies the condition $\beta_0 h << 1$ is given approximately by

$$I_s = I_0 \left(1 - \frac{|z|}{h} \right)$$

The axis of the antenna coincides with the z axis of a coordinate system; its center and driving point is at $z = 0$.

a. Use this distribution function to determine **A**, **E**, and **B** in the far zone.

b. Determine and plot the "vertical" field pattern.

6. Using (IV.12.10) for the vector potential of a Hertzian dipole, determine the following, showing all steps:

a. The radiation function $K^2(\Theta)$ for the far-zone field; plot this function in a polar and in a rectangular plot.

b. The Poynting vector in the far zone.

c. The radiation resistance R^e.

d. The absolute directivity D.

Why is it unnecessary to specify a reference current by suitable subscripts in $K^2(\Theta)$ and R^e?

7. Using the formula (Problem 4a)

$$A_s^r = \frac{I_0}{2\pi\beta_0\nu_0} \frac{e^{-j\beta_0 R_0}}{R_0} \frac{\cos \left(\dfrac{\pi}{2} \cos \Theta \right)}{\sin^2 \Theta}$$

for the vector potential in the far zone of an infinitely thin center-driven antenna of half-length h, determine

    a. The radiation function $K_0^a(\Theta)$.

    b. The radiation resistance R_0^c

    c. The absolute directivity D.

(In evaluating R_0^c the integral

$$\int_0^{2\pi} \frac{1 - \cos u}{u}\, du = \overline{Ci}\, 2\pi = 2.438$$

is obtained. The function $\overline{Ci}\, x$ is tabulated in Volume II.)

    8. Determine in Problem 5

    a. The radiation function.

    b. The radiation resistance referred to input current.

    c. The absolute directivity of the antenna.

    d. What must be the input current if the antenna is to radiate 100 watts at a frequency of 1 megahertz when it is 0.2 of a wave length in half-length?

    9. A broadcast antenna is erected vertically over moist earth. Its length is $\frac{5}{8}$ of a wave length; the frequency is 1.5 megahertz. Measurements are to be made at 60 megahertz on a small-scale model. What must be the length of the model and the dielectric constant and conductivity of the earth over which it is erected? Is this physically realizable?

    10. A so-called Abraham dipole is the upper half of a Hertzian dipole erected vertically on a perfectly conducting half-space. Determine its vector potential, electromagnetic field, radiation function, and radiation resistance, and compare with those of the Hertzian dipole.

Chapter V

    1. Calculate and plot to appropriate scales the cross-sectional distribution of current in a No. 18 copper wire of radius $a = 0.51$ mm. at the following frequencies:

    a. $f = 60$ hertz.

    b. $f = 1.5$ megahertz.

    c. $f = 3,000$ megahertz.

    2. Calculate the internal resistance r^i and reactance x^i per unit length for No. 18 copper wire of radius $a = 0.51$ mm. at the following frequencies. Calculate the internal inductance l^i per unit length at 60 hertz only.

    a. $f = 60$ hertz.

    b. $f = 150$ megahertz.

    c. $f = 3,000$ megahertz.

    3. Calculate the internal resistance r^i and reactance x^i, per loop unit length of a coaxial cable consisting of a silvered copper inner conductor of No. 20 wire (radius $= 0.406$ mm.), (σ for silver $= 6.14 \times 10^7$ 1/ohm-meter) and a thinned copper outer conductor (inner radius $= 0.265$ cm.),

$$(\sigma \text{ for tin} = 0.87 \times 10^7 \text{ 1/ohm-meter})$$

Use $f = 3,000$ megahertz.

4. Calculate the internal resistance and reactance per loop unit length of a coaxial cable made entirely of copper. The inner conductor is No. 12 wire (radius 0.103 cm.); the outer conductor has an inner radius of 0.38 cm. (σ for copper is 5.8×10^7 1/ohm-meter.) Use $f = 150$ megahertz.

5. *a.* Calculate the thickness of copper at which the amplitude of the electric field tangent to a plane boundary is reduced to 1 per cent of the value at the boundary for 1 megahertz and for 1,000 megahertz. Use $\sigma = 5.80 \times 10^7$ 1/ohm-meter.

     b. Repeat for aluminum with $\sigma = 3.54 \times 10^7$ 1/ohm-meter.

     c. Repeat for sheet steel with $\sigma = 1.0 \times 10^7$ 1/ohm-meter and with $\nu_r = 10^{-3}$ ($\mu_r \doteq 10^3$).

6. *a.* Calculate the depth below the earth's surface at which the amplitude of the current density is reduced to $1/e$ of its value at the surface. Use $\sigma_e = 6 \times 10^{-4}$ 1/ohm-meter, $\epsilon_{er} = 4$, $\nu_r = 1$ for dry earth at 10 megahertz.

     b. Repeat for wet earth with $\sigma_e = 3 \times 10^{-2}$ 1/ohm-meter, $\epsilon_{er} = 10$, $\nu_r = 1$.

     c. Repeat for salt water using $\sigma_e = 4.3$ 1/ohm-meter, $\epsilon_{er} = 80$, $\nu_r = 1$.

7. Calculate the fraction of the total current above the depth at which the current density is $1/e$ of the value at the surface.

8. Determine the skin depth and the surface impedance of

     a. Salt water at 60 megahertz.

     b. Brass at 3,000 megahertz.

     c. Copper at 3,000 megahertz.

     d. Silver at 10,000 megahertz.

9. Compare the internal impedance per unit length of the copper wire in Problem V-1 with that of a dielectric rod with a uniform surface layer of copper 0.01 mm. thick if the outer radius is the same as the copper wire. Use the same frequencies as in Problem V-1.

10. Calculate the ratio of the electric field just outside and just inside an infinitely long aluminum tube of wall thickness 2 mm. and radius 8 cm. at (*a*) 1.5 megahertz; (*b*) 3,000 megahertz.

11. How thick must an iron tube be in order to serve as an effective shield at 60 hertz? (Use $\nu_r = 0.001$ for iron.)

12. Calculate the internal impedance per loop meter of a coaxial line made of copper operated at 3,000 megahertz. The inner conductor has a radius 0.406 mm., the outer conductor an inner radius of 0.254 cm. and a wall thickness of 1 mm.

13. Calculate the internal reactance of a circular condenser of radius 2 cm. and plate separation 1 m. at 300 kilohertz and 300 megahertz.

Chapter VI

1. Calculate the impedance of a rectangular loop of sides 1 meter by 2 meters made of No. 10 copper wire and driven at a frequency of 5 megahertz. If the current in the loop is 5 amp., what is the radiated power? What is the radiation efficiency?

2. Repeat Problem 1 if the same wire forms a circle instead of a rectangle.

3. Calculate the inductance of a helical coil of six turns with pitch equal to one-half the radius of the coil which is 5 cm. The radius of the wire is 0.05 cm.

4. Calculate the resistance and the inductance per unit length of a two-wire line forming the tank circuit of an U. H. F. oscillator. It is made of brass tubes $\frac{1}{4}$ in. in diameter and separated a distance of $\frac{1}{2}$ in. between centers. It oscillates at a frequency of 300 megahertz.

5. Calculate the impedance of a circular loop of wire 20 cm. in diameter in the presence of a completely closed identical loop, coaxial with it but at a distance of 10 cm. along the axis. The first loop is driven by a slice generator at 50 megahertz. Both loops are made of copper wire of radius 1 mm. What power is radiated if a current of 1 amp. is in the driven loop?

6. A square loop is constructed of a single turn of heavy copper wire (radius 1 mm.; $\sigma = 5.8 \times 10^7$ mho/meter). The square has sides 1 meter long. It is driven at the center of one side by a generator operated at 10 megahertz maintaining a potential difference of 1,000 volts r.m.s.

At a distance of 100 km. from the driven loop is an identical receiving loop but with a load of 1,000 ohms pure resistance connected at the center of one side instead of a generator. The two loops are in the same plane with parallel pairs of sides. Assume that they are isolated from all else.

    a. Determine the impedance seen by the generator in the driven loop and the current in the loop.

    b. What power and what fraction of the total power is radiated?

    c. Determine the magnitude of the current in and potential difference across the load in the receiving loop. (*Hint:* Note and verify that each loop is sufficiently small that quasi-near-zone formulas may be used for self-impedances. For mutual impedance, general formulas but with $f(s) = 1$ are required. In evaluating Z_{12}, note that with $\beta_0 R_{12}$ as large as it is, four of the eight integrals obtained are practically equal and opposite in pairs, and in the other four $e^{-j\beta_0 R_{12}}/R_{12}$ may be taken out from under the sign as sensibly constant because $\beta_0 s$ is small.)

7. Repeat Problem 6 if the center of each loop is 2.5 m. above a perfectly conducting half-space and the planes of the loop are perpendicular to the conducting surface.

8. Repeat Problem 7 if the planes of the loops are parallel to the conducting plane.

*9. Investigate the impedance of an equilateral triangle of wire driven at one apex if the conditions for the quasi-near zone apply.

10. A resonant section of two-wire line terminated in wire bridges for which $\Phi_0' = \Phi_s' = \beta_0 b/2$ (where $b = 1$ cm. is the distance between centers of wires) is 49 cm. long. It is driven at a frequency of 3,000 megahertz. The maximum current in the line is 0.4 amp. What power is radiated if the currents in the two wires are equal and opposite? Assume the radius a of the wire satisfies $a^2 \ll b^2$. Repeat for distances between centers of wires of 0.5 and 2 cm.

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